Matrix factorizations and families of curves of genus 15

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Introduction

The moduli spaces $\mathcal{M}_g$ of curves of genus $g$ is

- unirational for $g \leq 14$, [Severi, Sernesi, Chang-Ran, Verra],
- of general type for $g = 22$ and $g \geq 24$, [Harris-Mumford, Eisenbud-Harris, Farkas].

The cases in between are not fully understood:

- $\mathcal{M}_{23}$ has positive Kodaira dimension [Farkas],
- $\mathcal{M}_{15}$ is rationally connected [Bruno-Verra],
- $\mathcal{M}_{16}$ is uniruled [Chang-Ran, Farkas].

In this talk I report on an attempt to prove the unirationality of $\mathcal{M}_{15}$. 
By Brill-Noether theory,

\[ W^r_d(C) = \{ L \in \text{Pic}^d C \mid h^0(L) \geq r + 1 \} \]

has dimension at least

\[ \rho = g - (r + 1)(g - d + r), \]
By Brill-Noether theory,
a general curve of genus $15 = 5 \cdot 3$ has a finite number of smooth models of degree 16 in $\mathbb{P}^4$. 
By Brill-Noether theory, a general curve of genus $15 = 5 \cdot 3$ has a finite number of smooth models of degree 16 in $\mathbb{P}^4$. Let

$$\mathcal{H} \subset \text{Hilb}_{16t+1-15}(\mathbb{P}^4)$$

be the corresponding component of the Hilbert scheme, and let

$$\tilde{M}_{15} \subset \{(C, L) \mid C \in M_{15}, L \in W^4_{16}(C)\} \to M_{15}$$

be the component of the Hurwitz scheme, which dominates generically finite to one. So $\mathcal{H}//PGL(5)$ is birational to $\tilde{M}_{15}$. 
By Brill-Noether theory, a general curve of genus $15 = 5 \cdot 3$ has a finite number of smooth models of degree $16$ in $\mathbb{P}^4$. Let

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be the corresponding component of the Hilbert scheme, and let

$$\tilde{\mathcal{M}}_{15} \subset \{(C, L) \mid C \in \mathcal{M}_{15}, L \in W_{16}^{4}(C)\} \rightarrow \mathcal{M}_{15}$$

be the component of the Hurwitz scheme, which dominates generically finite to one. So $\mathcal{H} // PGL(5)$ is birational to $\tilde{\mathcal{M}}_{15}$. Our main result connects the moduli space $\tilde{\mathcal{M}}_{15}$ to a moduli space of certain matrix factorizations of cubic threefolds.
Main Results

Theorem

The moduli space $\widetilde{M}_{15}$ of curves of genus 15 together with a $g^4_{16}$ is birational to a component of the moduli space of matrix factorizations of type

$$\mathcal{O}^{18}(-3) \xrightarrow{\psi} \mathcal{O}^{15}(-1) \oplus \mathcal{O}^{3}(-2) \xrightarrow{\varphi} \mathcal{O}^{18}$$

of cubic forms on $\mathbb{P}^4$.

Theorem

$\widetilde{M}_{15}$ is uniruled.
Overview

1. Introduction
2. Review of matrix factorizations
3. The structure theorem
4. Constructions
5. Tangent space computations
6. Conclusion
Matrix factorizations [Eisenbud, 1980]

$R$ a regular local ring, $f \in \mathfrak{m}^2$. A *matrix factorization* of $f$ is a pair $(\varphi, \psi)$ of matrices satisfying

$$\psi \circ \varphi = f \ id_G \quad \text{and} \quad \varphi \circ \psi = f \ id_F.$$

$M = \text{coker} \ \varphi$ is a maximal Cohen-Macaulay $R/f$-module.
Matrix factorizations [Eisenbud, 1980]

$R$ a regular local ring, $f \in m^2$. A **matrix factorization** of $f$ is a pair $(\varphi, \psi)$ of matrices satisfying

$$\psi \circ \varphi = f \ id_G \quad \text{and} \quad \varphi \circ \psi = f \ id_F. $$

$M = \text{coker } \varphi$ is a maximal Cohen-Macaulay $R/f$-module.

