außerdem das Ideal $I(C_i) \subset \bar{K}[x_1, \ldots, x_n]$ durch Polynome aus $K[x_1, \ldots, x_n]$ erzeugt ist.
Eine alg. Menge $C \subset \mathbb{A}^n$ definiert über K heißt absolut irreduzibel, wenn $J = I(C) \subset \bar{K}[x_1, \ldots, x_n]$ ein Primideal ist, das von Polynomen $f_1, \ldots, f_r \in K[x_1, \ldots, x_n]$ erzeugt wird. In diesem Fall ist

$(f_1, \ldots, f_r) \subset K[x_1, \ldots, x_n]$

ebenfalls ein Primideal. und $\bar{K}[C] \cong \bar{K} \otimes_K K[C]$

Übung 33 Ist $C \subset \mathbb{A}^n$ definiert über K absolut irreduzibel

$$\mathrm{trdeg}_K K(C) = \mathrm{trdeg}_{\bar{K}} \bar{K}(C)$$

1.26 Df: Let $A \subset \mathbb{A}^n$ be an absolutely irreducible algebraic set defined over $K \subset \bar{K}$ alg. closed.
Then we call A an affine variety

$$\bar{K}(A) = Q(\bar{K}[A])$$

is called the field of rational functions on A. and $K(A) = Q(K[A])$ is a subfield of $\bar{K}(A)$ and is called the field of rational functions defined over K.
The Zariski topology is the subspace topology of $A \subset \mathbb{A}^n$ of the topological space $\mathbb{A}^n$ endowed with the Zariski-topology.
(In case $\bar{K} = \mathbb{C}$ we also have the euclidean topology $\mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n$)
A quasi affine variety is an non empty Zariski open subset $U \subset A$ in an affine variety $A \subset \mathbb{A}^n$

Exercise: The Zariski topology of A does not depend on the embedding $A \subset \mathbb{A}^n$, i.e. choose generators $z_1, \ldots, z_r$ of the $\bar{K}$-Algebra and the corresponding embedding $A \hookrightarrow \mathbb{A}^r$ gives the same topology.

Remark: (1) A quasi-affine variety can be isomorphic to an affine variety

Example $U = \mathbb{A}^1 \setminus \{0\} \longrightarrow V(XY-1) \subset \mathbb{A}^2$

$a \longmapsto (a, 1/a)$

$$\bar{K}[U] = \bar{K}[X, X^{-1}] \subset \bar{K}(X)$$

$$\begin{array}{cc} \uparrow & \uparrow^{X^{-1}} \\ K[X,Y] & Y \end{array}$$

(2) We will later see that
$\mathbb{A}^2 - \{0\} = \mathbb{C}^2 - \{0\}$ is a quasi-affine variety
which is not isomorphic to an affine variety

1.30 Df: Let $A \subset \mathbb{A}^n$ be an affine variety
A rational map $\Phi : A \dashrightarrow \mathbb{A}^m$ is given by an m-tupel
$(S_1, \dots, S_m)$ of elements $S_i \in K(A)$
If $S_i = \bar{g}_i / \bar{h}_i$, $\bar{g}_i, \bar{h}_i \in K[A]$ and $g_i, h_i \in K[X_1, \dots, X_n]$
representatives then $h_i \notin I(A)$
then Φ is a honest map on $U = A - (\cup V(h_i))$

$$f : U \to \mathbb{A}^m$$
$$a \mapsto (S_1(a), \dots, S_m(a)) = \left(\frac{g_1(a)}{h_1(a)}, \dots, \frac{g_m(a)}{h_m(a)} \right)$$

So Φ is only a map on a non empty Zariski open subset of A.
Since $K[A]$ might not be factorial we can find diff dec. of S_i
as fraction $S_i = \bar{g}_i / \bar{h}_i = \bar{g}_i' / \bar{h}_i'$ $\Leftrightarrow$ $\bar{g}_i \bar{h}_i' = \bar{g}_i' \bar{h}_i$
and consequently U is not unique
$U' = A - (\bigcup_{i=1}^{m} V(h_i'))$. But $U \cap U' \neq \phi$ and $f : U \to \mathbb{A}^n$
$f' : U' \to \mathbb{A}^n$ coincide on $U \cap U'$, $f|_{U \cap U'} = f'|_{U \cap U'}$
so we can define Φ as a map on $U \cup U'$.
A rational map $\Phi : A \dashrightarrow B \subset \mathbb{A}^n$, A absolutely irreducible is a rational map $\Phi : A \dashrightarrow \mathbb{A}^n$ with $\Phi(a) \in B$
for any honest map $f : U \to \mathbb{A}^n$ where Φ is defined.
Exercise: $\overline{f(U)} \subset B$. The Zariski closure of $f(U) \subset \mathbb{A}^m$ lies in an irreducible component of B.

