§1 The Algebra - Geometry Dictionary

Throughout the course K denotes a field and $\overline{K}$ an algebraically closed extension field of K.
For example $\mathbb{Q} \subset \mathbb{C}$

$\mathbb{A}^n(\overline{K}) = \{(a_1,\dots,a_n) \in \overline{K}^n\}$ affine n-space and $\mathbb{A}^n(K) = \{(a_1,\dots,a_n) \in K^n\}$ its subset of K-rational points of $\mathbb{A}^n$

<u>1.1 Def:</u> For $I \subset K[x_1,\dots,x_n]$ a subset we denote by
$$V(I) = \{a \in \mathbb{A}^n \mid f(a) = 0 \quad \forall f \in I\}$$
its <u>vanishing loci</u> (zero loci)
For a finite set $\{f_1,\dots,f_r\}$ we simplify notation
$$V(f_1,\dots,f_r) := V(\{f_1,\dots,f_r\})$$
For $f \in K[x_1,\dots,x_n]$ not constant we call
$$V(f) \subset \mathbb{A}^n$$
a <u>hypersurface</u>
A subset $A \subset \mathbb{A}^n$ is called an algebraic set defined over K if there is $I \subset K[x_1,\dots,x_n]$ such that $A = V(I)$

<u>1.2 Prop</u> Every algebraic set is a finite intersection of hypersurfaces

<u>Proof:</u> Suppose $X = V(I)$
Then we consider the ideal $J = \left\{ \sum_{i=1}^{r} g_i f_i \mid \begin{matrix} g_i \in K[x_1,\dots,x_n] \\ f_i \in I \end{matrix} \right\}$
the <u>ideal generated</u> by the set I
Then $V(J) = V(I)$ because for all $a \in X$
$$\left(\sum_{i=1}^{r} g_i f_i\right)(a) = \sum_{i=1}^{r} g_i(a)\, f_i(a) = 0 \quad \text{and} \quad I \subset J.$$

So w.l.o.g. we may assume that I is an ideal
By Hilbert's basis theorem ($K[x_1,\dots,x_n]$ is noetherian)
I is finitely generated, say
$$I = (f_1,\dots,f_r) = \left\{ \sum_{i=1}^{r} g_i f_i \mid g_i \in K[\underline{x}] \right\}$$
Then $V(I) = \bigcap_{j=1}^{r} V(f_j)$ $\square$

1.3 Let I, J and $(I_\lambda)_{\lambda\in\Lambda}$ be ideals in $K[x_1,..,x_n]$ then $I\cdot J = (g f \mid g\in I, f\in J)$ denotes the ideal generated by the products and

$\sum_{\lambda\in\Lambda} I_\lambda = (\bigcup_{\lambda\in\Lambda} I_\lambda)$ the ideal generated by the unions

Prop.: (i) $V(0) = \mathbb{A}^n$, $V(1) = \emptyset$

(ii) $V(I\cdot J) = V(I\cap J) = V(I)\cup V(J)$

(iii) $V(\sum_\lambda I_\lambda) = \bigcap_{\lambda\in\Lambda} V(I_\lambda)$

(iv) $I\subset J \Rightarrow V(I)\supset V(J)$

Proof: (i) (iii) and (iv) are obvious

(ii) $I\cdot J \subset I\cap J \subset I$ and $\subset J$, so $V(IJ)\supset V(I)\cup V(J)$ by (iv).

