Balancing in relative canonical resolutions and a unirational moduli space of K3 surfaces

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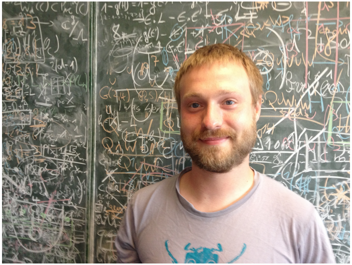
In this talk I report on recent work of two of my students



Christian Bopp

Michael Hoff

Relative canonical embedding

Throughout this talk I denote by

$$\pi : C \xrightarrow{|D|} \mathbb{P}^1$$

a smooth non-hyperelliptic curve of genus g together with a base point free complete pencil of divisors of degree $d \leq g-1$. The canonical embedding of C factors

$$\begin{array}{ccccc} C & & \xrightarrow{\quad\iota\quad} & & \mathbb{P}^{g-1} \\ \downarrow{\scriptstyle \pi} & \searrow & & \nearrow & \\ \mathbb{P}^1 & \xleftarrow[\pi]{} & X = \mathbb{P}(\mathcal{E}) & & \end{array}$$

through a $\mathbb{P}^{d-2}$-bundle over $\mathbb{P}^1$ associated to

$$\pi_* \omega_C \cong \omega_{\mathbb{P}^1} \oplus \mathcal{E},$$

where $\mathcal{E}$ is a vector bundle of rank $d-1$ and degree $f = g-d+1$ on $\mathbb{P}^1$, hence slope $\frac{g-d+1}{d-1}$.

Maroni invariant

The splitting type $\mathcal{E} = \mathcal{O}(e_1) \oplus \ldots \oplus \mathcal{O}(e_{d-1})$ is called the Maroni invariant of $(C, |D|)$.

Theorem (Ballico)

For $(C, |D|) \in \mathcal{W}^1_{g,d}$ general, $\mathcal{E}$ is balanced, i.e. $|e_i - e_j| \leq 1$.

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$\operatorname{Pic} X \cong \mathbb{Z}H \oplus \mathbb{Z}R$ of $X = \mathbb{P}(\mathcal{E})$ is generated by the hyperplane class H and the ruling R with intersection products

$$H^{d-1} = \sum_{i=1}^{d-1} e_i = f, H^{d-2} \cdot R = 1 \text{ and } R^2 = 0.$$

The canonical class of X is $\omega_X \cong \mathcal{O}_X(-(d-1)H + (f-2)R$.

Relative canonical resolution

Theorem (Schreyer, 1986)

$C \to \mathbb{P}^1$ a degree d cover by a curve C of genus g as above. Then as an $\mathcal{O} = \mathcal{O}_X$-module $\mathcal{O}_C$ has a locally free resolution of shape

$$0 \leftarrow \mathcal{O}_C \leftarrow \mathcal{O} \leftarrow \mathcal{O}(-2H) \otimes \pi^*\mathcal{N}_1 \leftarrow \mathcal{O}(-3H) \otimes \pi^*\mathcal{N}_2 \leftarrow \dots$$

$$\dots \leftarrow \mathcal{O}(-(d-2)H) \otimes \pi^*\mathcal{N}_{d-3} \leftarrow \mathcal{O}(-dH + (f-2)R) \leftarrow 0,$$

where the $\mathcal{N}_i$ are vector bundles on $\mathbb{P}^1$ of

$$\operatorname{rank} \mathcal{N}_i = \tfrac{d(d-2-i)}{(i+1)} \tbinom{d-2}{i-1}$$

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and slope $\mu_i = \frac{(i+1)(g-d-1)}{d}$ (Bopp-Hoff, 2015).

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Question

Are the vector bundles $\mathcal{N}_i$ balanced for $(C, |D|) \in \mathcal{W}^1_{g,d}$ general?

Yes, if $d | g-1$ (Bujokas, Patel, 2015).