Conversely, if $M$ is a MCM on $R/f$, then as $R$-module $M$ has a short resolution

$$0 \leftarrow M \leftarrow F \leftarrow G \leftarrow 0. $$

and multiplication with $f$ on this complex is null homotopic

$$0 \leftarrow M \leftarrow F \varphi \leftarrow G \varphi \leftarrow 0$$

which yields a matrix factorization $(\varphi, \psi)$. 
2-periodic resolutions

As an $R/f$-module, $M$ has the infinite 2-periodic resolution

$$0 \leftarrow M \leftarrow \overline{F} \leftarrow \overline{G} \leftarrow \overline{F} \leftarrow \overline{G} \leftarrow \ldots$$

where $\overline{F} = F \otimes R/f$ and $\overline{G} = G \otimes R/f$. In particular, this sequence is exact, and the dual sequence corresponding to the matrix factorization $(\psi^t, \varphi^t)$ is exact as well.
2-periodic resolutions

As an $R/f$-module, $M$ has the infinite 2-periodic resolution

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where $\overline{F} = F \otimes R/f$ and $\overline{G} = G \otimes R/f$.

The resolution of an arbitrary $R/f$-module $N$ is eventually 2-periodic. If

$$0 \leftarrow N \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \ldots \leftarrow F_c \leftarrow 0$$

is the finite resolution of $N$ as $R$-module then

$$0 \leftarrow N \leftarrow \overline{F}_0 \leftarrow \overline{F}_1 \leftarrow \overline{F}_2 \oplus \overline{F}_0 \leftarrow \overline{F}_3 \oplus \overline{F}_1 \leftarrow \ldots \leftarrow \overline{F}_{\text{ev}} \leftarrow \overline{F}_{\text{odd}} \leftarrow \ldots$$

is a $R/f$-resolution, where

$$F_{\text{ev}} = \bigoplus_{i \equiv 0 \mod 2} F_i \quad \text{and} \quad F_{\text{odd}} = \bigoplus_{i \equiv 1 \mod 2} F_i.$$
MCM-approximation

The high syzygy modules over a Cohen-Macaulay ring are MCM. In case of an hypersurface, $M = \text{coker}(\overline{F}_{\text{odd}} \to \overline{F}_{\text{ev}})$ is a MCM module. There is a natural surjection from $M$ to $N$ with kernel $P$,

$$0 \leftarrow N \leftarrow M \leftarrow P \leftarrow 0$$

where $P$ is a module of finite projective dimension

$$\text{pd}_{R/f} P < \infty.$$
The graded case: replace $R$ by $S = k[x_0, \ldots, x_n]$

If $f \in S$ is a homogeneous form of degree $d$ then we have to take the grading into account:

- A matrix factorization is now given by a pair

$$G \xrightarrow{\varphi} F \xrightarrow{\psi} G(d)$$

of maps between graded free $S$-modules.

- The $i$-th term in the (not necessarily minimal) eventually 2-periodic $S/f$-resolution obtained from an $S$-resolution $F_\bullet$ is

$$\overline{F}_i \oplus \overline{F}_{i-2}(-d) \oplus \ldots \oplus \overline{F}_0(-id/2)$$

or

$$\overline{F}_i \oplus \overline{F}_{i-2}(-d) \oplus \ldots \oplus \overline{F}_1(-(i-1)d/2)$$

in case $i$ is even or odd, respectively.
If $X = V(f) \subset \mathbb{P}^n$ is a smooth hypersurface then an MCM module

$$M = \text{coker } \varphi$$

sheafifies to a vector bundle

$$\mathcal{F} = \tilde{M}$$

on $X$ with no intermediate cohomology,

$$H^p(X, \mathcal{F}(t)) = 0 \text{ for all } p \text{ with } 0 < p < \dim X.$$

If $\det \varphi = \lambda f^r$ with $\lambda \in K$ a scalar, then

$$\text{rank } \mathcal{F} = r.$$
Section 3. The structure theorem

We begin now with the proof of the main theorem.

