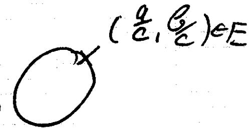
# Algebraic Geometry

= study of the geometry of the solutions of algebraic systems of equations

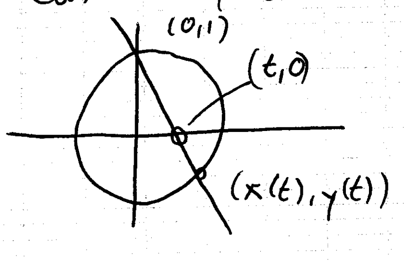



<u>Example</u> $x^2 + y^2 = 1$ defines the circle
$(a, b, c) \in \mathbb{Z}_{>0}$ in Pythagorean triple
$a^2 + b^2 = c^2$

<u>General setting</u> Let $\bar{k}$ be a fixed algebraically closed field $k \subset \bar{k}$ a subfield ( Typically : $\mathbb{Q} \subset \mathbb{C}$)
Given $f_1, .., f_r \in k[X_1, .., X_n]$, then
$$X = V(f_1 .. f_r) = \{ a = (a_1, .., a_n) \in \bar{k}^n \mid f_1(a) = 0, .., f_r(a) = 0 \}$$
the vanishing loci or zero loci
Let $\mathbb{A}^n = \mathbb{A}^n(\bar{k}) = \bar{k}^n$ a set (not the vectorspace denote the affine n-space)
$X = V(f_1 .., f_r) \subset \mathbb{A}^n$ is called an <u>algebraic</u> set
$X(k) = X \cap \mathbb{A}^n(k)$ is called the set of <u>$k$-rational points</u>
$k$ is called a <u>field</u> of <u>definition</u> of $X$

<u>Remark</u>: Since a polynomial $f \in \bar{k}[x_1, .., x_n]$ has only finitely many coefficients we take as a field of definition of $V(f_1, .., f_r)$ always a finitely generated extension ofield of a prime field.
$$\mathbb{Q}(\alpha_1, .., \alpha_n) \quad \text{or} \quad \mathbb{F}_p(\alpha_1, .., \alpha_n)$$

<u>Example</u> $(\frac{a}{c}, \frac{b}{c}) \in E(\mathbb{Q})$ is $\mathbb{Q}$ rational point of the circle
Can we find all ?




Consider the projection of $E$ from the point $(0,1)$ on the $x$-axis
The equation of the line through $(x_1, y_1) = (0,1)$, $(x_2, y_2) = (t, 0)$ is given by
$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\Leftrightarrow y - 1 = -\frac{1}{t}(x - 0)$$
Substituting $y = -\frac{x}{t} + 1$ into $x^2 + y^2 = 1$ gives
$$x^2 + \left(-\frac{x}{t} + 1\right)^2 = 1 \Leftrightarrow x^2 + \left(\frac{x^2}{t^2} - 2\frac{x}{t}\right) = 0$$
$$\Leftrightarrow x\left(x\left(1 + \frac{1}{t^2}\right) - \frac{2}{t}\right) = 0$$

Hence $x(t) = \frac{2}{t} \cdot \frac{1}{1+1/t^2} = \frac{2t}{t^2+1}$, $y(t) = \frac{t^2-1}{t^2+1}$

By using the equation of the line

Check: $\left(\frac{2t}{t^2+1}\right)^2 + \frac{(t^2-1)^2}{(t^2+1)^2} = \frac{(t^2+1)^2}{(t^2+1)^2} = 1$

$f: \mathbb{A}^1 \to E$, $t \mapsto (x(t), y(t)) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$

is a rational parametrization of the circle with $\lim_{t\to\infty} f(t) = (0,1)$.

Why is this better than the analytic parametrization $t \mapsto (\cos(t), \sin(t))$?

Because $f(\mathbb{Q}) \cup \{(0,1)\} = E(\mathbb{Q})$

What are the basic questions we would like to answer?

(A) Given $f_1, \dots, f_r \in K[x_1, \dots, x_n]$

Does there exist a solution $X = V(f_1, \dots, f_r) \neq \emptyset$?

Answer: Hilberts Nullstellensatz

$X = \emptyset \iff$ The ideal $I = (f_1, \dots, f_r) = \{\sum_{i=1}^{r} g_i f_i \mid g_i \in K[x_1, \dots, x_n]\}$ contains $1$

$\iff I = (1)$

"$\Leftarrow$" easy "$\Rightarrow$" is the essential direction

"Looking for "solution in $\overline{K}$ is important

$X = V(x^2+1)$, $X(\mathbb{R}) = \emptyset$ but $\langle x^2+1 \rangle \subsetneq \mathbb{R}[x]$

(B) When is $X$ a finite set?

Answer: $|X| < \infty \iff$ the quotient ring $k[x_1, \dots, x_n]/I$ is a finite dim'l $K$-Vectorsp.