Relative quadrics and $\mathcal{N}_1$

Theorem (Bopp-Hoff, 2015)

$(C, |D|) \in \mathcal{W}^1_{g,d} \to \mathcal{M}_g$ general with Brill-Noether number $\rho = \rho(g, d, 1) = g - 2(g - d + 1) > 0$. Then $\mathcal{N}_1$ is unbalanced iff

$$(d - \rho - \frac{7}{2})^2 > \frac{23}{4} - 2d.$$

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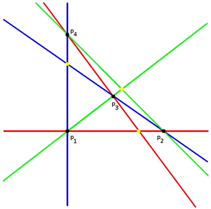
Conjecture (Bopp-Hoff, 2015)

For general $(C, |D|) \in \mathcal{W}^1_{g,d}$ and $\rho = \rho(g, d, 1) \leq 0$ the bundle $\mathcal{N}_1$ is balanced.

Cubic Resolvent

Let $C \subset X \to \mathbb{P}^1$ be a tetragonal defined over K. Then $C \subset X$ is a complete intersection of two quadric bundles and $\mathcal{N}_1 = \mathcal{O}(b_1) \oplus \mathcal{O}(b_2)$ with $b_1 + b_2 = g - 5$.

The cubic resolvent of the field extension of function fields of degree $[K(C) : K(t)] = 4$ can be identified with the rank 2 quadrics of the pencil in each fiber $\mathbb{P}^2_t \approx R$.



Proposition (Casnati, 1998)

The cubic resolvent corresponding to the normal subgroup $D_4 \subset S_4$ of index 3 is a trigonal curve C' of genus $g+1$ (if smooth) and Maroni invariant $(e_1, e_2) = (b_1 + 2, b_2 + 2)$.

Resolvents

Conjecture (Castryck-Zhou, 2017)

$C \subset X \to \mathbb{P}^1$ be a 5-gonal of genus g defined over K. $\mathcal{N}_2 = \mathcal{O}(b_1) \oplus \ldots \oplus \mathcal{O}(b_5) \cong \mathcal{H}om(\mathcal{N}_1, \mathcal{O}(f-2))$. The Cayley resolvent corresponding to the subgroup $F_{20} \subset S_5$ defines a 6-gonal curve of genus $3g+7$ (if smooth) with Maroni invariant $(b_1+4, \ldots, b_5+4)$.

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Castryk and Zhou have further experimental findings for various other resolvents associated to subgroups $G \subset S_d$.

A unbalanced case

Theorem (Bopp-Hoff, 2017)

Let $(C, |D|) \in \mathcal{W}^1_{9,6}$ *be a general curve of genus* 9 *together with a general pencil of degree* 6*. Then*

$$X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^1})$$

and the relative canonical resolution of $\mathcal{O}_C$ *as* $\mathcal{O} = \mathcal{O}_X$*-module has shape*

$$\mathcal{O} \leftarrow \begin{matrix} \mathcal{O}(-2H+R)^{\oplus 6} \\ \oplus \\ \mathcal{O}(-2H)^{\oplus 3} \end{matrix} \leftarrow \begin{matrix} \mathcal{O}(-3H+2R)^{\oplus 2} \\ \oplus \\ \mathcal{O}(-3H+R)^{\oplus 12} \\ \oplus \\ \mathcal{O}(-3H)^{\oplus 2} \end{matrix} \leftarrow \begin{matrix} \mathcal{O}(-4H+2R)^{\oplus 3} \\ \oplus \\ \mathcal{O}(-4H+R)^{\oplus 6} \end{matrix} \leftarrow \mathcal{O}(-6H+2R) \leftarrow 0$$

In particular, $\mathcal{N}_2 = \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 12} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ *is unbalanced.*

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Unlike a balanced case, it is not possible to prove this by exhibiting an example and a semi-continuity argument.
But we can study an example!