**Theorem**

The moduli space $\widetilde{M}_{15}$ of curves of genus 15 together with a $g^4_{16}$ is birational to a component of the moduli space of matrix factorizations of type

$$\mathcal{O}^{18}(-3) \xrightarrow{\psi} \mathcal{O}^{15}(-1) \oplus \mathcal{O}^{3}(-2) \xrightarrow{\varphi} \mathcal{O}^{18}$$

of cubic forms on $\mathbb{P}^4$. 
Postulation

For $C \subset \mathbb{P}^4$ be a smooth curve of degree $d = 16$ and genus $g = 15$. We have

- $S_C = S/I_C$, the homogeneous coordinate ring, and
- $H^0_*(\mathcal{O}_C) = \bigoplus_n H^0(\mathcal{O}_C(n))$, the ring of sections.
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- $H^0(\mathcal{O}_C) = \bigoplus_n H^0(\mathcal{O}_C(n))$, the ring of sections.

Proposition

As $S$-modules these rings have free resolution with Betti tables

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iff $C$ has maximal rank and $(C, L)$ is not a ramification point of $\tilde{\mathcal{M}}_{15} \to \mathcal{M}_{15}$. In particular a general curve $C$ lies on a unique cubic $X$. 
Syzygies $H_0^*(\mathcal{O}_C)$ of as $S_X$-module

From now on, $C \subset \mathbb{P}^4$ will always denote a general curve of degree 16 and genus 15.

The eventual 2-periodic resolution of $H_0^*(\mathcal{O}_C)$ as an $S_X = S/f$ has the shape

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This is not a minimal resolution.
Syzygies $H^0_*(\mathcal{O}_C)$ of as $S_X$-module

From now on, $C \subset \mathbb{P}^4$ will always denote a general curve of degree 16 and genus 15.

**Proposition**

*The minimal resolution of $H^0_*(\mathcal{O}_C)$ as an $S_X = S/f$ has the shape*

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From $C$ to a matrix factorization

Corollary

*A general curve $C$ determines a matrix factorization of shape*

$$
\begin{array}{c|ccc}
 & 0 & 1 & 2 \\
1 & 15 & . & . \\
2 & 3 & 18 & 15 \\
3 & . & . & 3 \\
\end{array}
$$
From $C$ to a matrix factorization

Corollary

A general curve $C$ determines a matrix factorization of shape

\[
\begin{array}{|ccc|}
\hline
 & 0 & 1 & 2 \\
1 & 15 & \cdot & \cdot \\
2 & 3 & 18 & 15 \\
3 & \cdot & \cdot & 3 \\
\hline
\end{array}
\]

Define $\mathcal{F}$ via

\[
0 \leftarrow \mathcal{F} \leftarrow \mathcal{O}_X^{18}(-3) \leftarrow \mathcal{O}_X^{15}(-4) \oplus \mathcal{O}_X^3(-5)).
\]

The composition

\[
\mathcal{O}_X^3(-2) \leftarrow \mathcal{F} \leftarrow \mathcal{O}_X^{18}(-3)
\]

is surjective with a summand $\mathcal{O}_X^3(-3)$ in the kernel, since there are only 5 linear forms on $\mathbb{P}^4$. 
From the matrix factorization back to $C$

Theorem (Structure Theorem)

*Given the matrix factorization associated to $C$ then the complex*

$$
0 \leftarrow \mathcal{O}_X^3(-2) \leftarrow \mathcal{F} \leftarrow \mathcal{O}_X^3(-3) \leftarrow 0
$$

*is a monad for the ideal sheaf $\mathcal{I}_{C/X}$ of $C \subset X$, i.e. $\alpha$ is surjective, $\beta$ injective and*

$$
\mathcal{I}_{C/X} \cong \ker \alpha / \text{im} \beta.
$$

$\mathcal{F}$ *is a rank 7 vector bundle on the cubic $X$, because*

$$
deg \det \begin{pmatrix} 18 & 15 \\ 3 & 3 \end{pmatrix} = 15 + 3 \cdot 2 = 7 \cdot 3.
$$
Proof of the main theorem

Since it is an open condition on matrix factorizations of shape

\[
\begin{array}{c|ccc}
   & 0 & 1 & 2 \\
1 & 15 & . & . \\
2 & 3 & 18 & 15 \\
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\]

to lead to a monad of a smooth curve of genus 15 and degree 16, this completes the proof of the main theorem.

