Szygies of some degenerate K3 surfaces and Green's conjecture in positive characteristic

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Instruments of Algebraic Geometry
Bucharest, 18 September 2017

A family of reducible K3 surfaces

Let $a \geq b \geq 2$ be two integers and consider in $\mathbb{P}^{a+b+1}$ with coordinates $x_0, \ldots, x_a, y_0, \ldots, y_b$ the scheme $X_e(a, b)$ defined by the 2×2 minors of

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_{a-1} \\ x_1 & x_2 & \ldots & x_a \end{pmatrix} \text{ and } \begin{pmatrix} y_0 & y_1 & \ldots & y_{b-1} \\ y_1 & y_2 & \ldots & y_b \end{pmatrix}$$

and the entries of the $(a-1) \times (b-1)$ matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ \vdots & \vdots & \vdots \\ x_{a-2} & x_{a-1} & x_a \end{pmatrix} \begin{pmatrix} 0 & 0 & e_2 \\ 0 & -e_1 & 0 \\ 1 & 0 & o \end{pmatrix} \begin{pmatrix} y_0 & y_1 & \ldots & y_{b-2} \\ y_1 & y_2 & \ldots & y_{b-1} \\ y_2 & y_3 & \ldots & y_b \end{pmatrix}$$

for parameters e_1, e_2 in our ground field $\mathbb{K}$.

Geometry of $X_e(a, b)$

If $t^2 - e_1 t + e_2 = (t - t_1)(t - t_2) \in \mathbb{K}[t]$ has distinct nonzero roots, then $X_e(a, b)$ is the union of two rational normal surface scrolls defined by the 2×2 minors of

$$m_\ell = \begin{pmatrix} x_0 & x_1 & \dots & x_{a-1} & y_0 & \dots & y_{b-1} \\ x_1 & x_2 & \dots & x_a & t_\ell y_1 & \dots & t_\ell y_b \end{pmatrix}.$$

Its hyperplane section consist of two rational normal curves of degree $a + b$ intersecting in $a + b + 2$-points, i.e canonically embedded stable nodal curves of genus $g = a + b + 1$. If $a \geq b$ then their expected Clifford index is b.
In case $t_1 = t_2 \neq 0$ the scheme $X_e(a, b)$ is a double structure on the rational normal scroll defined by m_ℓ, hence a K3-carpet.

Gröbner basis and Schreyer resolution

Theorem

1. *The defining equations of $X_e(a,b)$ form a Gröbner basis with initial terms*

$$\begin{aligned} &x_i x_j \quad \textit{for } 1 \le i \le j \le a-1, \\ &x_i y_j \quad \textit{for } 2 \le i \le a, 0 \le j \le b-2, \\ &y_i y_j \quad \textit{for } 1 \le i \le j \le b-1. \end{aligned}$$

2. *The "Schreyer" algorithm computes a free resolution defined over $\mathbb{Z}[e_1, e_2]$, which for the initial ideal specalises to the minimal free resolution of the monomial ideal.*

Except for Borel fixed ideals the "Schreyer" algorithm rarely computes the minimal resolution even for monomial ideals. The next few slides sketch the argument which proves the minimality in our special case.

Steps of the Schreyer algorithm and Induced orders

Given a Gröbner basis $\langle f_1, \ldots, f_n \rangle \subset K[x_0, \ldots, x_r]$ we sort them by the degree refined by the reverse lexicographic of their initial forms. Next we compute the monomial ideal

$$M_i = \langle in(f_1), \ldots, in(f_i) \rangle : in(f_i).$$

For each monomial generator $x^\alpha \in M_i$ Buchberger's test for Gröbner basis provides a syzygy

$$g \in \ker(S^n \to S), e_i \mapsto f_i$$

with lead term $x^\alpha e_i$ with respect to the induces monomial order on S^n defined by

$$x^\alpha e_i > x^\beta e_j \Leftrightarrow x^\alpha in(f_i) > x^\beta in(f_j) \text{ or equality and } i > j.$$

These syzygies form a Gröbner basis for $\ker(S^n \to S)$. The algorithm proceeds with syzygies among the generators of the M_i and gives a finite free resolution

$$S \leftarrow F_1 \leftarrow F_2 \leftarrow \ldots \leftarrow F_c \leftarrow 0.$$

Names of syzygies

It is convenient to use in addition to the induced orders also recursively defined names for each generator $e_k \in F_p$ in the free resolution

$$S \leftarrow F_1 \leftarrow F_2 \leftarrow \ldots \leftarrow F_c \leftarrow 0.$$

Definition

The generator $e_i \in F_1 = S^n$ gets as name the monomial $in(f_i)$,

$$name(e_i) := in(f_i).$$

For $p > 1$ and a generator $e_k \in F_p = S^{n_p}$ with $in(\varphi_p(e_k)) = x^\beta e_\ell \in F_{p-1}$ we define

$$name(e_k) := name(e_\ell), x^\beta.$$

Thus a name of a generator of F_p consist of a sequence of p monomials.

