28 Abusing notation we write for two terms
$\lambda x^{\alpha} > \mu x^{\beta}$, $\lambda, \mu \in K - \{0\}$ if $x^{\alpha} > x^{\beta}$.

<u>Proposition</u> $f, g \in K[x_1,..,x_n]$, "$>$" global monomial order
Then

(1) $\operatorname{in}(f \cdot g) = \operatorname{in}(f)\operatorname{in}(g)$

(2) $\operatorname{in}(f+g) \leq \max(\operatorname{in}(f), \operatorname{in}(g))$ and equality holds unless $\operatorname{in}(f) + \operatorname{in}(g) = 0$

<u>Proof</u> (1) $\operatorname{in}(f \cdot g) = \operatorname{in}(f)\operatorname{in}(g)$ because every term m of f satisfies $\operatorname{in}(f) \geq m$

<u>Proof of the division thm:</u>

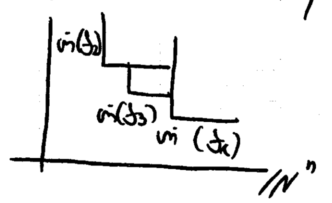
Let $f_1,..,f_r \in K[x_1,..,x_n]$ and "$>$" a global monomial order.
For every $f \in K[x_1..,x_n]$ there are unique $g_1,..,g_r$ in $K[x_1..,x_n]$ and a remainder $h \in K[x_1..,x_n]$ such that

(1) $f = g_1 f_1 + \dots + g_r f_r + h$

(2) (a) No term of $g_i \operatorname{in}(f_i)$ is divisible by an $\operatorname{in}(f_j)$ for some $j < i$

(b) No term of h is divisible by one of the $\operatorname{in}(f_i)$

<u>Existence</u>: Since 2a) and b) induce a partition of the monomials or equivalently exponent vectors



We can write $f = \sum_{i=1}^{r} \tilde{g}_i \operatorname{in}(f_i) + \tilde{h}$ uniquely

Then look at $f' = f - \left(\sum_{i=1}^{r} \tilde{g}_i f_i + \tilde{h}\right)$
Since we have a partition the initial term of f and $\operatorname{in}\left(\sum_{i=1}^{r} \tilde{g}_i f_i + \tilde{h}\right) = \max\{\operatorname{in}(\tilde{g}_i)\operatorname{in}(f_i), \operatorname{in}(\tilde{h})\}$ coincide.
Hence $\operatorname{in}(f') < \operatorname{in}(f)$ and induction applies

$$f' = g'_1 f_1 + \dots + g'_r f_r + h'$$

and then $f = (\underbrace{\tilde{g}_1 + g_1'}_{=g_1}) f_1 + \cdots + (\underbrace{\tilde{g}_r + g_r'}_{=g_r}) f_r + (\underbrace{\tilde{h} + h'}_{=h})$

<u>Remark</u>: It would be good enough to write

$$in(f) = \begin{cases} m \; in(f_i) & , \; in(f) \in \Delta_i \\ m & , \; in(f) \in \bar{\Delta} \end{cases}$$

for term m, where

$\Delta_i = (\exp(f_i) + \mathbb{N}^n) - \bigcup_{j<i} \exp(f_j) + \mathbb{N}^n$

$\bar{\Delta} = \mathbb{N}^n - \bigcup \Delta_i$ and to subtract $in(f_i)$ resp. m

More precisely for the induction

Consider $\mathcal{F} = \{ f \mid f$ has no presentation as in 2a) and b)$\}$

and look at

$M = \{ in(f) \mid f \in \mathcal{F} \}$

We have to prove $M = \emptyset$. If not then by Dixon's lemma this set has a minimal element.

Doing one division step arives at a contradiction

<u>Rmk</u>: Notice that the algorithm as presented in the proof only depends on $in(f_1), \dots, in(f_r)$ but not on the global monomial order.

The existence of global monomial order guarantees termination.