We now could study the moduli space \( \mathcal{M}_X(7; c_1F, c_2F, c_3F) \) of vector bundles on the cubic threefold \( X \).
Different approach: construct auxiliary modules $N$, whose syzygies would lead to a desired matrix factorization.

Possible shape of Betti tables $\beta(N)$ are

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 \\
0 & a & . & . & . & . \\
1 & b & c & d & . & . \\
2 & . & . & e & f & h \\
3 & . & . & . & . & i \\
\end{array}
\]
or

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 \\
0 & a & b & . & . & . \\
1 & . & c & d & e & . \\
2 & . & . & f & h & . \\
\end{array}
\]

A computation shows: There are 39 of the tables in the Boij-S"oderberg cone with codim $\beta(N) \geq 3$, in all case we have equality.
Section 4. Constructions

Different approach: construct auxiliary modules \( N \), whose syzygies would lead to a desired matrix factorization.

Possible shape of Betti tables \( \beta(N) \) are

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2 & . & . & . & f & h \\
\end{array}
\]

with \((a + d + h, b + e + i, c + f) = (3, 15, 18)\) or \((15, 3, 18)\) for the first case.
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with $(a + d + h, b + e + i, c + f) = (3, 15, 18)$ or $(15, 3, 18)$ for the first case, and $(a + d + h, b + e, c + f) = (18, 15, 3)$ or $(18, 3, 15)$ in the second case.
Section 4. Constructions

Different approach: construct auxiliary modules $N$, whose syzygies would lead to a desired matrix factorization.

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A computation shows: There are 39 of the tables in the Boij-Söderberg cone with codim $\beta(N) \geq 3$, in all case we have equality.
## Four candidate tables

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How to think about $N$?
How to think about $N$?

- In all cases we will assume that

$$\mathcal{L} = \tilde{N}$$

is a line bundle on an auxiliary curve $E$ of degree $d_E = \deg \beta(N)$.
How to think about $N$?

- In all cases we will assume that

\[ \mathcal{L} = \sim N \]

is a line bundle on a auxiliary curve $E$ of degree $d_E = \deg \beta(N)$.

- Since $\text{pd}_S(N) \leq 4$, $N \subset H^0_*(\mathcal{L}) = \bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{L}(n))$ and local cohomology measures the difference

\[ 0 \to N \to H^0_*(\mathcal{L}) \to H^1_m(N) \to 0. \]
How to think about $N$?

- In all cases we will assume that
  
  $$\mathcal{L} = \tilde{N}$$

  is a line bundle on a auxiliary curve $E$ of degree $d_E = \deg \beta(N)$.

- Since $\text{pd}_S(N) \leq 4$, $N \subset H^0_*(\mathcal{L}) = \bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{L}(n))$ and local cohomology measures the difference

  $$0 \rightarrow N \rightarrow H^0_*(\mathcal{L}) \rightarrow H^1_\mathfrak{m}(N) \rightarrow 0.$$ 

- Since $H^1_\mathfrak{m}(N)$ is dual to $\text{Ext}^4_S(N, S(-5))$ the 4-th map in the resolution gives us an idea about $N$. 

How to think about $N$?

- In all cases we will assume that

$$\mathcal{L} = \tilde{N}$$

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The genus $g_E$ and the degree $d_\mathcal{L} = \deg \mathcal{L}$ are however not yet determined. Their choice is motivated by a dimension count.
Example 1.

