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Tate Resolutions

The Tate resolution of a coherent sheaf \mathcal{F} on \mathbb{P}^n is a double infinite free complex over an exterior algebra, which encodes the cohomology of \mathcal{F} . Applications include

- Beilinson monads
- Chow forms, resultants
- Boij-Söderberg theory
- direct image complexes (local or affine case)

Today, work in progress with David Eisenbud and Daniel Erman

- Extension of this theory to products of projective spaces.
- Application include direct image complexes in the global case: computation of Rπ_∗F for a morphism π : X → Y between projective varieties and F ∈ coh(X)

- Introduction

Overview

- 1. Review of Tate resolutions on \mathbb{P}^n
- 2. Construction of the Tate resolution
- 3. Beilinson monads
- 4. Exactness property of the Tate resolution

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5. Open Questions

└ 1. Review of Tate resolutions on \mathbb{P}^n

Koszul pair

• K a ground field, W an (n + 1)-dimensional vector space

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• $S = Sym W = K[x_0, ..., x_n]$ coordinate ring of \mathbb{P}^n

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- $V = W^*$ with dual basis e_0, \ldots, e_n
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The Koszul complex proves

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$$E = Ext_{\mathcal{S}}(K, K)$$

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The Koszul complex proves

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$$E = Ext_S(K, K)$$
 and $S = Ext_E(K, K)$.

I. Review of Tate resolutions on ℙⁿ

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The Koszul complex proves

• $E = Ext_S(K, K)$ and $S = Ext_E(K, K)$.

E is Gorenstein with $\omega_E = Hom_K(E, K) = \Lambda W$ free,

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The Koszul complex proves

• $E = Ext_S(K, K)$ and $S = Ext_E(K, K)$.

E is Gorenstein with $\omega_E = Hom_K(E, K) = \Lambda W$ free, so injective=projective over *E*

└ 1. Review of Tate resolutions on \mathbb{P}^n

BGG-Functors

 $M = \oplus_d M_d$ graded *S*-module

$$\mathbf{R}(M):\ldots \rightarrow Hom_{\mathcal{K}}(E,M_d) \rightarrow Hom_{\mathcal{K}}(E,M_{d+1}) \rightarrow \ldots$$

with differential

$$arphi \mapsto \{ oldsymbol{e} \mapsto \sum_{i=0}^n x_i arphi(oldsymbol{e}_ioldsymbol{e}) \}$$

 $P = \bigoplus_d P_d$ graded *E*-module

 $\mathbf{L}(P):\ldots\to P_1\otimes S\to P_0\otimes S\to P_{-1}\otimes S\to\ldots$

with differential $p \otimes s \mapsto \sum_{i=0}^{n} pe_i \otimes x_i s$.

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with differential $p \otimes s \mapsto \sum_{i=0}^{n} pe_i \otimes x_i s$. Note deg $e_i = -1$

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└ 1. Review of Tate resolutions on \mathbb{P}^n

BGG-Functors

$$\mathbf{R}: grmod(S) \rightarrow lincplx(E)$$

and

$$L: grmod(E) \rightarrow lincplx(S)$$

extend to a pair of adjoint functors

$$cplx(S) \stackrel{L,R}{\longleftrightarrow} cplx(E)$$

Theorem (Bernstein, Gelfand, Gelfand 1978)

$$D^b(S) \cong D^b(E)$$
 and $D^b(\mathbb{P}^n) \cong \underline{mod} E$

Tate resolution

$M = \oplus_d M_d$ and $\mathcal{F} = \widetilde{M}$ corresponding coherent sheaf. If $r \ge reg M$ then

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Tate resolution

 $M = \bigoplus_d M_d$ and $\mathcal{F} = \widetilde{M}$ corresponding coherent sheaf. If $r \ge reg M$ then $\mathbf{R}(M_{\ge r})$ is acyclic

 $0 \rightarrow P \rightarrow Hom(E, M_r) \rightarrow Hom(E, M_{r+1}) \rightarrow \ldots$

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Theorem (Eisenbud, Fløystad, S., 2003)

▶ (Reciprocity) M an S-module, P an E-module. $0 \rightarrow P \rightarrow \mathbf{R}(M)$ is an injective resolution \Leftrightarrow $\mathbf{L}(P) \rightarrow M \rightarrow 0$ is a projective resolution.