To see equality. Let $a\in V(IJ)$ with $a\notin V(I)$
We have to show $a\in V(J)$
By assumption $\exists\, g\in I$ such that $g(a)\neq 0$
for every $f\in J$ we have
$0 = (g\cdot f)(a) = g(a) f(a) \in \bar{K}$ is a field
$\Rightarrow f(a) = 0 \ \forall f\in J$ and we have $a\in V(J)$ $\square$

One can rephrase (i) – (iii) as follows
"The algebraic closed sets defined over K form the closed sets of a topology on $\mathbb{A}^n$" the K-Zariski-topology

Recall: A topology on a set X is a subset $\mathcal{T}\subset 2^X$ such that

(1) $\emptyset, X\in\mathcal{T}$

(2) $U, V\in\mathcal{T} \Rightarrow U\cap V\in\mathcal{T}$

(3) $U_\lambda\in\mathcal{T}\ \forall\lambda\in\Lambda \Rightarrow \bigcup_{\lambda\in\Lambda} U_\lambda\in\mathcal{T}$

The element $U\in\mathcal{T}$ are called the open sets of the topology and there complements $X\setminus U = A$ are called the closed sets of the topology. For $B \subset X$ arbitrary $\overline{B} = \bigcap_{\substack{A \supset B \\ \text{closed}}} A$ is called the closure of B with respect to T.

The Zarisky topology T on $\mathbb{A}^n$ is the topology whose closed sets are algebraic defined over $\overline{K}$.

<u>Remark:</u> In case $\overline{K} = \mathbb{C}$ we have also the ordinary (or euclidean) topology on $\mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n$.
Since polynomials are continous functions every Zariski open subset of $\mathbb{A}^n(\mathbb{C})$ is also open in $\mathbb{C}^n$.
The converse is far from being true.
Since $\overline{K}[X_1]$ is a PID every closed set of $\mathbb{A}^1$ different from $\mathbb{A}^1$ consists of finitely many points.
$I \subset K[X_1]$, $I = (f)$ with $(f) \neq (0)$, then

$$V(I) = \{ p_1, \ldots, p_d \mid p_i \in \overline{K} \text{ are the roots of } f \}$$

Hence the Zariski-topology on $\mathbb{A}^1$ is the cofinite topology whose open sets $\neq \emptyset$ are complements of finite sets.
In particular, any two non empty Zariski open subsets intersect in a non-empty (open) set.

<u>Exercise 2:</u>

(1) Let $X = V(I) \subsetneq \mathbb{A}^n$ be an (proper) algebraic set and suppose that K is an infinite field.
Proove that there exists a K-rational point $a \in \mathbb{A}^n(K) \setminus X$

(2) Let $X, Y \subsetneq \mathbb{A}^n$ be algebraic sets and $U = \mathbb{A}^n \setminus X$, $V = \mathbb{A}^n \setminus Y$ their complements.
Proove $U \cap V \neq \emptyset$.

<u>1.4. Def:</u> Let $A \subset \mathbb{A}^n$ be an arbitrary subset.
Then

$$I(A) = \{ f \in \overline{K}[X_1, \ldots, X_n] \mid f(a) = 0 \ \forall a \in A \} \subset \overline{K}[X_1, \ldots, X_n]$$

is called the vanishing ideal of A.

Prop $A, B, (A_\lambda)_{\lambda\in\Lambda}$ subsets of $\mathbb{A}^n$

Then (1) $I(\emptyset) = \bar{K}[x_1, \dots, x_n]$, $I(\mathbb{A}^n) = 0$

(2) $I(\bigcup_{\lambda\in\Lambda} A_\lambda) = \bigcap_{\lambda\in\Lambda} I(A_\lambda)$

(3) $A \subset B \Rightarrow I(A) \supset I(B)$

(4) $I(A) + I(B) \subset I(A \cap B)$

(5) $I(\{(a_1, \dots, a_n)\}) = (x_1 - a_1, \dots, x_n - a_n)$

(6) $V(I(A)) \supset A$ with equality of A is algebraic set. In general

$\bar{A} = V(I(A))$ is the Zariski closure of A

Proof: (1) $I(\emptyset) = (1)$ is obvious

$I(\mathbb{A}^n) = 0$ use that for $f \in \bar{K}[x_1, \dots, x_n] \setminus \{0\}$ there is $a \in \mathbb{A}^n$ with $f(a) \neq 0$

(Special and crucial case of Exercise 2.(1))

(2) $f \in I(\bigcup_{\lambda\in\Lambda} A_\lambda) \Leftrightarrow \forall \lambda \in \Lambda\ \forall a \in A_\lambda : f(a) = 0$

$\Leftrightarrow \forall \lambda \in \Lambda\ f \in I(A_\lambda) \Leftrightarrow f \in \bigcap_{\lambda\in\Lambda} I(A_\lambda)$

(3) Clear: A impose fewer conditions on f' than B.