The easiest case is perhaps \( d_E = 11 \) with Betti table

\[
\begin{array}{lcccc}
 & 0 & 1 & 2 & 3 \\
0 & 5 & 9 & . & . \\
1 & . & 3 & 13 & 6 \\
2 & . & . & . & 0 \\
\end{array}
\]

It is natural to assume that \( h_0 \mathcal{O}_E(1) = 5 \). Riemann-Roch implies \( h_1 \mathcal{O}_E(1) = g_E - d_E + 4 = g_E - 7 \).

Parameter count:

\[
\dim \{ (E, \mathcal{O}_E(1)) \} = 4 \quad g_E - 3 - 5 \cdot h_1 \mathcal{O}_E(1) = 32 - g_E
\]

\[
\dim \{ X \mid X \supset E \} = 34 - (3d_E + 1 - g_E) = g_E
\]

Finally \( h_1 (L) = 0 \) can be read off the Betti table, so \( L \) is non-special and we obtain further \( g_E \) parameters.

Altogether we get \( g_E + 32 \) parameters, and to obtain (at least) 42 motivates the choice \( g_E = 10 \).
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$$\dim\{(E, \mathcal{O}_E(1))\} = 4g_E - 3 - 5 \cdot h^1\mathcal{O}_E(1) = 32 - g_E$$

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Finally $h^1(\mathcal{L}) = 0$ can be read of the Betti table, so $\mathcal{L}$ is non-special and we obtain further $g_E$ parameters. Altogether we get $g_E + 32$ parameters, and to obtain (at least) 42 motivates the choice $g_E = 10$. 
Example 1.

\[ g_E = 10 \Rightarrow h^1(\mathcal{O}_E(1)) = 3, \] so \( E \) has a plane model \( E' \) of degree \( 18 - 11 = 7 \) with \( \delta = \binom{6}{2} - 10 = 5 \) double points. So we can choose \( 5+10 \) points in \( \mathbb{P}^2 \),

\[ E' \in |7h - \sum_{i=1}^{5} 2p_i - \sum_{j=1}^{10} q_j|, \]

and take \( \mathcal{L} = \omega_E(q_1 + q_2 + q_3 - (q_4 + \ldots + q_{10})) \).
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and take \( \mathcal{L} = \omega_E(q_1 + q_2 + q_3 - (q_4 + \ldots + q_{10})) \). By checking an example with \textit{Macaulay2} over a finite field we conclude:

**Theorem (Family 1)**

There exists a 42-dimensional unirational family of tuples

\[ (E, \mathcal{O}_E(1), X, \mathcal{L}) \text{ with } (d_E, g_E, d_\mathcal{L}) = (11, 10, 14) \]

such that \( N = H^0_*(\mathcal{L}) \) leads to a matrix factorization of desired shape. The general one gives a smooth curve \( C \subset \mathbb{P}^4 \) of degree 16 and genus 15.
Example 2.

In case of

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we have $N \subset H^0_*(\mathcal{L})$ with cokern $K(-1)$. The resolution of $N$ and $H^0_*(\mathcal{L})$ differ by a Koszul complex on 5 linear forms.
Example 2.

In case of

\[
\begin{array}{ccccc}
0 & 1 & 2 & 3 & 4 \\
0 & 2 & . & . & . \\
1 & 1 & 9 & . & . \\
2 & . & . & 14 & 9 & 1 \\
\end{array}
\]

we have \( N \subset H^0_*(\mathcal{L}) \) with cokern \( K(-1) \). The resolution of \( N \) and \( H^0_*(\mathcal{L}) \) differ by a Koszul complex on 5 linear forms. Thus the Betti table is \( H^0_*(\mathcal{L}) \) is

\[
\begin{array}{ccccc}
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So $E$ has a model in $\mathbb{P}^3$ and to pass from $H^0_*(\mathcal{L})$ to $N$ amounts to the choice a point in a $\mathbb{P}^1$. 
Example 2.