Tate resolution

 $M = \bigoplus_d M_d$ and $\mathcal{F} = \widetilde{M}$ corresponding coherent sheaf. If $r \ge reg M$ then $\mathbf{R}(M_{\ge r})$ is acyclic

$$\mathbf{T}(\mathcal{F}): \ldots \to T^{r-1}(\mathcal{F}) \to Hom(E, M_r) \to Hom(E, M_{r+1}) \to \ldots$$

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double infinite minimal complex of free E modules.

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double infinite minimal complex of free E modules.

Theorem (Eisenbud, Fløystad, S., 2003)

(Reciprocity) M an S-module, P an E-module.
0 → P → R(M) is an injective resolution ⇔
L(P) → M → 0 is a projective resolution.
T^d(F) = ∑ⁿ_{i=0} Hⁱ(ℙⁿ, F(d − i)) ⊗ ω_E(i − d)

2. Construction of the Tate resolution

Cox ring

- $\blacktriangleright \mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} = \mathbb{P}(W_1) \times \cdots \times \mathbb{P}(W_t)$
- ► $W = W_1 \oplus \ldots \oplus W_t$ and $S = Sym \ W = K[x_{1,0}, \ldots, x_{t,n_t}]$ the \mathbb{Z}^t -graded Cox ring of \mathbb{P}

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- $E = \Lambda V$ exterior algebra, $\omega_E = \Lambda W$

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- deg $x_{i,j} = (\delta_{i1}, \dots, \delta_{in}) \in \mathbb{Z}^t$ and deg $e_{i,j} = -\deg x_{i,j}$
- c = (c₁,..., c_t) a (multi)-degree, |c| = ∑_i c_i denotes the total degree.

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2. Construction of the Tate resolution

What should be the shape of the Tate resolution?

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Example (Künneth case)

 $\blacktriangleright \mathbb{P} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$

-2. Construction of the Tate resolution

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Example (Künneth case)

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$$\mathbb{P} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$$
, $\mathcal{F} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 := p_1^* \mathcal{F}_1 \otimes p_2^* \mathcal{F}_2$

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$$\mathbf{T}(\mathcal{F}) = \mathbf{T}(\mathcal{F}_1) \otimes_{\mathcal{K}} \mathbf{T}(\mathcal{F}_2)$$
 over $E \cong E_1 \otimes_{\mathcal{K}} E_2$

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For arbitrary $\mathcal{F} \in coh(\mathbb{P})$ we should have

$$\mathbf{T}^{d}(\mathcal{F}) = \sum_{\substack{0 \leq i \leq n \\ |a| = d}} \sum_{\substack{a \in \mathbb{Z}^{t} \\ |a| = d}} \mathcal{H}^{|i|}(\mathbb{P}, \mathcal{F}(a - i)) \otimes_{K} \omega_{E}(i - a)$$

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no longer finitely generated, but free!

2. Construction of the Tate resolution

An example on $\mathbb{P}^1 \times \mathbb{P}^1$

Consider $\omega_E \to \omega_E(-2,0) \oplus \omega_E^4(-1,-1) \oplus \omega_E(0,-2)$ defined by the matrix

$$m = (e_0 e_1, e_0 f_0, e_1 f_0, e_0 f_1, e_1 f_1, f_0 f_1)^t$$

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where $V = V_1 \oplus V_2 = \langle e_0, e_1, f_0, f_1 \rangle$.