Names of syzygies

Example

Consider the resolution of the monomial ideal $\langle wy, wz, xy, xz \rangle \subset S = K[w, x, y, z]$

$$S \leftarrow S^4 \leftarrow S^4 \leftarrow S \leftarrow 0$$

with $\varphi_2 = \begin{pmatrix} -z & -x & 0 & 0 \\ y & 0 & -x & 0 \\ 0 & w & 0 & -z \\ 0 & 0 & w & y \end{pmatrix}$ and $\varphi_3 = \begin{pmatrix} -x \\ z \\ -y \\ w \end{pmatrix}$

The name of the generators of $F_2 = S^4$ are

$$\begin{array}{ll} name(e_1) = \{wx, y\} & name(e_2) = \{xy, w\} \\ name(e_3) = \{xy, w\} & name(e_4) = \{xz, y\} \end{array}$$

and for $e_1 \in F_3$ we have $name(e_1) = \{xz, y, w\}$.

Syzygies of $X_e(a, b)$

The ideal of $X_e(a, b)$ has $n = \binom{a+b-1}{2}$ generators. For $1 \leq k \leq n-1$ the monomial ideals M_i are very simple:

$in(f_k)$	range	M_k
$x_i x_j$	$1 \leq i \leq j \leq a-1$	$\langle x_1, \ldots, x_{j-1} \rangle$
$x_i y_j$	$2 \leq i \leq a-1, 0 \leq j \leq b-2$	$\langle x_1, \ldots, x_{a-1}, y_0, \ldots, y_{j-1} \rangle$
$x_a y_j$	$0 \leq j \leq b-2$	$\langle x_2, \ldots, x_{a-1}, y_0, \ldots, y_{j-1}, x_1^2 \rangle$
$y_i y_j$	$1 \leq i \leq j \leq b-2$	$\langle x_2, \ldots, x_{a-1}, y_1, \ldots, y_{j-1}, x_1^2 \rangle$
$y_i y_{b-1}$	$1 \leq i < b-1$	$\langle x_2, \ldots, x_{a-1}, y_1, \ldots, y_{b-2}, x_1^2 \rangle$

The last one is more complicated. For f_n whose initial form is $in(f_n) = y_{b-1}^2$ we get

$$M_n = \langle y_1, \ldots, y_{b-2}, x_1^2, x_1 x_2, \ldots, x_{a-1}^2, x_2 y_0, \ldots, x_a y_0, \rangle$$

Sketch of the minimality

Proposition

The Schreyer resolution of the ideal $\langle in(f_1), \ldots, in(f_{n-1})\rangle$ above is the minimal free resolution.

Proof. Let G denote the Schreyer resolution. The names of the generators of G_p are an initial monomial of an f_k followed by a decreasing sequence of distinct elements of M_k of length $p-1$, since each M_k is generated by a regular sequence of monomials.
We can recover the total degree of a generator as the degree of the product of the monomials of the name, which we call the name product for short. So the generators of G_p have degree $p+1$ and $p+2$. To see the minimality we utilise that the $\mathbb{Z}^{a+b+2}$-grading of the monomial ideal induces a $\mathbb{Z}^{a+b+2}$ -grading on G and that the multidegree of a generator coincides with the multidegree of its name product.

Sketch of the minimality

$in(f_k)$	range	M_k
$x_i x_j$	$1 \leq i \leq j \leq a-1$	$\langle x_1, \ldots, x_{j-1} \rangle$
$x_i y_j$	$2 \leq i \leq a-1, 0 \leq j \leq b-2$	$\langle x_1, \ldots, x_{a-1}, y_0, \ldots, y_{j-1} \rangle$
$x_a y_j$	$0 \leq j \leq b-2$	$\langle x_2, \ldots, x_{a-1}, y_0, \ldots, y_{j-1}, x_1^2 \rangle$
$y_i y_j$	$1 \leq i \leq j \leq b-2$	$\langle x_2, \ldots, x_{a-1}, y_1, \ldots, y_{j-1}, x_1^2 \rangle$
$y_i y_{b-1}$	$1 \leq i < b-1$	$\langle x_2, \ldots, x_{a-1}, y_1, \ldots, y_{b-2}, x_1^2 \rangle$

Since each name product of a generator of G_p of degree $p+2$ is divisible by x_1^2 and some y_j and the only products names of generators of G_{p+1} of degree $p+2$ which are divisible by x_1^2 are monomials in $K[x_1, \ldots, x_{a-1}]$ there is no constant terms in the differential $G_p \leftarrow G_{p+1}$. $\square$

The proof of the minimality statement in the Theorem uses the same ideas.