Example (1) $\mathbb{A}^1 \to V(X^2 + Y^2 - 1) \subset \mathbb{A}^2$, $t \mapsto \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1} \right)$
is a rational map
(2) Let A, B, C be affine varieties and $\Phi : A \dashrightarrow B$, $\Psi : B \dashrightarrow C$
be rational maps.

In general $\Psi \circ \Phi$ is not defined. The problem is that for $U \subset A$ open such th. $f : U \to B$ is defined and $V \subset B$ open such that $\Psi : V \to C$ is defined we may have
$f(U) \subset B - V$

Def: A rational map $\Phi : A \to B$ between affine varieties is called <u>dominant</u> if for $U \subset A$ a domain of definition of Φ. $\varphi(U)$ is Zariski dense in B.

In that case for $\Psi : B \dashrightarrow C$ and $V \subset B$ a domain of definition of Ψ, $\psi = \Psi|V$ we have $\varphi(U) \cap V \neq \emptyset$ and $U' = \varphi^{-1}(\varphi(U) \cap V) \subset A$ is Zariski open and $\varphi|U'$ can be composed with ψ

So if $\Phi : A \dashrightarrow B$ dominant then $\Psi \circ \Phi : A \dashrightarrow C$ is defined for every $\Psi : B \dashrightarrow C$

<u>Example</u>: $A \subset \mathbb{A}^n$ affine $f \in \bar{K}(A)$. Then f defines a rational map $A \dashrightarrow \mathbb{A}^1$, $a \mapsto f(a)$ and f is dominant unless f is constant, $f \in \bar{K}$.

Uses the exercise

$f = \bar{g}/\bar{h}$, $U = A - V(h)$, $f : U \to \mathbb{A}^1$ is defined on U.

Likewise $f \in K[A] \subset K(A)$ then $f : A \to \mathbb{A}^1$ defines a morphism.

Given a dominant rational map $\Phi : A \dashrightarrow B$ between varieties. $A \xrightarrow[(f_1, f_m)]{} B \subset \mathbb{A}^m$

every $g \in K(B)$ pulls back, $g = \bar{a}/\bar{b}$, $\bar{a}, \bar{b} \in K[B]$

$$\Phi^* g = = g(f_1, , f_m) = \frac{\bar{a}(f_1, , f_m)}{\bar{b}(f_1, , f_m)} \in K(A)$$

Since $\bar{b}(f_1, , f_m) \neq 0$ which holds because Φ is dominant and $b \notin I(Y) \subset K[Y_1 \dots Y_m]$, b represent. and $\varphi(U) \not\subset B \cap V(b) \subsetneq B$.

<u>Thm</u>: Let A, B be affine varieties.

There exists a bijection

$$\{ A \underset{\text{dominant}}{\overset{\Phi}{\dashrightarrow}} B \} \longleftrightarrow \left\{ \begin{matrix} \text{field extension} \\ \text{of } \bar{K} \text{ algebras} \end{matrix} \ \bar{K}(B) \hookrightarrow \bar{K}(A) \right\}$$

defined by $\Phi \longmapsto g \mapsto \Phi^* g = g \circ \Phi$

<u>Proof</u>: Let $i : \bar{K}(B) \to \bar{K}(A)$ a ring homomorphism and letter a morphism of $\bar{K}$-algebras

Unless this is the zero map.