-2. Construction of the Tate resolution

An example on $\mathbb{P}^1 \times \mathbb{P}^1$

Consider $\omega_E \to \omega_E(-2,0) \oplus \omega_E^4(-1,-1) \oplus \omega_E(0,-2)$ defined by the matrix

$$m = (e_0e_1, e_0f_0, e_1f_0, e_0f_1, e_1f_1, f_0f_1)^t$$

where $V = V_1 \oplus V_2 = \langle e_0, e_1, f_0, f_1 \rangle$.

 $L(\text{image } m) \rightarrow M \rightarrow 0$

is the minimal free resolution of the module of global sections $M = \sum_{(a,b)\in\mathbb{Z}^2} H^0(\mathcal{F}(a,b))$ of a rank 3 vector bundle \mathcal{F} with cohomology as indicated on the next slide.

2. Construction of the Tate resolution

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$$\left(\sum_{i=0}^2 \dim H^i(\mathbb{P}^1 imes \mathbb{P}^1, \mathcal{F}(\pmb{a}, \pmb{b})) \cdot h^i
ight)_{-3 \leq \pmb{a}, \pmb{b} \leq 3}$$

	(28h	18 <i>h</i>	8h	2	12	22	32 \	
	20 <i>h</i>	13 <i>h</i>	6 <i>h</i>	1	8	15	22	
	12 <i>h</i>	8h	4 <i>h</i>	0	4	8	12	
:	4 <i>h</i>	3h	2h	h	0	1	2	$\in \mathbb{Z}[h]^{7 imes 7}$
	4 <i>h</i> ²	2 <i>h</i> ²	0	2h	4h	6h	8h	
	12 <i>h</i> ²	7 <i>h</i> ²	2 <i>h</i> 2	3h	8h	13 <i>h</i>	18 <i>h</i>	
	$\setminus 20h^2$	12 <i>h</i> ²	4 <i>h</i> ²	4 <i>h</i>	12 <i>h</i>	20 <i>h</i>	28h/	

The injective resolution of $P = \ker m$ has total Betti numbers

2. Construction of the Tate resolution

High truncations

M finitely gen. \mathbb{Z}^t -graded module, $\mathcal{F} = \widetilde{M}$ sheaf on \mathbb{P} . Then $\exists b: \forall c \ge b$

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-2. Construction of the Tate resolution

High truncations

M finitely gen. \mathbb{Z}^t -graded module, $\mathcal{F} = \widetilde{M}$ sheaf on \mathbb{P} . Then $\exists b: \forall c \ge b$

1. $M_{\geq c}(c)$ has a linear resolution, i.e.

$$(0 \leftarrow M_{\geq c}(c) \leftarrow) F_0 \leftarrow F_1 \leftarrow \cdots$$

with $F_k = \bigoplus_a S^{\beta_{k,a}}(-a)$ satisfies $\beta_{k,a} \neq 0$ only if k = |a|, 2. $M_c = H^0(\mathbb{P}, \mathcal{F}(c))$ and $H^p(\mathbb{P}, \mathcal{F}(c)) = 0$ for p > 0.

We call such $b \in \mathbb{Z}^t$ sufficiently positive for *M*.

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High truncations

M finitely gen. \mathbb{Z}^t -graded module, $\mathcal{F} = \widetilde{M}$ sheaf on \mathbb{P} . Then $\exists b: \forall c \ge b$

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2. Construction of the Tate resolution

BGG and positive quadrants

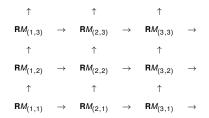
Reciprocity still works: **R** and **L** respect the finer grading. $\mathbf{R}(M_{\geq c}(c))$ gives a part of the Tate resolution in a positive quadrant.