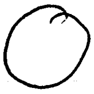
If $\dim_{K\text{-VS}} K[x_1, \dots, x_n]/I = N < \infty$, then $1, \bar{x}_i, \bar{x}_i^2, \dots, \bar{x}_i^N$ must be $K$-linear dependent in $K[x]/I$

(C) If $X = V(f_1, \dots, f_n)$ is an infinite set, what is the dimension of the solution space?

Answer: Suppose $(f_1, \dots, f_r)$ is a prime ideal

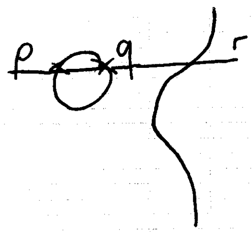
Then $K[x]/I$ is an integral domain and

$\dim X = \operatorname{tr\,deg}_K Q(k[x]/I)$ (Quotient field)

 $x^2+y^2-1$ $\quad \mathbb{C}[x,y]/(x^2+y^2-1) = \mathbb{C}(x)[y]/(y^2+x^2-1)$

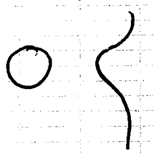
$\cup\ \mathbb{C}(x)$ is degree 2 alg extension.

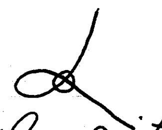
(D) Given $f_1,\dots,f_r \in \mathbb{Z}[x_1,\dots,x_n]$ is $X(\mathbb{Q}) \neq \emptyset$?
$X(\mathbb{Z})$ Diophantine Geometry
Example $X: y^2 = x^3+ax+b \in \mathbb{Q}[x,y]$, $a,b \in \mathbb{Q}$




$p,q \in X(\mathbb{Q}) \Rightarrow r \in X(\mathbb{Q})$
Elliptic curve
Mordell: $X(\mathbb{Q}) \cup \{\infty\}$ is a finitely generated abelian group

(E) Can we use rational functions to parametrize $X$?
Example: $y^2 = x^3 - x$ no $\qquad y^2 = x^3 + x^2$ yes

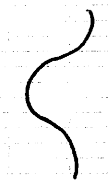




Exercise 1

Singular point
Remember $K[x,y]$ is factorial

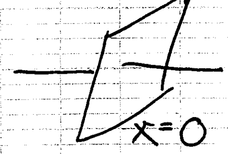
$y^2 = x^3 + x$



no singular points

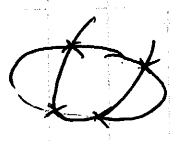
We will answer this question completely in case $X = 1$
For $\dim X > 1$ this question is part of birational geometry the research area of Vladi Lazić

(F) For some systems, e.g. $xy = 0$, $xz = 0$ the solution set decomposes



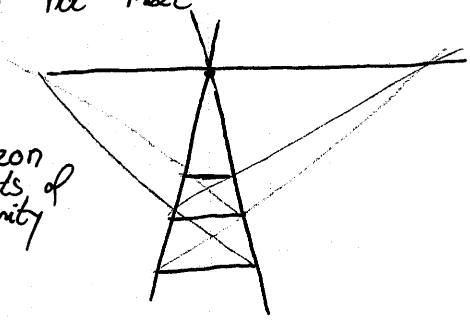

When is $X$ irreducible?
Answer: $I = \langle f_1,\dots,f_r \rangle$ is a prime ideal $\Rightarrow X$ irreducible

(G) For $n$ polynomials in $n$ Variables
Taking our clue from linear algebra, we expect for sufficiently general polynomials only finitely many solutions
For two conic degree two equations $4 = 2\cdot 2$ solutions



(H) For special system of equations we might have fewer or infinitely many solutions ?
Parallel lines do not meet

$\mathbb{P}^n = \mathbb{A}^n \cup$ Horizon
points of infinity

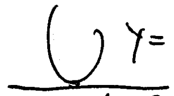


H Horizon

(I) Thm (Bézout), $f, g \in K[X,Y]$ be polynomials of degree $d$ and $e$ such that $C = V(f)$ and $D = V(g)$ have no common component. Then

$$\sum_{p \in \mathbb{P}^2} \underbrace{i(C,D,p)}_{\text{intersection multiplicity}} = de$$

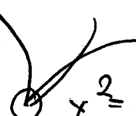
<u>Example</u>



$y = x^2$
$y = 0$
$i$ double zero

How to define $i(C,D,p)$

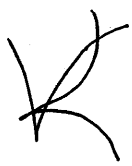
(1) $y^3 = x^2$



$y^2 = x^3$

dynamical point of view

(2)



$K[X,Y]/\langle f,g \rangle$

$p = (0,0) \in \mathbb{A}^2$, $i(C,D,0) = \dim_{K\text{-VS}} K[X,Y]/\langle f,g \rangle$

$\langle y^2 - x^3, x^2 - y^3 \rangle = \langle x^2 - y^3, y^2 - xy^3 \rangle = \langle x^2 - y^3, y^2 \rangle$
$= \langle x^2, y^2 \rangle$