Exercise Sheet 2 (22. 11. 2018)

Exercise 1. *Let $\mathcal{F}$ be an invertible sheaf on $(X, \mathcal{O}_X)$. Show that $\mathcal{F} \otimes \mathcal{F}^\vee \simeq \mathcal{O}_X$.*

Exercise 2 (Geometric Nakayama's lemma)**.** *Let X be a scheme, $\mathcal{F}$ be a coherent sheaf. Let $U \subset X$ be an open neighborhood of a point $P \in X$, and let $s_1, \cdots, s_n \in \mathcal{F}(U)$ be sections such that their images $s_1|_P, \cdots, s_n|_P \in \mathcal{F}|_P := \mathcal{F}_P \otimes (\mathcal{O}_{X,P}/\mathfrak{m}_{X,P})$ generate the geometric fiber $\mathcal{F}|_P$. Show that there is an affine open neighborhood $P \in \operatorname{Spec} A \subseteq U$ such that $s_1|_{\operatorname{Spec} A}, \cdots, s_n|_{\operatorname{Spec} A}$ generate $\mathcal{F}|_{\operatorname{Spec} A}$ in the following sense:*

(1) *$s_1|_{\operatorname{Spec} A}, \cdots, s_n|_{\operatorname{Spec} A}$ generate $\mathcal{F}|_{\operatorname{Spec} A}$ as an A-module;*
(2) *for any $Q \in \operatorname{Spec} A$, $s_1, \cdots, s_n$ generate the stalk $\mathcal{F}_Q$ as an $\mathcal{O}_{X,Q}$-module.*

Exercise 3. *Let X be a scheme, and $\mathscr{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras, that is, a sheaf of rings which is a quasi-coherent $\mathcal{O}_X$-module simultaneously.*

(1) *Show that there is a unique scheme Y and a morphism $f : Y \to X$ such that for every affine open $U \subseteq X$, $f^{-1}(U) \simeq \operatorname{Spec} \mathscr{A}(U)$, and for every inclusion $V \hookrightarrow U$ of open affines of X, the morphism $f^{-1}(V) \hookrightarrow f^{-1}(U)$ corresponds to the restriction homomorphism $\mathscr{A}(U) \to \mathscr{A}(V)$. The scheme Y is called the global Spec, or the relative Spec, and denoted by* $\mathbf{Spec}\ \mathscr{A}$.
(2) *Let $\mathcal{E}$ be a locally free sheaf of rank r on a scheme X. Take $\mathscr{A} = \operatorname{Sym} \mathcal{E}^\vee$ be the sheaf associated to the presheaf $U \mapsto \operatorname{Sym}(\mathcal{E}^\vee(U))$. Show that this is a sheaf of $\mathcal{O}_X$-algebra.*
(3) *Let $\mathscr{A} = \operatorname{Sym} \mathcal{E}^\vee$ be as above. Check that the morphism $\mathbf{Spec}\ \mathscr{A} \to X$ gives a vector bundle of rank r. This is called the* total space of a locally free sheaf $\mathcal{E}$ *of a finite rank, or the* vector bundle associated to a locally free sheaf $\mathcal{E}$.

Exercise 4. *Let X be a noetherian scheme, and $\mathscr{S}$ be a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras, that is, $\mathscr{S} \simeq \bigoplus_{d \geq 0} \mathscr{S}_d$ where $\mathscr{S}_d$ is the homogeneous part of degree d. Assume that $\mathscr{S}_0 = \mathcal{O}_X$, and $\mathscr{S}_1$ is a coherent $\mathcal{O}_X$-module, and that $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as an $\mathcal{O}_X$-algebra. Complete the details of the following construction.*

(1) *For each open affine open subset $U = \operatorname{Spec} A \subseteq X$, let $\mathscr{S}(U)$ be the graded A-algebra $\Gamma(U, \mathscr{S}|_U)$. We have a natural morphism $\pi_U : \operatorname{Proj} \mathscr{S}(U) \to U$. Check that this is compatible with a further localization, that is, $\operatorname{Proj} \mathscr{S}(U_f) \simeq \pi_U^{-1}(U_f)$ where $U_f = \operatorname{Spec} A_f \subseteq U$ for some element $f \in A$.*
(2) *Let U, V be two affine open subsets of X. Check that $\pi_U^{-1}(U \cap V) \simeq \pi_V^{-1}(U \cap V)$. In particular, the schemes $\operatorname{Proj} \mathscr{S}(U)$ glue together and form a scheme* $\mathbf{Proj}\ \mathscr{S}$, *together with a morphism* $\pi : \mathbf{Proj}\ \mathscr{S} \to X$.
(3) *Check that the invertible sheaves $\mathcal{O}(1)$ on each $\operatorname{Proj} \mathscr{S}(U)$ are also compatible under the above construction. They give rise to an invertible sheaf $\mathcal{O}(1)$ on* $\mathbf{Proj}\ \mathscr{S}$.
(4) *Let $\mathcal{E}$ be a locally free sheaf of rank $(r+1)$ on X. Take $\mathscr{S} = \bigoplus_{d \geq 0} \operatorname{Sym}^d \mathcal{E}^\vee$ be the symmetric algebra. Check that this is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. We denote by* $\mathbb{P}(\mathcal{E}) := \mathbf{Proj}\ \mathscr{S}$

be the projective space bundle*, together with a natural projection* $\pi : \mathbb{P}(\mathcal{E}) \to X$*. If* $\mathcal{E}$ *is free over a sufficiently small open affine subset* $U \simeq \operatorname{Spec} A$*, then* $\pi^{-1}(U) \simeq \mathbb{P}^r_A = \operatorname{Proj} A[x_0, \cdots, x_r]$*.*