Idea of proof: Syzygy schemes

Consider a syzygy s:

$$\begin{matrix} \mathcal{O}(-2H+R)^{\oplus 6} \\ \oplus \\ \mathcal{O}(-2H)^{\oplus 3} \end{matrix} \leftarrow \mathcal{O}(-3H+2R)$$

$h^0(\mathcal{O}_X(H-R)) = 4$ and $h^0(\mathcal{O}_X(H-2R)) = 0$. So this syzygy is really only a relation among 4 quadrics in $H^0((\mathcal{J}_{C/X}(2H-R))$, and by [S. 1991] there exists a skew symmetric 5×5 matrix $\psi = (\psi_{ij})$ with entries as indicate

$$\begin{pmatrix} & H-R & H-R & H-R & H-R \\ H-R & & H & H & H \\ H-R & H & & H & H \\ H-R & H & H & & H \\ H-R & H & H & H & \end{pmatrix}$$

such that 4 of the 5 pfaffians are the given quadrics. All five of them define a codimension 3 Gorenstein subscheme which turns out to be a K3 surface $Y \subset X$.

The K3 surfaces

Indeed, by the Buchsbaum-Eisenbud complex,

$$\omega_Y = \mathcal{E}xt^3_X(\mathcal{O}_Y, \omega_X) \cong \mathcal{O}_Y \text{ and } h^1(\mathcal{O}_Y) = 0$$

$|\mathcal{O}(H)|$ embeds Y into $\mathbb{P}^8 = \mathbb{P}H^0(\mathcal{O}_X(1))$. Hence $|H|$ cuts out on Y a family of curves of genus 8. The pencil $|R|$ gives a pencil of elliptic curves on Y of degree 5 and we also have $C \subset Y$. Thus the intersection products of these curves on Y are

$$\begin{array}{c} \\ H \\ C \\ R \end{array} \begin{array}{c} \begin{array}{ccc} H & C & R \end{array} \\ \begin{pmatrix} 14 & 16 & 5 \\ 16 & 16 & 6 \\ 5 & 6 & 0 \end{pmatrix} \end{array} = M.$$

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Since we have pencil of syzygies

$$\mathcal{N}_2 = \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 12} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$$

in our (randomly choosen) example, we expect that each each pair $(C, |D|) \in \mathcal{W}^1_{9,6}$ give rise to a pencil of such K3 surfaces.

A dimension count

Let $\mathcal{F}^M$ denote the moduli space of with M lattice polarized $K3$ surfaces, so $Y \in \mathcal{F}^M$ is a tuple $Y = (Y, \mathcal{O}_Y(H), \mathcal{O}_Y(C), \mathcal{O}_Y(R))$

$$\begin{array}{ccccc} \mathcal{P} & \xleftarrow{\cong ?} & \mathcal{S} & & \\ \downarrow{\scriptstyle \mathbb{P}^9} & \searrow^{\varphi} & \downarrow{\scriptstyle \mathbb{P}^1} & & \\ \mathcal{F}^M & & \mathcal{W}^1_{9,6} & \longrightarrow & \mathcal{M}_9 \end{array} \qquad \begin{array}{ccccc} (Y, C) & \xleftarrow{\text{syz. scheme}} & (C, |D|, s) & & \\ \downarrow & \searrow & \downarrow & & \\ Y & & (C, |D|) & \xrightarrow{W^1_6(C)} & C \end{array}$$

with $\varphi(Y, C) = (C, |\mathcal{O}_C(R)|)$

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$$
\begin{array}{ccccc}
26 & \xleftarrow{\cong ?} & ? & & \\
{\scriptstyle 9}\downarrow & \searrow & \downarrow{\scriptstyle 1} & & \\
17 = 20 - 3 & & 25 & \xrightarrow{1} & 3g - 3 = 24
\end{array}
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Key Lemma

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Lemma (Bopp-Hoff)

$(Y, C) \in \mathcal{P}$ general, $(C, |D|) \in \mathcal{W}^1_{9,6}$ its image under φ. Then

