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Algebraic Geometry   Summer Term 2018

Exercise Sheet 6. Hand in by Friday, June 1.

Exercise 1

Given polynomials $f_1, \dots, f_r \in \mathbb{Z}[x_1, \dots, x_n]$, consider the ideal $I_0 \subset \mathbb{Q}[x_1, \dots, x_n]$ generated the f_i's and for a finite prime field $\mathbb{F}_p = \mathbb{Z}/(p)$ the ideal $I_p \subset \mathbb{F}_p[x_1, \dots, x_n]$ generated by the polynomials $\overline{f_j}$ obtained by coefficient reduction mod p. Prove that for all but finitely many primes p the initial ideals $in(I_0)$ and $in(I_p)$ are minimally generated by the same set of monomials.

Exercise 2

Let $P = K[x_1, \dots, x_n]$ denote the polynomial ring over a field K.

(1) Prove the division algorithm for a system of polynomial vectors $f_1, \dots, f_r \subset P^s$ with respect to a global monomial order $>$ on P^s.
(2) Define Gröbner basis and formulate Buchberger's criterion for submodues $I \subset P^s$!

Exercise 3

Consider the ideal $I \subset K[x_1, \dots z_5]$ generated by the 3×3 minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \end{pmatrix}$$

Compute a free resolution of P/I as a P-module and the dimension $\dim V(I)$.

Exercise 4
Let $R = K[x_1, \ldots, x_n]/I$ be a finitely generated (commutative) K-algebra and consider an R-module homomorphism $\varphi\colon R^r \to R^s$ defined by an $s \times r$ matrix $A = (a_{ij})$ with entries in R. Design an algorithm which computes generators of $\ker \varphi$.