It is not the zero map because $\bar{K} \xrightarrow{=} \bar{K}$

Suppose $B \hookrightarrow \mathbb{A}^m$ and $\bar{K}[B] = \bar{K}[Y_1, \dots, Y_m]/I(B)$
Then $\bar{Y}_i \in \bar{K}[B] \subset \bar{K}(B)$ are rational functions
then $f_j = i(\bar{Y}_j) \in \bar{K}(A)$. and

$$\begin{array}{ccl} A & \dashrightarrow & \mathbb{A}^m \\ \| & & \cup \\ A & \dashrightarrow & B \end{array}$$

which induces a rational map
Indeed if $g \in I(B)$

$\Phi^* g = g \circ \Phi = g(f_1, \dots, f_m)$

If $U \subset A$ is a domain of definition of $A \xrightarrow{(f_1, \dots, f_m)} \mathbb{A}^m$ we have

$$(\Phi^* g)(a) = g(f_1(a), \dots, f_m(a)) = g(i(Y_1), \dots, i(Y_m))(a)$$
$$= i(\underbrace{g(\bar{Y}_1, \dots, \bar{Y}_m)}_{=0})(a) = 0$$

So g vanishes on $f(U)$, i.e. $f(U) \subset V(g)$ $\forall g \in I(B)$
$\Rightarrow f(U) \subset B$ $\square$

Def: Two affine varieties A, B are called birational if there exists rational maps, also dominant
$\Phi : A \dashrightarrow B$, $\Psi : B \dashrightarrow A$
such that $\Psi \circ \Phi : A \dashrightarrow A$ with $\Psi \circ \Phi|_U = id_U$
for any domain of definition U of $\Psi \circ \Phi$
and $\Phi \circ \Psi : B \dashrightarrow B$ with $\Phi \circ \Psi|_V = id_V$ for any domain of definition of $\Phi \circ \Psi$.

Cor: A, B are birational iff $\bar{K}(A) \cong \bar{K}(B)$ as extensions of fields of $\bar{K}$.

Example: $A = V(X \cdot Y - 1) \subset \mathbb{A}^1$

$\bar{K}[A] = \bar{K}[X, Y]/(XY-1) \cong \bar{K}[X, X^{-1}] \subset \bar{K}(X)$

$\bar{K}(X) = Q(\bar{K}[X]) = \bar{K}(\mathbb{A}^1)$

Thm: The category of finitely generated field extensions of $\bar{K}$ with $\bar{K}$ alg closed with objects these fields and morphisms field injections over $\bar{K}$ is equivalent to the category whose objects are birational equivalence classes of affine varieties $[A]$ and morphisms represented by dominant rational maps

$\mathrm{Hom}(\mathbb{C}A\mathbb{J}, \mathbb{C}B\mathbb{J}) = \{\text{dominant rat. map } A \dashrightarrow B\}$

Proof: Let $L = \bar{k}(\bar{Y}_1, \dots, \bar{Y}_m)$ be a finitely generated extension field.

Consider $\bar{k}[Y_1, \dots, Y_m] \xrightarrow{f} L$, $Y_i \mapsto \bar{Y}_i$. Then $\mathrm{Im}(f)$ is a subring of the field.

Hence a domain and $I = \mathrm{Ker} f \subset \bar{k}[Y_1, \dots, Y_m]$ is a prime ideal.

Then $Y = V(I) \subset \mathbb{A}^m$ is an affine variety with coordinate ring $\bar{k}[Y_1] = \bar{k}[Y_1 \dots Y_n]/I \cong \mathrm{Im} f$

$\bar{k}(Y) \cong Q(\mathrm{Im} f) = L$ because $\bar{Y}_1, \dots, \bar{Y}_m \in Q(\mathrm{Im} f)$ hence contains $\bar{k}(\bar{Y}_1, \dots, \bar{Y}_m) = L$.

Example: (Lüroth)

(1) $\bar{k}(x) \supset L \supset \bar{k}(f)$ $\quad f \in \bar{k}(x)$ not constant

$\mathbb{A}^1 \overset{f}{\dashrightarrow} \mathbb{A}^1 \qquad \bar{k}(f) = \bar{k}(Y)$

Lüroth: $L \cong \bar{k}(g)$ for some g so with $\bar{k}(x)$ and $\bar{k}(f)$ purely transcendental exts L is also transcend. ext.

$\mathbb{A}^n \overset{\text{dom}}{\dashrightarrow} \mathbb{A}^n$

$\bar{k}(x_1, \dots, x_n) \supset L \supset \bar{k}(f_1, \dots, f_n)$ purely transcendental

Question: Is L also purely transcendental?

$L = \bar{k}(X)$ is $X \approx \mathbb{A}^n$ again?

Answer: True for $n = 1, 2$

false for $n \geq 3$.