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2. Construction of the Tate resolution

BGG and positive quadrants

Reciprocity still works: **R** and **L** respect the finer grading. **R**($M_{\geq c}(c)$) gives a part of the Tate resolution in a positive quadrant. Say t = 2 and c = (1, 1) sufficiently positive

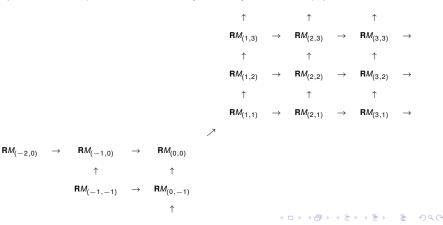


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-2. Construction of the Tate resolution

BGG and positive quadrants

Reciprocity still works: **R** and **L** respect the finer grading. $\mathbf{R}(M_{\geq c}(c))$ gives a part of the Tate resolution in a positive quadrant. Say t = 2 and c = (-2, -2) sufficiently positive

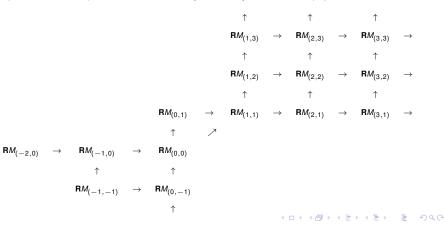


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2. Construction of the Tate resolution

BGG and positive quadrants

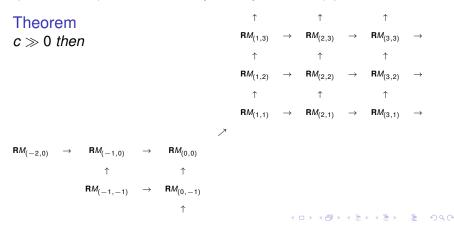
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-2. Construction of the Tate resolution

BGG and positive quadrants

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-2. Construction of the Tate resolution

Projective dimension of high truncations

Corollary M a graded module over the Cox ring S. If $c \gg 0$ then

 $pd M_{\geq c} = \dim S - t.$

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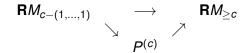
-2. Construction of the Tate resolution

Projective dimension of high truncations

Corollary *M* a graded module over the Cox ring S. If $c \gg 0$ then

 $pd M_{\geq c} = \dim S - t.$

Proof.



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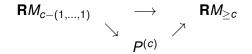
-2. Construction of the Tate resolution

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Proof.



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Hence $LP^{(c)} \rightarrow M_{\geq c} \rightarrow 0$ has length dim S - t.

2. Construction of the Tate resolution

Construction

↑ ↑ ↑ **R***M*_(1,3) **R***M*_(2,3) RM(3,3) \rightarrow \rightarrow \rightarrow ↑ ↑ **R***M*_(1,2) **R***M*_(2,2) RM_(3,2) \rightarrow \rightarrow \rightarrow ↑ ↑ ↑ **R***M*_(1,1) **R**M_(2,1) **R**M_(3,1) \rightarrow \rightarrow \rightarrow ↗

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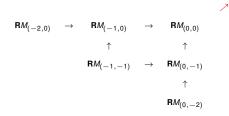
$$\mathbf{R}M_{(-1,-1)} \rightarrow \mathbf{R}M_{(0,-1)}$$

$$\uparrow$$

$$\mathbf{R}M_{(0,-2)}$$

2. Construction of the Tate resolution

Construction

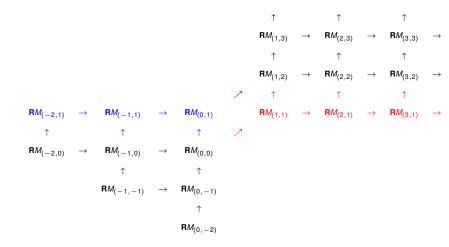


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2. Construction of the Tate resolution

Construction



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-2. Construction of the Tate resolution

Construction

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R M _(-2,0)	\rightarrow	R M _(-1,0)	\rightarrow	R <i>M</i> _(0,0)
		↑		\uparrow
		$RM_{(-1,-1)}$	\rightarrow	$RM_{(0,-1)}$
				\uparrow
				R <i>M</i> _(0,-2)

