Unirational moduli, Hurwitz spaces and random curves

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Introduction

The moduli spaces $\mathcal{M}_g$ of curves of genus g is

- unirational for $g \leq 14$, [Severi, Sernesi, Chang-Ran, Verra],
- of general type for $g = 22$ and $g \geq 24$, [Harris-Mumford, Eisenbud-Harris, Farkas].

The cases in between are not fully understood:

- $\mathcal{M}_{23}$ has positive Kodaira dimension [Farkas],
- $\mathcal{M}_{15}$ is rationally connected [Bruno-Verra] ,
- $\mathcal{M}_{16}$ is uniruled [Chang-Ran, Farkas].

Introduction

In this talk I will report on unirationality proofs for moduli spaces. The emphasis will lie on the construction technique, trying to point out, where (from my point of view) the methods fail for the next cases. We will focus on

- Hurwitz schemes $\mathcal{H}_{g,d} = \{C \xrightarrow{f} \mathbb{P}^1\} \to W^1_{g,d}$ of degree d covers of $\mathbb{P}^1$ by curves of genus g,
- Severi varieties $\mathcal{V}_{g,d} \to W^2_{g,d}$ of degree d nodal plane curves of geometric genus g,
- further spaces $W^r_{g,d}$ for $r \geq 3$.

Brill-Noether theory

A general curve C of genus g has a linear system g^r_d of dimension r of divisors of degree d if and only if the Brill-Noether number

$$\rho = \rho(g, r, d) = g - (r + 1)(g + r - d)$$

is non-negative. Moreover, in this case, the Brill-Noether scheme

$$W^r_d(C) = \{L \in \mathrm{Pic}^d(C) \mid h^0(L) \geq r + 1\}$$

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has dimension ρ. Recall some notation from [ACGH]:

$$\mathcal{M}^r_{g,d} = \{C \in \mathcal{M}_g \mid \exists L \in W^r_d(C)\},$$

$$W^r_{g,d} = \{(C, L) \mid C \in \mathcal{M}^r_{g,d}, L \in W^r_d(C)\}$$

and

$$G^r_{g,d} = \{(C, L, V) \mid (C, L) \in W^r_{g,d}, V \subset H^0(L), \dim V = r + 1\}.$$

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$$G^r_{g,d} = \{(C, L, V) \mid (C, L) \in W^r_{g,d}, V \subset H^0(L), \dim V = r+1\}.$$

Then we have natural morphisms

$$\mathcal{H}_{g,d} \to G^1_{g,d} \to W^1_{g,d} \to \mathcal{M}^1_{g,d} \subset \mathcal{M}_g.$$

Color coding

Color coding indicates where $W^1_{g,d}$ is known to be unirational, uniruled or not unirational.

Results are due to

- Petri ($d \leq 5$) (1923) or B. Segre ($d = 5$) (1928)
- Harris and Mumford (1982)
- Chang and Ran (1984)
- Eisenbud and Harris (1987)
- Mukai ($g \leq 9$) (1995)
- Farkas (2000), Verra (2005)
- Geiß (2012)
- Bini, Fontanari and Viviani (2012)
- Farkas and Verra (2013)
- Casalaina-Martin, Kass and Viviani (2014)
- Damadi, Schreyer and Tanturri (2016)

g / d	**2**	**3**	**4**	**5**	6	7	8	9	10	11	12	13	14
45				P	G								
|				|									
40				P	G								
|				|									
36				P	G								
35				P	G								
34				P									
33				P	G								
32				P									
31				P	G								
30				P	G								
29				P									
28				P	G								
27				P	G								
26				P	G								EH
25				P	G								HM
24				P	G							EH	EH
23				P	G							HM	HM
22				P	G						F	F	F
21				P	G								
|				|	|							
|				|	|							
16				P	G								
15				P	G			V					
14				P	G		V						FV
13				P	G	ST		ST				FV	CKV
12				P	G	G	ST	S			FV	CKV	CKV
11				P	G	G	CR			FV	CKV	CKV	CKV
10				P	G	G			FV	CKV	CKV	CKV	BFV
9				P	G	G	DS	M	M	M	M	M	M
8				P	|	G	M	M	M	M	M	M	M
7				P	|	M	M	M	M	M	M	M	M
6													
|													
1													

