UNIVERSITÄT DES SAARLANDES
Fachrichtung Mathematik
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Algebraic Geometry

Summer Term 2018

Exercise Sheet 1

Exercise 1.1

Prove that $V(y^2 - x^3 + x) \subset \mathbb{A}^2$ has no rational prarametization, but $V(y^2 - x^2 - x^3)$ has one.

Exercise 1.2

Let $X = V(I) \subsetneq \mathbb{A}^n$ be an proper algebraic subset and K an infinite field.

(1) Prove, that there is an K-rational point $a \in \mathbb{A}^n(K) \setminus X$.
(2) Let $X, Y \subsetneq \mathbb{A}^n$ be two algebraic sets and let $U = \mathbb{A}^n \setminus X, V = \mathbb{A}^n \setminus Y$ their complements. Prove: $U \cap V \neq \emptyset$.

Exercise 1.3**

Compute the $\mathbb{Q}$-Zariski closure of the point $(\pi, e) \in \mathbb{A}^2(\mathbb{C})$ and get famous.

Schanuel's conjecture (1960) implies that $\overline{\{(\pi, e)\}}^{\mathbb{Q}} = \mathbb{A}^2(\mathbb{C})$, which means that π and e are algebraically independent over $\mathbb{Q}$.

Exercise 1.4

Find algebraic sets $A, B \subset {}^2$ such that

$$I(A) + I(B) \subsetneq I(A \cap B).$$

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Exercise Sheet 2

Exercise 2.1

(1) Find a polynomial $f \in \mathbb{F}_q[x_1, \ldots, x_n]$, which vanishes on all points of $\mathbb{A}^n(\mathbb{F}_q)$, where $\mathbb{F}_q$ denotes the field with q elements.
(2) Compute $I(\mathbb{A}^n(\mathbb{F}_q))$.

Exercise 2.2

Let $K = \overline{K}$ and let $A = \{p_1, \ldots, p_d\}$ be a finite set of points. Proof that

$$K[A] = K^A$$

is the set of all functions $\{A \to K\}$. In particular,

$$\dim_{K-Vect} K[A] = d.$$

Hint: Construct polynomials f_p such that $f_p(q) = \delta_{p,q}$ for all $q \in A$.

Exercise 2.3

Prove, that the parabola $A = V(y - x^2)$ and the hyperbola $B = V(xy - 1) \subset \mathbb{A}^2$ are not isomorphic.

Exercise 2.4

Let $I \subset K[x_1, \ldots, x_n]$ be an ideal. Prove that $A = V(I) \subset {}^2$ is finite if and only if $d = \dim_{K-Vect} K[x_1, \ldots, x_n] < \infty$, and that d is a bound for the number of points of A.

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Exercise Sheet 3

Exercise 3.1

Let $K = \mathbb{F}_p(t)$ and let $\overline{K}$ be the algebraic closure of K. Consider $f = x^p - ty^p \in K[x, y]$. Prove that f generates a radical ideal in $K[x, y]$, but does not generate a radical ideal in $\overline{K}[x, y]$.

Exercise 3.2

(1) Let $f \in K[x_1, \ldots, x_n$ be a homogeneous polynomial, and let g be an irreducible factor. Prove that g is homogeneous as well.

(2) Consider $q = x^2+y^2 \in \mathbb{Q}[x, y]$. Prove that q generates a prime ideal in $\mathbb{Q}]x, y]$, but does not generate a prime ideal in $\mathbb{C}[x, y]$.

Exercise 3.3

Let C be an algebraic subset of $\mathbb{A}^n$ defined over K. Prove that if C is absolutely irreducible, then

$$trdeg_K K(C) = \dim C = trdeg_{\overline{K}} \overline{K}(C).$$

Exercise 3.4

Let $\varphi : C \dashrightarrow \mathbb{A}^m$ be an rational map on a variety, and let $U \subset C$ be a domain of definition of φ. Prove that the Zariski closure

$$\overline{\varphi(U)} \subset \mathbb{A}^m$$

of $\varphi(U)$ is irreducible.