\uparrow		\uparrow		\uparrow	
R M _(1,3)	\rightarrow	R M _(2,3)	\rightarrow	$RM_{(3,3)}$	\rightarrow
¢		↑		\uparrow	
R M _(1,2)	\rightarrow	R M _(2,2)	\rightarrow	$RM_{(3,2)}$	\rightarrow

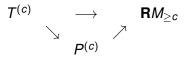
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$$\nearrow$$

-2. Construction of the Tate resolution

Construction Step 2

Given *M* and *b*, sufficiently positive for *M*, consider free resolutions $T^{(c)}$ of $P^{(c)}$,



for all $c \ge b$. We have a directed system $\{T^{(c')} \to T^{(c)} | c' \ge c\}$. Define

$$T' = \lim_{\leftarrow} T^{(c)}$$

and finally the **Tate resolution** of $\mathcal{F} = \widetilde{M}$ as the subcomplex of homogeneous elements:

$$\mathbf{T}(\mathcal{F}) = \{ f \in T' | f \text{ is homogeneous } \} \subset T'.$$

2. Construction of the Tate resolution

First Main Theorem

Proposition

The Tate resolution $\mathbf{T}(\mathcal{F})$ exact. For each multidegree a the space of homogeneous elements $\mathbf{T}(\mathcal{F})_a$ of multidegree a is finite dimensional.

Theorem

The Tate resolution of a coherent sheaf \mathcal{F} on \mathbb{P} has terms

$$\mathbf{T}(\mathcal{F})^{d} \cong \sum_{\substack{a \in \mathbb{Z}^{t} \\ |a|=d}} \sum_{0 \leq i \leq n} H^{|i|}(\mathcal{F}(a-i)) \otimes_{K} \omega_{E}(i-a)$$

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-3. Beilinson Monads

The derived category $D^b(\mathbb{P})$

$$U_k = \ker(H^0(\mathbb{P}^{n_k}, \mathcal{O}(1)) \otimes \mathcal{O} \to \mathcal{O}(1))$$

tautological rank n_k subbundle on \mathbb{P}^{n_k} . Set

$$U^{a} = \Lambda^{a_{1}} U_{1} \boxtimes \cdots \boxtimes \Lambda^{a_{t}} U_{t}$$

Of course, U^a is nonzero if and only if $0 \le a \le n$.

Theorem (Beilinson, xyz)

 $\{U^a|0 \le a \le n\}$ forms a full strongly exceptional series for the derived category $D^b(\mathbb{P})$, which is right orthogonal to the strongly exceptional series $\{\mathcal{O}(a)|0 \le a \le n\}$ in the sense that

$$H^{p}$$
RHom $(\mathcal{O}(c), U^{a}) = H^{p}(U^{a}(-c)) = \begin{cases} K & \text{if } a = c \text{ and } p = |a|, \\ 0 & \text{otherwise.} \end{cases}$

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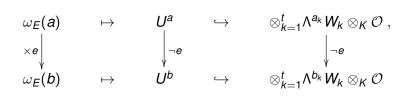
-3. Beilinson Monads

The **U**-functor

Consider the additive functor on the category of direct sums of finitely generated free graded *E*-modules defined by

 $\mathbf{U}: \omega_E(\mathbf{a}) \mapsto U^{\mathbf{a}}$

on objects. For the morphism given by the multiplication with $e \in E_{b-a} = \bigotimes_{k=1}^{t} \Lambda^{a_k - b_k} V_k$ we define **U** by the diagram



where the right hand maps are given by contraction.

-3. Beilinson Monads

Beilinson Monad

Applying ${\bm U}$ to the Tate resolution, we obtain a bounded complex

 $U(\mathcal{F}) := U(T(\mathcal{F})).$

This is the Beilinson monad for \mathcal{F} .

