Exercise Sheet 1 (08. 11. 2018)

Exercise 1. *Let X be a topological space, A be an abelian group. The constant presheaf $\mathcal{F}$ is defined by $U \mapsto A$ for every nonempty open subset $U \subseteq X$, together with the restriction maps given by the identity map id_A.*

(1) *What is the stalk of $\mathcal{F}$ at a point $P \in X$?*

(2) *Explain why it is not a sheaf in general.*

(3) *Find its sheafification $\mathcal{F}^+$ by definition, and compare with the constant sheaf we treated in the class.*

Exercise 2. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.*

(1) *Show that $(\ker \varphi)_P = \ker(\varphi_P)$ and $(\operatorname{im} \varphi)_P = \operatorname{im}(\varphi_P)$ for each point $P \in X$.*

(2) *Show that φ is injective (resp., surjective) if and only if the induced map φ_P is injective (resp., surjective) for every $P \in X$.*

(3) *Show that $\operatorname{im} \varphi \simeq \mathcal{F}/\ker \varphi$.*

(4) *Show that $\operatorname{coker} \varphi \simeq \mathcal{G}/\operatorname{im} \varphi$.*

Exercise 3. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ be a left exact sequence of sheaves on a topological space X, and let $f : X \to Y$ be a continuous map between two topological spaces.*

(1) *Let $U \subseteq X$ be an open subset. Show that the functor $\Gamma(U, -)$ is left exact.*

(2) *Show that the functor f_* is left exact.*

Exercise 4 (Extension by zero)**.** *Let $Z \subset X$ be a closed subset, and $U = X \setminus Z$ be the complement open set. Let $i : Z \hookrightarrow X$, $j : U \hookrightarrow X$ be the inclusions.*

(1) *Let $\mathcal{F}$ be a sheaf on Z. Show that the stalk of the sheaf $i_*\mathcal{F}$ on X at $P \in X$ is $\mathcal{F}_P$ if $P \in Z$, and 0 otherwise (if there is no confusing, we sometimes omit i_*.)*

(2) *Let $\mathcal{G}$ be a sheaf on U. Let $j_!\mathcal{G}$ be the sheaf associated to the presheaf $V \mapsto \mathcal{F}(V)$ if $V \subseteq U$, and 0 otherwise. Show that the stalk of the sheaf $j_!\mathcal{G}$ at P is $\mathcal{G}_P$ if $P \in U$, and 0 otherwise. The sheaf $j_!\mathcal{G}$ is called the sheaf obtained by* extending $\mathcal{G}$ by zero *outside U.*

(3) *Show that there is a natural bijection of sets $\operatorname{Hom}_X(j_!\mathcal{G}, \mathcal{F}) = \operatorname{Hom}_U(\mathcal{G}, \mathcal{F}|_U)$ for any sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on U, that is, j^{-1} is a right adjoint of $j_!$ and $j_!$ is a left adjoint of j^{-1}.*

Exercise 5. *Let A be a ring, and $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ be its spectrum. Check that*

(1) *For any $\mathfrak{p} \in \operatorname{Spec} A$, the stalk $\mathcal{O}_\mathfrak{p}$ of $\mathcal{O}_{\operatorname{Spec} A}$ is isomorphic to the local ring $A_\mathfrak{p}$.*

(2) *The global section $\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is isomorphic to A.*

Exercise 6. *Describe affine schemes $\operatorname{Spec} \mathbb{Z}, \operatorname{Spec} \mathbb{R}[x], \operatorname{Spec} \mathbb{C}[x, y]$ as topological spaces.*

Exercise 7. *Show that the projective n-space $\mathbb{P}^n_k$ $(n > 0)$ over a field k is not affine. (Hint: what is the global section of the structure sheaf $\mathcal{O}_{\mathbb{P}^n_k}$?)*