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Algebraic Geometry Summer Term 2018

Exercise Sheet 9. Hand in by Friday, June 22.

Exercise 1 Show that Hilbert's syzygy theorem holds for $R = K[[x_1, \dots, x_n]]$: Every finitely generated R-module M has a finite free resolution of length at most n, by finitely generated free R-modules.

Exercise 2
Consider $K[x, y]$ with the local monomial order $>_{ldrlex}$, that is $>_{(-1,-1)}$ refined by $>_{drlex}$, and the ideals

$$I = (x^3 - y^3, x^2y^2 + xy^3),\ J = (x^3 - y^3, x^2y^2),$$

$$\text{and } K = (x^3 - y^4, x^2y^2).$$

Compute that $\{x^3 - y^3, x^2y^2 + xy^3, xy^4 - y^5, y^6\}$, $\{x^3 - y^3, x^2y^2, y^5\}$, and $\{x^3 - y^4, x^2y^2, y^6\}$ are Gröbner bases for I, J and K, respectively.

Exercise 3 (Weierstrass Preparation Theorem).
If $f \in K[[x_1, \dots, x_n]]$ is a power series, show:

(1) By a triangular change of coordinates, we can achieve that f is x_n-general, that is $f(0, x_n) \in K[[x_n]]$ is non-zero.
(2) If f is x_n-general, there exist a local monomial order on $K[x_1, ..., x_n]$ such that such that $in(f) = in(f(0, x_n))$.
(3) If f is x_n-general, then (f) is generated by a Weierstrass polynomial

$$p = x_n^d + a_1x_n^{d-1} + \dots + a_d \in K[[x_1, \dots, x_{n-1}]][x_n]$$

with $p(0, x_n) = x_n^d$ that is there exists a unit $u \in k[[x_1, \dots, x_n]]$ with $f = up$.

Exercise 4
Using the Weierstrass' preparation theorem, prove that $K[[x_1, \ldots, x_n]]$ is factorial.