Syzygies of small K3 carpets in arbitrary characteristic

For $(a,b) = (6,6)$ the non-minmal resolution has Betti table

0	1	2	3	4	5	6	7	8	9	10	11
1	.	.	.	.	.	.	.	.	.	.	.
.	55	320	930	1688	2060	1728	987	368	81	8	.
.	.	39	280	906	1736	2170	1832	1042	384	83	8
.	.	.	1	8	28	56	70	56	28	8	1

The crucial constant strand

$$0 \leftarrow \mathbb{Z}^8 \leftarrow \mathbb{Z}^{1736} \leftarrow \mathbb{Z}^{1728} \leftarrow 0$$

has a surjective first map, which leads to a 1728×1728 matrix M with determinant

$$\det M = 2^{1312}\, 3^{72}\, 5^{120}.$$

Syzygies of small K3 carpets in arbitrary characteristic

In characteristic 0 or characteristic $p \neq 2, 3, 5$ the Betti table is

	0	1	2	3	4	5	6	7	8	9	10	11
	1	.	.	.	.	.	.	.	.	.	.	.
	.	55	320	891	1408	1155	.	.	.	.	.	.
	.	.	.	.	.	.	1155	1408	891	320	55	.
	.	.	.	.	.	.	.	.	.	.	.	1

Syzygies of small K3 carpets in arbitrary characteristic

In characteristic 0 or characteristic $p \neq 2, 3, 5$ the Betti table is

0	1	2	3	4	5	6	7	8	9	10	11
1	.	.	.	.	.	.	.	.	.	.	.
.	55	320	891	1408	1155	.	.	.	.	.	.
.	.	.	.	.	.	1155	1408	891	320	55	.
.	.	.	.	.	.	.	.	.	.	.	1

For the exceptional primes $p = 2, 3, 5$ we get

1	.	.	.	.	.	.	.	.	.	.	.
.	55	320	900	1488	1470	720	315	80	9	.	.
.	.	9	80	315	720	1470	1488	900	320	55	.
.	.	.	.	.	.	.	.	.	.	.	1

1	.	.	.	.	.	.	.	.	.	.	.
.	55	320	891	1408	1162	48	7	.	.	.	.
.	.	.	.	7	48	1162	1408	891	320	55	.
.	.	.	.	.	.	.	.	.	.	.	1

Syzygies of small K3 carpets in arbitrary characteristic

For the exceptional primes $p = 2, 3, 5$ we get

1	.	.	.	.	.	.	.	.	.	.	.
.	55	320	900	1488	1470	720	315	80	9	.	.
.	.	9	80	315	720	1470	1488	900	320	55	.
.	.	.	.	.	.	.	.	.	.	.	1

1	.	.	.	.	.	.	.	.	.	.	.
.	55	320	891	1408	1162	48	7	.	.	.	.
.	.	.	.	7	48	1162	1408	891	320	55	.
.	.	.	.	.	.	.	.	.	.	.	1

1	.	.	.	.	.	.	.	.	.	.	.
.	55	320	891	1408	1155	120	.	.	.	.	.
.	.	.	.	.	120	1155	1408	891	320	55	.
.	.	.	.	.	.	.	.	.	.	.	1

by computing the Smith normal forms of the non-minimal map in the Schreyer resolution.

Syzygies of small K3 carpets in arbitrary characteristic

For the exceptional primes $p = 2, 3, 5$ we get

1	.	.	.	.	.	.	.	.	.	.	.
.	55	320	900	1488	1470	720	315	80	9	.	.
.	.	9	80	315	720	1470	1488	900	320	55	.
.	.	.	.	.	.	.	.	.	.	.	1

1	.	.	.	.	.	.	.	.	.	.	.
.	55	320	891	1408	1162	48	7	.	.	.	.
.	.	.	.	7	48	1162	1408	891	320	55	.
.	.	.	.	.	.	.	.	.	.	.	1

1	.	.	.	.	.	.	.	.	.	.	.
.	55	320	891	1408	1155	120	.	.	.	.	.
.	.	.	.	.	120	1155	1408	891	320	55	.
.	.	.	.	.	.	.	.	.	.	.	1

by computing the Smith normal forms of the non-minimal map in the Schreyer resolution. Can explain the case for $p = 2, 3$!