<u>Example</u>: $f_1 = x^2y - y^3$, $f_2 = x^3 \in K[x,y]$, with "$>_{lex}$"

Then $in(f_1) = x^2y$, For $f = x^3y$ we get

$f = x f_1 + 0 f_2 + xy^3$. So h is the remainder because

$xy^3 \notin (x^2y, x^3)$

On the other hand, if we take $f_1' = x^3$, $f_2' = x^2y - y^3$

$x^3y = y f_1' + 0 f_2 + 0$. So even the remainder h by division of f by $f_1 .., f_r$ depends on the order of $f_1 .., f_r$

Preliminary definition: ">" a global monomial order
$f_1, \dots, f_r \in K[x_1, \dots, x_n]$ form a Gröbner basis (GB) (or Gordon basis) if the remainder h of any $f \in K[x_1, \dots, x_r]$ divided by $f_1 \dots f_r$ does not depent on the ordering of $f_1, \dots, f_r$

In practise we give the following definition

23 Def: Let $I \subset K[x_1, \dots, x_n]$ be an ideal and ">" a global monomial order.
The initial ideal of I is

$$in(I) = in_>(I) \doteq (in(f) \mid f \in I).$$

$f_1, \dots, f_r \in I$ are a GB of I if

$$in(I) = (in(f_1), \dots, in(f_r))$$

Note that $in(I)$ is a monomial ideal, i.e an ideal generated by monomials

Gordon's proof of Hilbert's Basis theorem

Let $I \subset K[x_1, \dots, x_n]$. By Dixon's lemma $in(I)$ is finitely generated. Let $f_1, \dots, f_r \in I$ s. th.

$$in(I) = (in(f_1), \dots, in(f_r))$$

Let $f \in I$ be arbitrary and $f = g_1 f_1 + \dots + g_r f_r + h$ the expression from the division thm.
Then no term of h lies in $(in(f_1), \dots, in(f_r))$
On the other hand

$$h = f - \sum g_i f_i \in I.$$

So $in(h) \in in(I)$. Thus $in(h) = 0$, i.e $h = 0$ and $f \in (f_1, \dots, f_r)$, So $I = (f_1, \dots, f_r)$

Rmk: As any proof of Hilberts' basis thm Dixon's proof has two ingredients
(1) An induction on number of variables
(2) A division with remainder
The proof above seperates this two ingredients

2.11 Corollary (of Dixon's proof) (Thm of Macaulay)
The monomials $m \notin in(I)$ represent a K-vectorspace Basis of $K[x_1,..,x_n]/I$. In particular two elements $f, f' \in K[x_1,..,x_n]$ are congruent mod I iff theire rema[inders] h, h' of division by GB are equal.

Uniqueness:
$f = g_1 f_1 + \dots + g_r f_r + h$,
then
$in(f) = \max \{ in(g_1 f_1), .., in(g_r f_r), in(h) \}$
Since they are pairwise distinct. So
$f = 0$ if and only if $g_1 = 0, .., g_r = 0$, $h = 0$
as $in(g_i f_i) = in(g_i)\, in(f_i)$

Proof of Macaulay's thm
$f \equiv f' \mod I$ iff $f - f' \in I$ eq to $h - h' = 0$
By definition of $in(I)$ a remainder $h \in I$ iff $h = 0$
This proves that $m \notin in(I)$ are K-linearly independe[nt] and division with remainder prooves that they span $K[x_1,..,x_n]/I$ as a K-Vectorspace

How to detect GB?
Buchberger's criterion gives an answer and an algori[thm] to compute a GB from a generating set of an ideal
Let $f_1,.., f_r \in K[x_1,..,x_n]$ and $I = (f_1,.., f_r)$
For $1 \leq j < i \leq r$ consider the monomial
$$m = lcm(in(f_i), in(f_j))$$

$$\frac{m}{in(f_i)} f_i - \frac{m}{in(f_j)} f_j =: S(f_i, f_j)$$
In this expression the lead term cancels and dividing $f_1,.., f_r$ might lead to a new initial term in $in(I)$
One can do a little better in a way which point to a proof of Buchberger's thm.