The dimension count suggest to take $g_E = 11$. Riemann-Roch

$\Rightarrow h^1(\mathcal{O}_E(1)) = 1$, hence

$$\mathcal{O}_E(1) \cong \omega_E(-(p_1 + \ldots + p_6)).$$

Theorem (Family 2)

There exists a 42-dimensional uniruled family of tuples

$$(E, \mathcal{O}_E(1), X, \mathcal{L}, N)$$

with $(d_E, g_E, d_L) = (14, 11, 8)$

such the general tuple gives a smooth curve $C \subset \mathbb{P}^4$ of degree 16 and genus 15.
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Take the line bundle $\mathcal{L} = \omega_E(-h)$, where $h$ denotes the hyperplane class of the model $E \subset \mathbb{P}^3$ of degree 12, that is a Chang-Ran curve of genus 11.

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Take the line bundle $\mathcal{L} = \omega_E(-h)$, where $h$ denotes the hyperplane class of the model $E \subset \mathbb{P}^3$ of degree 12, that is a Chang-Ran curve of genus 11. I do not know how to choose a Chang-Ran curve together with 6 points unirationally.

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Take the line bundle \( \mathcal{L} = \omega_E(-h) \), where \( h \) denotes the hyperplane class of the model \( E \subset \mathbb{P}^3 \) of degree 12, that is a Chang-Ran curve of genus 11.

But over a finite field \( \mathbb{F}_q \) there are plenty of points in \( E(\mathbb{F}_q) \) which are easy to pick with a probabilistic method.

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There exists a 42-dimensional uniruled family of tuples

\[ (E, \mathcal{O}_E(1), X, \mathcal{L}, N) \text{ with } (d_E, g_E, d_\mathcal{L}) = (14, 11, 8) \]

such the general tuple gives a smooth curve \( C \subset \mathbb{P}^4 \) of degree 16 and genus 15.
Section 3. Tangent space computations

All what is needed to conclude from family 1 that $\widetilde{M}_{15}$ is unirational, is to prove that the map gives an isomorphism on tangent spaces in a random example. Since the association

$$(N, X) \mapsto (M, X)$$

might not be surjective, this is a nontrivial assertion. So we want to study the natural map

$$Ext^1_{S_X}(N, N)_0 \rightarrow Ext^1_{S_X}(M, M)_0.$$
5. Tangent space diagram

The relevant diagram is

\[ \begin{array}{cccc}
\text{Ext}^1_{S_X}(M, P) & \to & \text{Ext}^1_{S_X}(M, M) & \to \\
\uparrow & & \uparrow & \\
\text{Ext}^1_{S_X}(M, N) & \to & \text{Ext}^2_{S_X}(M, P)
\end{array} \]

\[ \begin{array}{c}
\text{Ext}^1_{S_X}(N, N) \\
\uparrow \\
\text{Hom}_{S_X}(P, N)
\end{array} \]

deduced from the MCM approximation

\[ 0 \leftarrow N \leftarrow M \leftarrow P \leftarrow 0. \]
5. Tangent space diagram

The relevant diagram is

\[
\begin{array}{ccc}
\text{Ext}^1_{S_X}(M, P) & \rightarrow & \text{Ext}^1_{S_X}(M, M) \\
\parallel & \text{IR} & \rightarrow \\
0 & \text{Ext}^1_{S_X}(M, N) & \rightarrow \text{Ext}^2_{S_X}(M, P) \\
\text{Ext}^1_{S_X}(N, N) & \uparrow & \\
\text{Hom}_{S_X}(P, N) & \rightarrow & 0
\end{array}
\]

deduced from the MCM approximation

\[0 \leftrightarrow N \leftrightarrow M \leftrightarrow P \leftrightarrow 0.\]
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The relevant diagram is

\[
\begin{array}{ccc}
\text{Ext}^1_{S_X}(M, P) \to & \text{Ext}^1_{S_X}(M, M) \xrightarrow{\text{id}} & \text{Ext}^1_{S_X}(M, N) \to \text{Ext}^2_{S_X}(M, P) \\
\parallel & & \parallel \\
0 & & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Ext}^1_{S_X}(N, N) \uparrow & & \text{Ext}^1_{S_X}(N, N) \\
\uparrow & & \uparrow \\
\text{Hom}_{S_X}(P, N) & & \text{Hom}_{S_X}(P, N)
\end{array}
\]

deduced from the MCM approximation

\[
0 \leftrightarrow N \leftrightarrow M \leftrightarrow P \leftrightarrow 0.
\]

\[
\dim \text{Ext}^1_{S_X}(M, M)_0 = \dim \text{Ext}^1_{S_X}(N, N)_0 = 32 \text{ as expected,}
\]

\[
\text{Hom}_{S_X}(P, N)_0 \leftrightarrow \text{Ext}^1_{S_X}(N, N)_0, \text{ but}
\]
Dimensions of the families