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Mukai's Theorem

A general canonical curve C of genus $g = 7, 8, 9$ arises as transversal intersection of a linear space with a homogeneous variety:

7	$C = \mathbb{P}^6 \cap \mathrm{Spinor}^{10} \subset \mathbb{P}^{15}$	isotropic subspaces of $Q^8 \subset \mathbb{P}^9$
8	$C = \mathbb{P}^7 \cap \mathbb{G}(2,6)^8 \subset \mathbb{P}^{14}$	Grassmannian of line in $\mathbb{P}^5$
9	$C = \mathbb{P}^8 \cap \mathbb{L}(3,6)^6 \subset \mathbb{P}^{13}$	Lagrangian subspaces of $(\mathbb{C}^6, \omega)$

$\Rightarrow$ the moduli spaces $\mathcal{M}_{g,g}$ of g-pointed curves of genus g and the universal Picard varieties $\mathrm{Pic}_g^d \to \mathcal{M}_g$ are unirational for $g \leq 9$.

$\Rightarrow$ $\mathcal{M}^1_{g,d}$ and $\mathcal{H}_{g,d}$ are unirational for $g \leq 9$ and $d \geq g$.

Petry's Theorem on 5-gonal curves

Let $C \to \mathbb{P}^1$ be given by a complete linear series of degree 5. The canonical image of C lies on a 4-dimension scroll X

$$C \subset X = \mathbb{P}(\mathcal{E}) \subset \mathbb{P}^{g-1}$$

of a rank 4 bundle $\mathcal{E} = \mathcal{O}(e_1) \oplus \ldots \oplus \mathcal{O}(e_4)$ degree $f = g - 4$ over $\mathbb{P}^1$.

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$$0 \to \mathcal{O}_X(-5H + (f-2)R) \to$$

$$\oplus_{j=1}^{5} \mathcal{O}_X(-3H + b_j R) \xrightarrow{\psi} \oplus_{i=1}^{5} \mathcal{O}_X(-3H + a_i R) \to$$

$$\mathcal{O}_X \to \mathcal{O}_C \to 0$$

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where the middle matrix $\psi = (\psi_{ij})$ is skew-symmetric with entries $\psi_{ij} \in H^0(\mathcal{O}_X(H - (b_j - a_i)R))$ and $a_i + b_i = f - 2$. The other maps have entries the 4×4 pfaffians of ψ (in accordance with the Buchsbaum-Eisenbud structure theorem)

$\Rightarrow \mathcal{H}_{g,5}$ and $M^1_{g,5}$ are unirational for all $g \geq 7$.

Florian Geiß' approach to 6-gonal curves

No structure theorem for Gorenstein rings in codimension 4.
Not even for the Betti table

	0	1	2	3	4
0	1	.	.	.	.
1	.	9	16	9	.
2	.	.	.	.	1

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Let $|L_1| = g^1_6$. Consider in addition an $|L_2| = g^2_d$ and the embedding

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Think of C as a family of 6 points in $\mathbb{P}^2$. The ideal of six points in $\mathbb{P}^2$ is generated by cubics, and they are linked via two cubics to three points. Thus C is linked to a trigonal curve E via two hypersurface of bi-degree $(a_1, 3), (a_2, 3)$. E might be easier to construct.

The case $W^1_{10,6}$

Since $10 > 3 \cdot 3$ a curve of genus 10 has a 1-dimensional family of g^2_9. So $C \subset \mathbb{P}^1 \times \mathbb{P}^2$ has intersection numbers $C.A = 6$, $C.B = 9$ with the two generators A, B of the Picard group.