(4) $h \in I(A) + I(B)$, then $h = f + g$ with $f \in I(A)$ $g \in I(B)$. Let $a \in A \cap B$, then $h(a) = f(a) + g(a) = 0$ So $h \in I(A \cap B)$

Remark: The inclusion $I(A) + I(B) \subset I(A \cap B)$ might be strict. Ex:

$A = \{(a, 0) \mid a \neq 0\} \subset \mathbb{A}^2$, $B = \{(0, b) \mid b \neq 0\}$

then $I(A) = (y)$, $I(B) = (x)$ and

$I(A) + I(B) = (x, y) \subsetneq I(A \cap B) = I(\emptyset) = (1)$.

(5) We note $f \in \bar{K}[x_1, \dots, x_n]$ in its „Taylorexpansion" at the point $(a_1, \dots, a_n)$

$$f = \sum_{\alpha\in\mathbb{N}^n} c_\alpha (x_1 - a_1)^{\alpha_1} \cdots (x_n - a_n)^{\alpha_n}$$

is a finite sum, i.e. all but finitely

many c_α are zero because f is a polynomial
$f(a) = 0$ is eq to $c_0 = 0$
Hence $f \in (x_1 - a_1, \dots, x_n - a_n)$

(6) $\bar{A} := V(I(A)) \supset A$ is clear
In case A is algebraic say $A = V(f_1, \dots, f_r)$ for $f_1, \dots, f_r \in \bar{K}[x_1, \dots, x_n]$ and $a \notin A$ then there is f_j with $f_j(a) \neq 0$. Since $f_1, \dots, f_r \in I(A)$ this implies $a \notin V(I(A))$ so $\bar{A} = A$ in this case.
A arbitrary and $B \supset A$ algebraic set, then
$I(B) \supset I(A)$
Hence $B = V(I(B)) \supset V(I(A)) \supset A$, so $V(I(A))$ is the smallest Zariski closed subset containing A.

__Remark__: If we consider instead of the Zariski topology the K-Zariski topology weird things happen:
Example: $\mathbb{Q} \subset \mathbb{C}$, $\pi \in \mathbb{A}^1(\mathbb{C}) = \mathbb{C}$
$I_{\mathbb{Q}}(\{\pi\}) = \{ f \in \mathbb{Q}[x] \mid f(\pi) = 0\} = (0)$
because π is transcendent

__Exercise__** Compute the $\mathbb{Q}$-Zariski closure of $\{(\pi, e)\}$ in $\mathbb{A}^2(\mathbb{C})$ and get famous.
Schanuel's conjecture (1960) $\Rightarrow$ $\overline{(\pi, e)}^{\mathbb{Q}} = \mathbb{A}^2(\mathbb{C})$
i.e π and e are algebraically independent over $\mathbb{Q}$.

__1.5 Def./Prop.__ $I \subset R$ ein Ideal in einem kommutativen Ring mit 1. Dann heißt
$$\mathrm{rad}(I) = \{ f \in R \mid \exists\, n > 0 : f^n \in I \} \subset R$$
das Radikalideal von I. I nennt man Radikalideal, wenn $I = \mathrm{rad}(I)$.

__Bew__: zz $\mathrm{rad}(I)$ ist Ideal. $f, g \in \mathrm{rad}(I)$, es ex. n, m mit $f^n, g^m \in I$.
$(f + g)^{n+m-1} \in I$ und $f + g \in \mathrm{rad}(I)$