1. *$Y \subset X$ and its $\mathcal{O}_X$-resolution is an Buchsbaum-Eisenbud complex with skew matrix as in the example above. In particular $\mathcal{N}_2(C, |D|)$ is unbalanced:*

   $$\mathcal{N}_2(C, |D|) = \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus a} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 16-2a} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus a} \text{ with } a \geq 1$$

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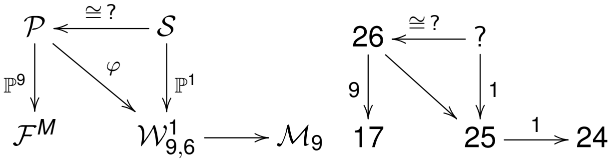
2. $\dim \varphi^{-1}(C, |D|) \leq a - 1. \quad \Longrightarrow \quad a \geq 2$

Key Lemma



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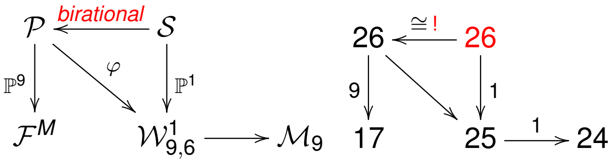
2. $\dim \varphi^{-1}(C, |D|) \leq a - 1. \quad \Longrightarrow \quad a \geq 2$

Existence of an example defined over $\mathbb{Q}$ with $a = 2$
$\Longrightarrow$ φ is dominant and $a = 2$ holds generically.

Key Lemma



Lemma (Bopp-Hoff)

$(Y, C) \in \mathcal{P}$ *general,* $(C, |D|) \in \mathcal{W}^1_{9,6}$ *its image under* φ*. Then*

1. $Y \subset X$ *and its* $\mathcal{O}_X$*-resolution is an Buchsbaum-Eisenbud complex with skew matrix as in the example above. In particular* $\mathcal{N}_2(C, |D|)$ *is unbalanced:*

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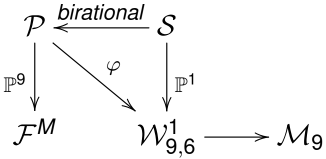
Unirationality



Corollary

$\mathcal{P}$ and $\mathcal{F}^M$ are unirational.

Proof: $\mathcal{W}^1_{9,6}$ is unirational by [Segre, 1926]: A general curve of genus $g = 9$ has a plane model of degree 9 with one triple and 16 double points. The projection from the triple points gives the the pencil of degree 6. Indeed,

$$\binom{9-1}{2} - 3 - 16 = 9 \text{ and } \binom{9+2}{2} - 6 - 16 \cdot 3 = 1.$$

Thus $17 = 1 + 16$ general points in $\mathbb{P}^2$ specifies a pair $(C, |D|)$ uniquely. $\square$

The plane curve

Take the triple point defined by the ideal (x_1, x_2), the sixteen points by their Hilbert-Burch matrix which we choose randomly with 1 digit coefficients. This leads to a plane curve C defined by the form

$$\begin{aligned} &354918011361065076985282828 2x_0^6x_1^3 \\ +&1243784112296986265985587761 7x_0^5x_1^4 \\ +&1512833175486892569402593632 2x_0^4x_1^5 \\ &\vdots \\ +&2606113043968937878067116160x_1x_2^8 \\ -&1042162038287194453776245414 4x_2^9 \end{aligned}$$

and (logarithmic) height $= 2534.47$, i.e. binary Information about 2534 bits.

The family of K3 containing C

It is no surprise that C is contained in a K3 surface. Actually the linear system $|\omega_C(-D)|$ embeds

$$C \hookrightarrow \mathbb{P}^3$$

and $C \subset \mathbb{P}^3$ is contained in a net (a $\mathbb{P}^2$) of quartics. The general such quartic has Picard rank 2 generated by the hyperplane class and the curve C. (Reality check: $25 + 2 - 9 = 18$)

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Inside this $\mathbb{P}^2$ the rational curve of our Y's form a Noether-Lefschetz component, which turns out to be a nodal cubic curve.