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$$h^0\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(a, 3) = (a + 1)10, h^0\mathcal{O}_C(a, 3) = 6a + 3 \cdot 9 + 1 - 10$$

hence $\quad h^0\mathcal{J}_C(a, 3) \geq 4a - 8 \geq 2$ for $a = 3$.

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$$(C+E).A = (3A+3B)^2.A = 9 = 6+3,$$

$$(C+E).B = (3A+3B)^2.B = 18 = 9+9$$

and $g_C - g_E = \frac{1}{2}(C-E).(4A+3B) = (6-3)\cdot 2 = 6$. Thus E is a genus $10-6=4$ curve of bi-degree $(3,9)$ in $\mathbb{P}^1 \times \mathbb{P}^2$ which are easy to construct.

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$h^0\mathcal{J}_E(3,3) = 40 - (3\cdot(3+9)+1-4) = 7 > 2$ we can reverse the construction. Moreover we can impose that the two hypersurfaces pass to 5 general points in $\mathbb{P}^1 \times \mathbb{P}^2$.
$\Rightarrow W^1_{10,6,5} \to \mathcal{M}_{10,5}$ are both unirational.

Why does this approach to $W^1_{g,6}$ fails for $g \gg 0$?

A general curve C of genus g has a g^2_d if only if

$$3 \cdot (g - d + 2) \leq g \Leftrightarrow d \geq \frac{2g+6}{3}.$$

$$h^0\mathcal{O}_C(a,3) \approx 6a + 2g + 6 + 1 - g \leq 10(a+1) - 1$$

$$\Leftrightarrow g + 8 \leq 4a$$

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The two hypersurface have bi-degree $(a_1, 3), (a_2, 3)$ with

$$a_1 \approx a_2 \approx g/4 + 2$$

and

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$(C+E).B = (a_1A + 3B).(a_2A + 3B).B = 3(a_1 + a_2) = d_C + d_E$.
For $g \gg 0$ the plane model E has larger degree than the plane model of C:

$$d_E \approx 3/2g - 2/3g = 5/6g > 2/3g \approx d_C.$$

The approach fails at the point where E has to be chosen special within its Hilbert scheme to achieve $h^0\mathcal{J}_E(a_1, 3) > 0$.

Verra's case: $W^1_{14,8} \to \mathcal{M}_{14}$ are both unirational

$h^0(D) - h^0(K-D) = 8+1-14 \Rightarrow h^0(K-D) = 7$. Betti numbers of $C \subset \mathbb{P}^6$

$\beta(S_C) =$

	0	1	2	3	4	5
0	1	.	.	.	.	.
1	.	5	.	.	.	.
2	.	8	45	56	25	.
3	.	.	.	.	.	2

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$$\beta(S_C) = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 1 & . & . & . & . & . \\ 1 & . & 5 & . & . & . & . \\ 2 & . & 8 & 45 & 56 & 25 & . \\ 3 & . & . & . & . & . & 2 \end{array}$$

$$C \sim_{2^5} E, \quad 32 = \deg C + \deg E = 18 + 14$$

$$g_C - g_E = \frac{1}{2}(C-E).(5\cdot 2 - 7)H = \frac{3}{2}(18-14) = 6 \Rightarrow g_E = 8$$

$$\beta(S_E) = \begin{array}{c|cccccc} \hline 0 & 1 & . & . & . & . & . \\ 1 & . & 7 & . & . & . & . \\ 2 & . & . & 35 & 56 & 35 & 8 \end{array}$$

$$\mathcal{O}_E(H) = \omega_E(p_1 + \ldots + p_4 - (p_5 + \ldots + p_8))$$

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0	1	.	.	.	.	.
1	.	7	.	.	.	.
2	.	.	35	56	35	8

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Mukai: $\mathcal{M}_{8,8}$ unirational; $W^1_{14,8} \approx \mathbb{G}(5,7)$-bundle over Pic^{14}_8 also.