1.12 Def $I, J \subset R$ ideals in a ring.
The quotient ideal (or colon ideal)
$$I : J = \{ r \in R \mid rJ \subset I \}$$
If $J = (f)$ is a principal ideal we write
$$I : f = I : (f)$$

Notation Let $f_1, \dots, f_r \in K[x_1, \dots, x_n]$, $>$ global m. order
Then for $i = 2, \dots, r$ consider
$$M_i = (\mathrm{in}(f_1), \dots, \mathrm{in}(f_{i-1})) : \mathrm{in}(f_i)$$
The minimal monomial generator correspond to the minimal ways to leave
$$\Delta_i = \exp(f_i) + \mathbb{N}^n \setminus (\exp(f_j) + \mathbb{N}^n)$$

Thm (Buchberger)
Let $f_1, \dots, f_r \in K[x_1, \dots, x_n]$ and $>$ global monomial order $f_1, \dots, f_r$ form a GB for $I = (f_1, \dots, f_r)$ if and only if for each monomial generator $m \in M_i$ the remainder of $m \cdot f_i$ divided by $f_1, \dots, f_r$ (in this order) is zero

Remark In the first step of the division algorithm we look at an $\mathrm{in}(f_j)$ with $j < i$ such that $m \cdot \mathrm{in}(f_i)$ is a multiple of $\mathrm{in}(f_j)$ in other words we look at
$$m\ \mathrm{in}(f_i) - \lambda\, m\ \mathrm{in}(f_j) \quad , \lambda \in K$$
which up to scalar is the S polynomial $S(f_i, f_j)$
Buchberger formulated his criterion with S-polynomials
Since there are usually more S-pairs $\binom{r}{2}$ then altogether minimal generator of the M_i's
Our formulation is a bit sharper

Example Consider the ideal of the 2×2 minors of
$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix}$$ and $>_{lex}$ and $x_1 > x_2 > \dots > x_4 > y_1 >$

There are $r = 6 = \binom{4}{2}$ minors with initial term

$x_1 y_2$	$M_1 = 0$	
$x_1 y_3$	$M_2 = (y_2)$	
$x_1 y_4$	$M_3 = (y_2, y_3)$	
$x_2 y_3$	$M_4 = (x_1)$	
$x_2 y_4$	$M_5 = (x_1, y_3)$	
$x_3 y_4$	$M_6 = (x_1, x_2)$	

There are 8 Buchberger tests to do which is less than 15, the number of S-poly

Beweis: Buchbergers Kriterium:
Eine Richtung ist einfach:
Ist $g_1, .., g_r$ eine GB für $I = (g_1, .., g_r)$ und $g \in I$ beliebiges Element, dann ist der Rest $h = 0$, da mit g und $g_1, .., g_r \in I$ auch $h \in I$
$in(h) \notin (in(g_1), .., in(g_r))$ nach Bed (2)(b)
Für die andere Richtung gibt uns das Kriterium für jedes und jeden minimalen Erzeuger m von M_i einen Ausdruck

$$m g_i = \sum_{j=1}^{r} g_j^{(m,i)} g_j$$

Der Vektor $G^{(i,m)} := (-g_1^{(m,i)}, .., m - g_i^{(m,i)}, .. - g_r^{(m,i)})$ liegt im Kern der Abb.

$$P^r \to P, \ (a_1, .., a_r) \mapsto \sum a_i g_i$$

wobei $P = K[x_1, .., x_n]$
Def: Sei R ein Ring und $g_1, .., g_r \in R^s$ Elemente in einem freien R-modul vom Rang s.
Ein Element $(a_1, .., a_r) \in Ker(R^r \xrightarrow{(g_1, .., g_r)} R^s)$ nennt man eine Syzygie zwischen $g_1, .., g_r$
$Ker(R^r \to R^s)$
heißt Syzygien-Modul
Beispiel: $\left(\frac{in\, g_j}{m}, -\frac{in\, g_i}{m}\right) \in Ker\left(P^2 \xrightarrow{(in\, g_i,\, in\, g_j)} P\right)$
und $m = lcm(in\, g_i, in\, g_j)$ ist eine Syzygie zwischen $in\, g_i$ und $in\, g_j$ und dieser erzeugt den Syzygien-Modul