Equation of the NL-cubic:

$$
\begin{aligned}
&\vdots \\
-&1651210573464703096321949510978589506851186669756329185244236855777200555941419323471067596 \\
&2195661737604354096814195707430112886494118049559052920465867468839800983590873338359097 \\
&3617183034366403904894962932115383212745871639770202292572381598075702819278061759693871547 \\
&4587177688347221425507504861987464847372219354391530870327777026416593592214851873436663696 \\
&1840946234144502740328670734214583251466411407092867821906923267971114951736733636115657218 \\
&1036718953298871557639464604423537423871935606279947066851675527636891043068058572397516262 \\
&0494106797434175535690709818736873856281545510996998952477935952166416107911286979634332 \\
&4470448764936177141159799651681005501704679966836316155592944849271167255966220278943293396 \\
&9235897668517509123823454513701440453797237763644676230072402586264827071245732303682321469 \\
&2438801486508269797841328535275898055981897489048311838913132436182387846630492297352805 \\
&0550750237757265214630635989466831348718586908523332086675330959818432100529087259514273985 \\
&3348774217852079326887454174809968797227683061330754210990137725278149727439146205467569842 \\
&1630198625144583910294712240800821815237589813853650794407353324549585817666243255372812 \\
&6499246119021225715330248439117349407882741566283284500654039143380140685619775499824102449 \\
&946478464985541\,x_2^3
\end{aligned}
$$

which has (logarithmic) height $= 21241.81$.

Equation of the NL-cubic:

$$
\vdots
$$

$$
\begin{aligned}
-&1651210573464703096321949510978589506851186669756329185244236855777200555941419323471067 59\\
&6219566173760435409681419570743011288649411804955905292046586746883980098359087333835909 73\\
&6171830343664039048949629321153832127458716397702022925723815980757028192780617596938715 47\\
&4587177688347221425507504861987464847372219354391530870327777026416593592214851873436663 69\\
&6184094623414450274032867073421458325146641140709286782190692326797111495173673363611565 721\\
&8103671895329887155763946460442353742387193560627994706685167552763689104306805857239751 62\\
&6204941067974341755356907098187368738562815455109969989524779359521664161079112869796343 32\\
&4470448764936177141159799651681005501704679966836316155592944849271167255966220278943293 39\\
&6923589766851750912382345451370144045379723776364467623007240258626482707124573230368232 14\\
&6924388014865082697978413285352758980559818974890483118389131324361823878466304922973528 05\\
&0550750237757265214630635989466831348718586908523332086675330959818432100529087259514273 98\\
&5334877421785207932688745417480996879722768306133075421099013772527814972743914620546756 98\\
&4216301986251445839102947122408008218152375898138536507944073533245495858176662432553728 12\\
&6499246119021225715330248439117349407882741566283284500654039143380140685619775499824102 44\\
&9946478464985541\,x_2^3
\end{aligned}
$$

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16 pts	plane mod.	adj. syst.	rel. can. eqs.	5×5	K3 surf.
227.2	2534.4	2225.0	61387.5	21699.1	71717.7

The nodal point

For $(C, |D|)$ general, the node of the cubic corresponds to a K3 surface of Picard rank 4 and lattice

$$\begin{array}{c|cccc} & h & C & L_1 & L_2 \\ h & 4 & 10 & 1 & 1 \\ C & 10 & 16 & 0 & 0 \\ L_1 & 1 & 0 & -2 & 0 \\ L_2 & 1 & 0 & 0 & -2 \end{array} = N$$

where $h \sim H_i - R_i$ and $H_i \sim C - L_i$, hence $R_i \sim C - L_i - h$.

Theorem (Bopp-Hoff)

$\mathcal{W}^1_{9,6}$ is birational to the universal family $\mathcal{P}^N$ of genus 9 curves over the lattice polarized moduli space of K3 surfaces $\mathcal{F}^N$.

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Thank you!