Sernesi, Chang-Ran unirationality of $\mathcal{M}_g$ for $g = 11, 12, 13$ via space curves

g/d	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
25	-														
24															
23															
22															
21															
20															
19															
18															
17															
16													?		
15															
14															
13											**CR**				
12										**Se**	Sc				
11										**CR**					
10											FV				
9															
8															
7															
6															
5															
4															
3															
2															
1															
0															

Space curves via Hartshorne-Rao modules, $W^1_{12,9}$

$h^0(D) - h^0(K-D) = 9+1-12 \Rightarrow h^0(K-D) = 4$, study $C \subset \mathbb{P}^3$. Hartshorne-Rao module

$$M = \oplus_n H^1(\mathbb{P}^3, \mathcal{J}_C(n)).$$

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C maximal rank $\Rightarrow$ expected syzygies:

$\beta(M) =$

	0	1	2	3	4
2	5	12	4	.	.
3	.	.	4	.	.
4	.	.	9	16	6

and

$\beta(\Gamma_* \mathcal{O}_C) =$

	0	1	2
0	1	.	.
1	.	.	.
2	5	12	4
3	.	.	2

$\beta(S_C) =$

	0	1	2	3
0	1	.	.	.
1	.	.	.	.
2	.	.	.	.
3	.	2	.	.
4	.	9	16	6

Construction of points in $W^1_{13,9}$

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2	5	12	4	.	.
3	.	.	4	.	.
4	.	.	9	16	6

$\beta(\Gamma_* \mathcal{O}_C) =$

0	1	.	.
1	.	.	.
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3	.	.	2

Construction

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Construction

1. Choose a general map $\mathcal{O}^{12}(-3) \xleftarrow{\varphi} \mathcal{O}^4(-4)$
   and compute generators $0 \leftarrow \ker(\varphi^t) \leftarrow \mathcal{O}(2)^8 \oplus \mathcal{O}(1)^8$

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2. Choose a point in $\mathbb{G}(5,8)$ and obtain the presentation

$$0 \leftarrow M \leftarrow S^5(-2) \xleftarrow{\psi} S^{12}(-3)$$

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2. Choose a point in $\mathbb{G}(5,8)$ and obtain the presentation

$$0 \leftarrow M \leftarrow S^5(-2) \xleftarrow{\psi} S^{12}(-3)$$

3. Choose a point in $\mathbb{G}(2,4)$ and obtain a locally free resolution

$$0 \leftarrow \mathcal{J}_C \leftarrow \mathcal{F} \leftarrow \mathcal{O}^4(-4) \oplus \mathcal{O}^2(-5) \leftarrow 0$$

where $\mathcal{F} = \widetilde{\ker(\psi)}$ is a rank 7 vector bundle on $\mathbb{P}^3$.

Strong maximal rank space curves of diameter ≤ 3

g/d	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
25															-19
24															
23														-17	
22															
21													-15		
20													-12		
19												-13			
18												-10			
17											-11	-7			
16											-8	-4	0?		
15										-9	-5	-1			
14										-6	-2				
13									-7	-3	**CR**				
12										**Se**	Sc				
11								-5	-1	**CR**					
10									2	6	FV				
9							-3	1	5	**9**					
8							0	4	**8**						
7						-1		**7**							
6					-2	2	**6**	10							
5					1	**5**	9								
4				0	**4**	8	12								
3				**3**		11	15								
2			**2**		10	14	18								
1		**1**	5	9	13										
0	**0**	4		12	16										