(5) *Let* Y *be a closed subscheme of* X*, and let* $\mathscr{I}$ *be the ideal sheaf. Take* $\mathscr{S} = \bigoplus_{d \geq 0} \mathscr{I}^d$*, where* $\mathscr{I}^0 = \mathcal{O}_X$*. Check that* $\mathscr{S}$ *is a quasi-coherent sheaf of* $\mathcal{O}_X$*-algebras. The scheme* $\widetilde{X} := \mathbf{Proj}\ \mathscr{S}$*, together with a natural morphism* $\pi_{\mathscr{I}} : \widetilde{X} \to X$*, is called the* blowing-up of X with respect to $\mathscr{I}$*. Compute it when* X *is an affine* n*-space, and* Y *is the origin of* X*.*

Exercise 5. *Let* $\mathcal{F}, \mathcal{G}$ *be coherent sheaves on* X*. Suppose that* $\mathcal{F}$ *is globally generated, and there is a surjection* $\mathcal{F} \to \mathcal{G} \to 0$*. Show that* $\mathcal{G}$ *is also globally generated.*

Exercise 6. *Let* A *be a ring,* X *be a closed subscheme of* $\mathbb{P}^N_A$*. We define the* homogeneous coordinate ring $S(X)$ *of* X *for the given embedding to be* $A[x_0, \cdots, x_N]/I$*, where* $I = \Gamma_*(\mathscr{I}_X)$ *is the (largest) ideal defining* X*.*

A scheme X *is* normal *if all the local rings* $\mathcal{O}_{X,P}$ *are integrally closed. A closed subscheme* $X \subseteq \mathbb{P}^N_A$ *is* projectively normal *(or,* arithemetically normal*) for the given embedding, if its homogeneous coordinate ring* $S(X)$ *is an integrally closed domain.*

Now assume that $A = k$ *is an algebraically closed field, and that* X *is a connected, normal, closed subvariety of* $\mathbb{P}^N_A$*.*

(1) *Let* $S(X)$ *be the homogeneous coordinate ring of* X*, and let* $R(X) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(n))$ *be the section ring. Show that* $S(X)$ *is an integral domain, and that* $R(X)$ *is its integral closure.*
(2) *Show that* $S(X)_d = R(X)_d$ *for all sufficiently large* d*.*
(3) *Show that* $S(X)^{(d)} := \bigoplus_n S(X)^{(d)}_n = \bigoplus_n S(X)_{nd}$ *is integrally closed for sufficiently large* d*. This implies that the* d*-uple embedding of* X *is projectively normal when* d *is large enough.*
(4) *Show that a closed subscheme* $X \subseteq \mathbb{P}^N_A$ *is projectively normal if and only if it is normal, and the natural map on global sections* $\Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) \to \Gamma(X, \mathcal{O}_X(n))$ *is surjective for every* $n \geq 0$*.*
(5) *Let* $X = \{[s^4 : s^3t : st^3 : t^4] \mid [s : t] \in \mathbb{P}^1\} \subseteq \mathbb{P}^3$ *be a rational quartic curve in* $\mathbb{P}^3$*. Let* $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^3}(1)$ *be the very ample line bundle. Compute* $S(X)$ *and* $R(X)$*, and conclude that* X *is not linearly normal, that is,* $\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \to \Gamma(X, \mathcal{O}_X(1))$ *is not surjective.*

Exercise 7 (Riemann-Roch problem)**.** *Let* k *be an algebraically closed field,* X *be a nonsingular projective variety over* k*. Let* $\mathcal{L}$ *be an invertible sheaf on* X*. We want to describe the number* $\dim \Gamma(X, \mathcal{L}^n)$ *as an integer-valued function of* n*.*

(1) *Assume that* $\mathcal{L}$ *is very ample, and* $i : X \hookrightarrow \mathbb{P}^N_k$ *is the corresponding embedding in a projective space. Show that* $\dim \Gamma(X, \mathcal{L}^n) = P_X(n)$ *for sufficiently large* n*, where* P_X *is the Hilbert polynomial of* X*.*
(2) *Show that* $\dim \Gamma(X, \mathcal{L}^n)$ *is a polynomial function for* n *large enough when* $\mathcal{L}$ *is ample.*
(3) *Show that* $\dim \Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)$ *is a polynomial function for* n *large enough when* $\mathcal{L}$ *is ample and* $\mathcal{F}$ *is coherent (Hint: use Hilbert's syzygy theorem for* $i_*(\mathcal{F} \otimes \mathcal{L}^n)$*).*