Resonance or Poncelet phenomena

Theorem

Suppose t_1/t_2 is a primitive k-th root of unity and $a, b \geq k+1$.

1. *$X_e(a, b)$ is contained in a rational normal scroll of type $S(a_0, \dots, a_{k-1}, b_0, \dots, b_{k-1})$ with*

$$a_i = |\{0 \leq j \leq a \mid j \equiv i \mod k\}| - 1$$

   and

$$b_i = |\{0 \leq j \leq b \mid j \equiv i \mod k\}| - 1.$$

2. *The map $S(a_0, \dots, a_{k-1}, b_0, \dots, b_{k-1}) \to \mathbb{P}^1$ induces a fibration of $X_e(a, b)$ into $2k$-gons.*
3. *If $a, b \geq 2k^2$ then $X_e(a, b)$ has non-zero graded Betti number precisely in the range where a general $2k$-gonal curve of genus $g = a + b + 1$ has non-zero Betti numbers.*

Interpretation of the Betti numbers of $X(6,6)$

The extra syzygies for the primes $p = 2, 3$ are explained by the resonance:

1	.	.	.	.	.	.	.	.	.	.	.
.	55	320	900	1488	1470	720	315	80	9	.	.
.	.	9	80	315	720	1470	1488	900	320	55	.
.	.	.	.	.	.	.	.	.	.	.	1

1	.	.	.	.	.	.	.	.	.	.	.
.	55	320	891	1408	1162	48	7	.	.	.	.
.	.	.	.	7	48	1162	1408	891	320	55	.
.	.	.	.	.	.	.	.	.	.	.	1

$t^2 - 2t + 1 \equiv t^2 - 1 \mod 2$ and $t^2 - 2t + 1 \equiv t^2 + t + 1 \mod 2$ leading to 4-gonal respectively 6-gonal curves.

However the case $p = 5$ is a different phenomena.

Conjectural exceptional characteristics for Green's Conjecture for general smooth curves of genus $g \leq 15$

genus	char($\Bbbk$)	extra syzygies
7	2	$\beta_{2,4} = 1$
9	3	$\beta_{3,5} = 6$
11	2, 3	$\beta_{4,6} = 28,\ 10$
12	2	$\beta_{4,6} = 1$
13	2, 5	$\beta_{5,7} = 64,\ 120$
15	2, 3, 5	$\beta_{6,8} = 299,\ 390,\ 315$

Refined (generic) Green Conjecture

Let $C \subset \mathbb{P}^{g-1}$ a be a smooth canonically embedded curve and let *strand*$_2(S_C)$ denote the second linear strand

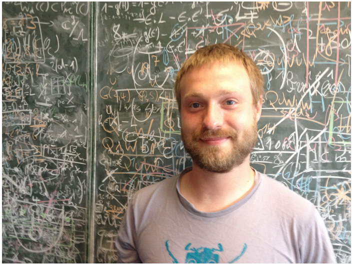
$$0 \leftarrow S(-3)^{\beta_{1,3}} \xleftarrow{\varphi_2} S(-4)^{\beta_{2,4}} \xleftarrow{\varphi_3} \dots \xleftarrow{\varphi_{g-3}} S(-(g-1))^{\beta_{g-3,g-1}} \leftarrow 0$$

of a minimal free resolution of the coordinate ring S_C (here $S(-(i+2))^{\beta_{i,i+2}}$ sits in homological degree i). Then

(a) $H_i($*strand*$_2(S_C))$ is a module of finite length for all $i \leq p$ if and only if $p < \mathrm{Cliff}(C)$.

(b) If C is general inside the gonality stratum $\mathcal{M}^1_{g,k} \subset \mathcal{M}_g$ with $2 < k < \lceil \frac{g+2}{2} \rceil$ then $H_{k-2}($*strand*$_2(S_C))$ is supported on the rational normal scroll swept out by the unique g^1_k on C.

Experimental evidence for the Green's conjecture in positive characteristic performed by



Christian Bopp.