Space curves

25	-	-	-	-	-	-	-	-	-	ci	◯				-19
24	-	-	-	-	-	-	-	-	-	q	q			q	
23	-	-	-	-	-	-	-	-	-	◯	◯	cm		-17	
22	-	-	-	-	-	-	-	-	-	◯			q		
21	-	-	-	-	-	-	-	-	-	q			-15		
20	-	-	-	-	-	-	-	-	cm	◯	cm	q	-12		
19	-	-	-	-	-	-	-	-	◯	ci		-13			
18	-	-	-	-	-	-	-	-	q		q	-10			
17	-	-	-	-	-	-	-	-	◯	cm	-11	-7			
16	-	-	-	-	-	-	-	ci	◯	q	-8	-4	0?		
15	-	-	-	-	-	-	-	q		-9	-5	-1			
14	-	-	-	-	-	-	-	◯	q,cm	-6	-2				
13	-	-	-	-	-	-	-	◯	-7	-3	**CR**				
12	-	-	-	-	-	-	cm	q		**Se**	Sc				
11	-	-	-	-	-	-	◯	-5	-1	**CR**					
10	-	-	-	-	-	-	q,ci		2	6	FV				
9	-	-	-	-	-	ci	-3	1	5	**9**					
8	-	-	-	-	-	q	0	4	**8**						
7	-	-	-	-	-	-1	3	**7**							
6	-	-	-	-	-2	2	**6**	10							
5	-	-	-	-	1	**5**	9								
4	-	-	-	0	**4**	8	12								
3	-	-	-	**3**	7	11	15								
2	-	-	**2**	6	10	14	18								
1	-	**1**	5	9	13										
0	**0**	4	8	12	16										
g/d	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

Plane nodal models $\mathcal{N}_{d,g}$

g/d	2	3	4	5	6	7	8	9	10	11	12	13
26	-	-	-	-	-	-	-	-31	-28	-25	-22	
25	-	-	-	-	-	-	-	-29	-26	-23	-20	
24	-	-	-	-	-	-	-	-27	-24	-21		
23	-	-	-	-	-	-	-	-25	-22	-19		
22	-	-	-	-	-	-	-	-23	-20	-17		
21	-	-	-	-	-	-	-24	-21	-18	-15		
20	-	-	-	-	-	-	-22	-19	-16	-13		
19	-	-	-	-	-	-	-20	-17	-14			
18	-	-	-	-	-	-	-18	-15	-12			
17	-	-	-	-	-	-	-16	-13	-10			
16	-	-	-	-	-	-	-14	-11	-8			
15	-	-	-	-	-	-15	-12	-9	-6		0Sc	
14	-	-	-	-	-	-13	-10	-7				
13	-	-	-	-	-	-11	-8	-5		1CR		
12	-	-	-	-	-	-9	-6	-3	0Se			
11	-	-	-	-	-	-7	-4	-1	2G			
10	-	-	-	-	-8	-5	-2	1				
9	-	-	-	-	-6	-3	0					
8	-	-	-	-	-4	-1	2					
7	-	-	-	-	-2	1	4					
6	-	-	-	-3	0	3						
5	-	-	-	-1	2	**5**						
4	-	-	-	1	**4**	7						
3	-	-	0	**3**	6							
2	-	-	**2**	5	8							
1	-	**1**	4	7								
0	**0**	3	6	9								

Models in $\mathbb{P}^4$ and matrix factorizations; $W^1_{12,8}$

$|K - D|$ embeds $C \hookrightarrow \mathbb{P}^4$ as a curve of degree $\deg C = 22 - 8 = 14$. Postulation

$$\beta(\Gamma_* \mathcal{O}_C) = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & . & . & . \\ 1 & . & . & . & . \\ 2 & 2 & 14 & 15 & 2 \\ 3 & . & . & . & 2 \end{array}$$

In particular $h^0(\mathbb{P}^4, \mathcal{J}_C(3)) = 4$.