Theorem $U(\mathcal{F})$ is a monad for the sheaf \mathcal{F} in the sense that

$$H^p({f U}({\cal F}))\cong egin{cases} {\cal F} & ext{ for }p=0, ext{ and }\ 0 & ext{ for }p
eq 0. \end{cases}$$

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4. Exactness properties of the Tate resolution

Locally finite *E*-complexes

Definition

A complex T of graded free E-module with terms

$$\mathcal{T}^{d} = \sum_{\pmb{a} \in \mathbb{Z}^{t}} \mathcal{B}^{d}_{\pmb{a}} \otimes \omega_{\pmb{E}}(-\pmb{a})$$

with vector spaces B_a^d is **locally finite**, if for each $a \in \mathbb{Z}^t$ the vector space

$$\sum_{d\in\mathbb{Z}}B_a^c$$

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is finite dimensional.

4. Exactness properties of the Tate resolution

Strands, quadrants, regions

T a locally finite complex of graded free *E*-modules with terms $T^d = \sum_{a \in \mathbb{Z}^t} B^d_a \otimes \omega_E(-a)$. For $c \in \mathbb{Z}^t$ and disjoint subsets $I, J, K \subset \{1, \dots, t\}$ we call the subquotient complexes $T_c(I, J, K)$ with

$$T_{c}(I, J, K)^{d} = \sum_{\substack{a \in \mathbb{Z} \\ a_{i} < c_{i} \text{ for } i \in I \\ a_{i} = c_{i} \text{ for } i \in J \\ a_{i} \geq c_{i} \text{ for } i \in K}} B_{a}^{d} \otimes \omega_{E}(-a)$$

a proper region complex of *T* if $I \cup J \cup K \subsetneq \{1, ..., t\}$ $T_c(\emptyset, J, \emptyset)$ with $J \subsetneq \{1, ..., t\}$ a strand and $T_c(I, \emptyset, K)$ with $I \cup K = \{1, ..., t\}$ a quadrant complex.

4. Exactness properties of the Tate resolution

Corner complex T_{rc}

 $T_{\geq c} = T_c(\emptyset, \emptyset, \{1, \dots, t\})$ and $T_{<c} = T_c(\{1, \dots, t\}, \emptyset, \emptyset)$ denote the first and last quadrant complex and abbreviate

$$T_{c,k} = T_c(\{1,\ldots,k\}, \emptyset, \{k+1,\ldots,t\})$$

for some of the intermediate quadrant complexes. The **corner complex** $T_{\Gamma c}$ is the cone over the map

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which we get as composition

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of the maps in T from one quadrant to the next.

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4. Exactness properties of the Tate resolution

Second Main Theorem

Theorem

T be a locally finite complex of free E-modules. TFAE

- 1. Every strand of T is exact.
- 2. Every proper region complex of T is exact.
- 3. Every corner complex T_{rc} is exact.
- The corner complexes T_r are exact for every sufficiently large c.
- 5. The proper region complexes *T*_c(*I*, ∅, ∅) are exact for every sufficiently large *c*.
- $\mathbf{T}(\mathcal{F})$ satisfies 5. by construction.

4. Exactness properties of the Tate resolution

Example on $\mathbb{P}^1\times\mathbb{P}^1$

(28h	18 <i>h</i>	8h	2	12	22	32 \
20h	13 <i>h</i>	6 <i>h</i>	1	8	15	22
12 <i>h</i>	8h	4 <i>h</i>	0	4	8	12
4 <i>h</i>	3h	2h	h	0	1	
	2 <i>h</i> ²					
	7 <i>h</i> 2					
$(20h^2)$	12 <i>h</i> ²	4 <i>h</i> ²	4 <i>h</i>	12h	20 <i>h</i>	28h/

Total Betti numbers of $T_{r'0}$

	-5	-4	-3	-2	-1	0	1	2		
-1:	140	84	45	20	6					
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4. Exactness properties of the Tate resolution