Proposition

For a randomly chosen example,

\[ \dim \text{Hom}_{S^1}(P, N)_0 = \begin{cases} 
3 & \text{in case of family 1} \\
0 & \text{in case of family 2} 
\end{cases} \]

Hence family 1 leads to a 39-dimensional subvariety of \( \widetilde{\mathcal{M}}_{15} \) and family 2 dominates. In particular \( \widetilde{\mathcal{M}}_{15} \) is unirruled.
Altogether I managed to construct 20 families of pairs \((X, N)\) all of dimension at least 42, of which

- 17 families are unirational,
- 3 are (possibly) not, since they required the choice of additional points on the auxiliary curve \(E\).
- The three non-unirational families dominate.
- Most of the unirational families lead to 39-dimensional subvarieties of \(\tilde{\mathcal{M}}_{15}\). One has dimension 40, another one dimension 41.

Could this be just bad luck?
A conjecture
I think no.
A conjecture

I think no. A good explanation could be

**Conjecture**

*The maximal rationally connected fibration of $\tilde{M}_{15}$ has a three dimensional base.*
A conjecture and a complexity result

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Conjecture

The maximal rationally connected fibration of $\widetilde{M}_{15}$ has a three dimensional base.

Theorem

The probabilistic algorithm, which for a finite field $\mathbb{F}_q$ selects randomly curves of genus 15, has running time $O((\log q)^3)$. 

Thank you!
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**Conjecture**

*The maximal rationally connected fibration of $\widetilde{M}_{15}$ has a three dimensional base.*

**Theorem**

*The probabilistic algorithm, which for a finite field $\mathbb{F}_q$ selects randomly curves of genus 15, has running time $O((\log q)^3)$.*

- I expect that the algorithm picks points from a subset of $\mathcal{M}_{15}(\mathbb{F}_q)$ of density about 47%. The image of $\widetilde{M}_{15}(\mathbb{F}_q)$ should have density about 63%. The same should hold for the image of the $\mathbb{F}_q$-rational points of the parameter space in $\widetilde{M}_{15}(\mathbb{F}_q)$. 

Thank you!
A conjecture and a complexity result

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The maximal rationally connected fibration of $\mathcal{M}_{15}$ has a three dimensional base.

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The probabilistic algorithm, which for a finite field $\mathbb{F}_q$ selects randomly curves of genus $15$, has running time $O((\log q)^3)$.

- I expect that the algorithm picks points from a subset of $\mathcal{M}_{15}(\mathbb{F}_q)$ of density about 47%.
- A unirational description of $\mathcal{M}_{15}$ would lead to an algorithm with running time $O((\log q)^2)$.

Thank you!
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- For any fixed genus $g$ there exists an algorithm which selects points from a subset of $\mathcal{M}_g(\mathbb{F}_q)$ of positive density in running time $O((\log q)^3)$.
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The maximal rationally connected fibration of \( \tilde{\mathcal{M}}_{15} \) has a three dimensional base.

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The probabilistic algorithm, which for a finite field \( \mathbb{F}_q \) selects randomly curves of genus 15, has running time \( O((\log q)^3) \).

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- A unirational description of \( \mathcal{M}_{15} \) would lead to an algorithm with running time \( O((\log q)^2) \).
- For any fixed genus \( g \) there exists an algorithm which selects points from a subset of \( \mathcal{M}_g(\mathbb{F}_q) \) of positive density in running time \( O((\log q)^3) \).

Thank you!