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In particular $h^0(\mathbb{P}^4, \mathcal{J}_C(3)) = 4$. Fix $f \in H^0(\mathbb{P}^4, \mathcal{J}_C(3))$ and consider the cubic solid $X = V(f)$. Resolve $\Gamma_* \mathcal{O}_C$ as an $S_X = S/f$ module:

$$\beta_X(\Gamma_* \mathcal{O}_C) = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 1 & . & . & . & . & . \\ 1 & . & . & 1 & . & . & . \\ 2 & 2 & 14 & 15 & 2 & & \\ 3 & . & . & 2 & 15 & 15 & 2 \\ 4 & . & . & . & . & 2 & 15 \end{array}$$

Models in $\mathbb{P}^4$ and matrix factorizations; $W^1_{12,8}$, II

$\beta_X(\Gamma_*\mathcal{O}_C) =$

0	1	.	.	.	.	.
1	.	.	.	.	.	.
2	2	13	15	2		
3	.	.	2	15	15	2
4	.	.	.	.	2	15

Models in $\mathbb{P}^4$ and matrix factorizations; $W^1_{12,8}$, II

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The sheaf

$$\mathcal{F} = \mathrm{coker}(\mathcal{O}_X^2(-2) \oplus \mathcal{O}_X^{15}(-3) \xleftarrow{\psi} \mathcal{O}_X^{15}(-3) \oplus \mathcal{O}_X^2(-4))$$

is a rank 7 vector bundle on X.

Models in $\mathbb{P}^4$ and matrix factorizations; $W^1_{12,8}$, II

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0	1	.	.	.	.	.
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Theorem (S.-Tanturri)

There is a monad

$$0 \leftarrow \mathcal{O}_X^2(-2) \leftarrow \mathcal{F} \leftarrow \mathcal{O}_X^2(-2) \oplus \mathcal{O}_x^2(-3) \leftarrow 0$$

whose homology is $\mathcal{J}_{C/X}$. For fixed $\mathcal{F}$ there is a $\mathbb{G}(2,5)$ of choices which yield curves C' of desired degree and genus.

Models in $\mathbb{P}^4$ and matrix factorizations; $W^1_{12,8}$, III

Consider a module N with Betti numbers

$$\beta(N) = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & . & . & . & . \\ 1 & . & . & . & . & . \\ 2 & . & 5 & . & . & . \\ 3 & . & 2 & 15 & 11 & 2 \end{array}$$

Syzygies of N as an S_X-module yield a matrix factorization of desired shape.

$$\begin{array}{c|ccc} & 3 & 4 & 5 \\ \hline 3 & 15 & 2 & . \\ 4 & 2 & 15 & 2 \\ 5 & . & . & 15 \end{array}$$

Models in $\mathbb{P}^4$ and matrix factorizations; $W^1_{12,8}$, III

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$$\begin{array}{c|ccc} & 3 & 4 & 5 \\ \hline 3 & 15 & 2 & . \\ 4 & 2 & 15 & 2 \\ 5 & . & . & 15 \end{array}$$

N is the homogeneous coordinate ring S_E of a curve of degree $\deg E = 13$ and (arithmetic) genus $g_E = 10$.

Models in $\mathbb{P}^4$ and matrix factorizations; $W^1_{12,8}$, IV

Riemann-Roch for $\mathcal{O}_E(H)$:

$$5 - h^1(\mathcal{O}_E(H)) = 13 + 1 - 10 \Rightarrow h^1(\mathcal{O}_E(H)) = 1.$$

Hence

$$\mathcal{O}_E(H) = \omega_E(-(p_1 + \ldots + p_5).$$

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Easier way to relate C and E:

$$C \sim_{3^3} E, \quad 27 = 14 + 13 = \deg C + \deg E$$

and $g_C - g_E = \frac{1}{2}(C - E).((9-5)H = 2$

$$\Rightarrow g_E = g_C - 2 = 10.$$

Models in $\mathbb{P}^4$ and matrix factorization; $W^1_{13,9}$

$|K - D|$ embeds $C \hookrightarrow \mathbb{P}^4$ as a curve of degree 15.