Example on $\mathbb{P}^1\times\mathbb{P}^1$

(.			2	12	22	32\	
		•	1	8	15	22	
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4 <i>h</i> ²	2 <i>h</i> 2	0				.	
12 <i>h</i> 2		2 <i>h</i> ²				.	
$\setminus 20h^2$	12 <i>h</i> ²	4 <i>h</i> ²				. /	

Total Betti numbers of $T_{r'0}$

	-5	-4	-3	-2	-1	0	1	2		
-1:	140	84	45	20	6					
0:	•					1				
1:										
2:							4	15		
	1						• • •		æ	596

4. Exactness properties of the Tate resolution

Direct image complexes

 \mathcal{F} coherent sheaf on \mathbb{P} , and $T = \mathbf{T}(\mathcal{F})$ it's Tate resolution. For each proper subset $J = \{j_1, \ldots, j_s\} \subset \{1, \ldots, n\}$ with complement J' we have the projection

$$\pi^J \colon \mathbb{P} \to \mathbb{P}^{n_{j_1}} \times \cdots \times \mathbb{P}^{n_{j_s}} = \mathbb{P}^J.$$

Corollary

For $c \in \mathbb{Z}^t$ the strand $T_c(\emptyset, J', \emptyset)$ is exact, and after twist and shift

$$T_{\boldsymbol{c}}(\emptyset, J', \emptyset)(\boldsymbol{c})[|\boldsymbol{c}|] \cong T_{J} \otimes_{K} \omega_{E^{J'}}$$

is a flat extension of an minimal complex T_J of free E^J -modules such that

$$\mathbf{U}_J(T_J)\cong R\pi^J_*(\mathcal{F}(\boldsymbol{c}))\in D^b(\mathbb{P}^J)$$

Half plane complexes

The Tate resolution $\textbf{T}(\mathcal{F})$ has many exact subquotient complexes.

Question

What is the geometric meaning of say, the **half plane complexes**

 $T_c(I, \emptyset, K)$

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for $I \cup K = \{1, ..., n\} \setminus \{j\}$?

Double complexes

For simplicity, assume t = 2, hence $\mathbb{P} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$.

$$\mathcal{F} \cong \bigoplus_j \mathcal{F}_j \boxtimes \mathcal{G}_j \quad \Rightarrow \quad \mathbf{T}(\mathcal{F}) \text{ is a double complex.}$$

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Question Is the converse true?

Objects in $D^{b}(\mathbb{P}^{n})$ as image sheaves ?

In the case of an affine space Spec *A*, David and I proved that any bounded complex

$$\mathbf{0} o \mathbf{A}^{lpha_{\mathbf{0}}} o \ldots o \mathbf{A}^{lpha_{\mathbf{n}}} o \mathbf{0}$$

arises as $R\pi_*\mathcal{F}$ of a vector bundle \mathcal{F} on Spec $A \times \mathbb{P}^n$.

Question

Could it be that any object in $D^b(\mathbb{P}^n)$ arises as $R\pi_*\mathcal{F}$ for a coherent sheaf \mathcal{F} on a product \mathbb{P} for a suitable projection $\pi: \mathbb{P} \to \mathbb{P}^n$ onto a factor?

Tate resolution of elements in $D^{b}(\mathbb{P})$

Any object in $F \in D^b(\mathbb{P})$ can be represent by a bounded minimal complex

$$0 o F^k o F^{k+1} o \ldots o F^\ell o 0$$

with $F^j = \bigoplus_i U^{a^{ij}}$. So there exist a smallest complex *T* of free *E* module such that $\mathbf{U}(T) \cong F$

Question

How to compute the Tate resolution of *F*, i.e. an exact complex *T'* of free *E*-modules such that $\mathbf{U}(T'(c)[|c|]) = F(c)$ for every $c \in \mathbb{Z}^t$?

We have a nice simple Macaulay2 code in case of $\mathbb{P} = \mathbb{P}^n$ of a single factor.