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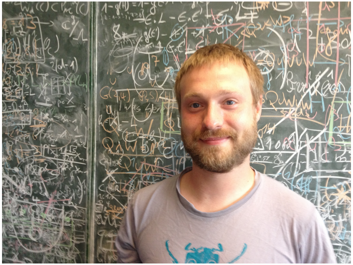

Christian Bopp.

Notation: Let φ be the first non-zero map in *strand*$_2(S_C)$. In the following let

$$X = \operatorname{supp}(\operatorname{coker} \varphi)$$

denote the support of the cokernel.

Betti tables of 500 random examples of genus 9 curves over $\mathbb{F}_3$

#	$(\deg X, \dim X)$	Betti table
350	$(6, 0)$	$\begin{array}{\|cccccccc} 1 & . & . & . & . & . & . & . \\ . & 21 & 64 & 70 & 6 & . & . & . \\ . & . & . & 6 & 70 & 64 & 21 & . \\ . & . & . & . & . & . & . & 1 \end{array}$
103	$(5, 4)$	$\begin{array}{\|cccccccc} 1 & . & . & . & . & . & . & . \\ . & 21 & 64 & 70 & 8 & . & . & . \\ . & . & . & 8 & 70 & 64 & 21 & . \\ . & . & . & . & . & . & . & 1 \end{array}$
31	$(10, 4)$	$\begin{array}{\|cccccccc} 1 & . & . & . & . & . & . & . \\ . & 21 & 64 & 70 & 10 & . & . & . \\ . & . & . & 10 & 70 & 64 & 21 & . \\ . & . & . & . & . & . & . & 1 \end{array}$
16	$(6, 3)$	$\begin{array}{\|cccccccc} 1 & . & . & . & . & . & . & . \\ . & 21 & 64 & 75 & 24 & 5 & . & . \\ . & . & 5 & 24 & 75 & 64 & 21 & . \\ . & . & . & . & . & . & . & 1 \end{array}$

Betti tables of 500 random examples of genus 11 curves over $\mathbb{F}_2$

#	$(\deg X, \dim X)$	Betti table
230	$(60, 0)$	1 . . . . 36 160 315 288 28 . . . . 28 288 315
76	$(6, 5)$	1 . . . . 36 160 315 288 30 . . . . . 30 288 315
⋮	$(6n, 5)$	1 . . . . 36 160 315 288 28+2n . . . . . 28+2n 288 315
2	$(48, 5)$	1 . . . . 36 160 315 288 44 . . . . . 44 288 315
1	$(60, 5)$	1 . . . . 36 160 315 288 50 . . . . . 50 288 315

Betti tables of 500 random examples of genus 11 curves over $\mathbb{F}_3$

#	$(\deg X, \dim X)$	Betti table
311	$(12, 0)$	1 . . . . 36 160 315 288 10 . . . . . 10 288 315
95	$(6, 5)$	1 . . . . 36 160 315 288 14 . . . . . 14 288 315
$\vdots$	$(6n, 5)$	1 . . . . 36 160 315 288 10+4n . . . . . 10+4n 288 315
4	$(36, 5)$	1 . . . . 36 160 315 288 34 . . . . . 34 288 315

Summary

To prove for a balanced carpet $X(a,a)$ Green's conjecture in characteristic 0 amounts to verify the determinant of $f(a) \times f(a)$ integer matrix M_a has a non-zero determinant, where

$$f(a) = \frac{4a^2}{a+1}\binom{2a-3}{a-3}.$$

By the resonance theorem we know that

$$2^{a\binom{2a-2}{a+1}}$$

is a factor of this determinant. For small a the absolute value of this determinant are

a	2	3	4	5	6	7
$f(a)$	0	9	64	350	1728	8085
$a\binom{2a-2}{a+1}$	0	3	24	140	720	3465
det	1	2^6	2^{54}	$2^{271}\,3^6$	$2^{1312}\,3^{72}\,5^{120}$	?

Any sensible proof should explain the factors.

Summary for balanced $X_e(a, a)$

a	$f(a)$	ann coker M_a
3	9	$2\,(t_1 t_2)\,(t_1 + t_2)$
4	64	$3\,(t_1 t_2)^2 (t_1 + t_2)^2$
5	350	$12 (t_1 t_2)^4 (t_1 + t_2)^3 (t_1^2 + t_1 t_2 + t_2^2)$
6	1728	$10 (t_1 t_2)^6 (t_1 + t_2)^4 (t_1^2 + t_1 t_2 + t_2)^2$

The polynomial prime factors are nicely explained by the resonance phenomenon. The integer factors are mysterious!

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Thank You!