$$\beta(\Gamma_*\mathcal{O}_C) = \begin{array}{c|cccc} 0 & 1 & . & . & . \\ 1 & . & . & . & . \\ 2 & 3 & 17 & 18 & 3 \\ 3 & . & . & . & 2 \end{array}; \quad h^0(\mathbb{P}^4, \mathcal{J}_C(3)) = 2.$$

Models in $\mathbb{P}^4$ and matrix factorization; $W^1_{13,9}$

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0	1	.	.	.
1	.	.	.	.
2	3	17	18	3
3	.	.	.	2

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As an S_X-module:

$\beta_X(\Gamma_* \mathcal{O}_C) =$

0	1	.	.	.	.	.
1	.	.		.	.	.
2	3	16	18	3		
3	.	.	3	18	18	3
4	.	.	.	.	3	18

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1	.	.		.	.	.
2	3	16	18	3		
3	.	.	3	18	18	3
4	.	.	.	.	3	18

Monad for $\mathcal{J}_{C/X}$ with a rank 9 bundle $\mathcal{F}$ on X:

$$0 \leftarrow \mathcal{O}_X^3(-2) \leftarrow \mathcal{F} \leftarrow \mathcal{O}_X^3(-2) \oplus \mathcal{O}_X^2(-3) \leftarrow 0,$$

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1	.	.		.	.	.
2	3	16	18	3		
3	.	.	3	18	18	3
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0	1	.	.	.
1	.	.	.	.
2	3	17	18	3
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0	1	.	.	.	.	.
1	.	.		.	.	.
2	3	16	18	3		
3	.	.	3	18	18	3
4	.	.	.	.	3	18

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a $\mathbb{G}(2,3) \cong \mathbb{P}^2$ of choices. $\Rightarrow$ $W^1_{13,9}$ is ruled. Why non-trivial?

Models in $\mathbb{P}^4$ and matrix factorization; $W^1_{13,9}$

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1	.	.	.	.
2	3	17	18	3
3	.	.	.	2

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0	1	.	.	.	.	.
1	.	.		.	.	.
2	3	16	18	3		
3	.	.	3	18	18	3
4	.	.	.	.	3	18

Monad for $\mathcal{J}_{C/X}$ with a rank 9 bundle $\mathcal{F}$ on X:

$$0 \leftarrow \mathcal{O}_X^3(-2) \leftarrow \mathcal{F} \leftarrow \mathcal{O}_X^3(-2) \oplus \mathcal{O}_X^2(-3) \leftarrow 0,$$

a $\mathbb{G}(2,3) \cong \mathbb{P}^2$ of choices. $\Rightarrow$ $W^1_{13,9}$ is ruled. Why non-trivial?
Start with a 13-nodal rational C, general C' will be smooth!

Hurwitz schemes $\mathcal{H}_{g,d} \to W^1_{g,d}$

g / d	2	3	4	5	6	7	8	9	10	11	12	13	14
45				P	G								
|				|									
40				P	G								
|				|									
36				P	G								
35				P	G								
34				P									
33				P	G								
32				P									
31				P	G								
30				P	G								
29				P									
28				P	G								
27				P	G								
26				P	G								EH
25				P	G								HM
24				P	G							EH	EH
23				P	G							HM	HM
22				P	G						F	F	F
21				P	G								
|				|	|							
16				P	G								
15				P	G			V					
14				P	G		V						FV
13				P	G	ST		ST				FV	CKV
12				P	G	G	ST	S			FV	CKV	CKV
11				P	G	G	CR			FV	CKV	CKV	CKV
10				P	G	G			FV	CKV	CKV	CKV	BFV
9				P	G	G	DS	M	M	M	M	M	M
8				P	|	G	M	M	M	M	M	M	M
7				P	|	M	M	M	M	M	M	M	M
6													
|													
1													

Color coding indicates where $W^1_{g,d}$ is known to be unirational, uniruled or not unirational.