Exercise Sheet 3 (06. 12. 2018)

Exercise 1. Let $X = \mathbb{P}^1_k \times \mathbb{P}^1_k = \operatorname{Proj} k[x,y,z,w]/(xy - zw)$ be a smooth quadric surface in $\mathbb{P}^3_k$. Let $D_1 = \{pt\} \times \mathbb{P}^1_k$, and let $D_2 = \mathbb{P}^1_k \times \{pt\}$.

(1) Show that there is a surjection $\mathbb{Z} \oplus \mathbb{Z} \to \operatorname{Cl} X$ (Hint : remove D_1 and D_2 from X. The complement is $X \setminus (D_1 \cup D_2) \simeq \mathbb{A}^2$, a spectrum of a UFD, hence has the trivial divisor class group.)

(2) Show that $\mathcal{O}(D_1)$ restricts to $\mathcal{O}$ on $D_1 \simeq \mathbb{P}^1_k$, and to $\mathcal{O}(1)$ on $D_2 \simeq \mathbb{P}^1_k$. Similarly, show that $\mathcal{O}(D_2)$ restricts to $\mathcal{O}(1)$ on D_1, and to $\mathcal{O}$ on D_2.

(3) Conclude that the homomorphism $\mathbb{Z} \oplus \mathbb{Z} \to \operatorname{Cl} X$ defined by $(a, b) \mapsto aD_1 + bD_2$ is an isomorphism.

Exercise 2. Show that $\mathbb{P}^2_k$ and $\mathbb{P}^1_k \times \mathbb{P}^1_k$ are birational but not isomorphic (Hint : what are the divisor class group of them?). Can you find two varieties, which are birational but not isomorphic, having the same divisor class group?

Exercise 3 (Generalized Euler sequence)**.** Let Y be a variety, and let $\mathcal{E}$ be a locally free sheaf of rank $n+1$ ($n \geq 1$) on Y. Let $\pi : X = \mathbb{P}(\mathcal{E}) \to Y$ be the projective bundle with the invertible sheaf $\mathcal{O}_X(1)$ followed from the construction. Let

Show that there is an exact sequence

$$0 \to \Omega_{X/Y} \to \mathcal{O}_X(-1) \otimes \pi^* \mathcal{E}^\vee \to \mathcal{O}_X \to 0.$$

Exercise 4. Let V be a vector space of dimension $n+1$, and let $Gr(k+1, V)$ be the Grassmannian; the space of sub-vector spaces of V of dimension $k+1$. Consider the incidence variety

$$\begin{array}{ccc} & \mathcal{I} = \{(p, \Lambda) \in \mathbb{P}(V^\vee) \times Gr(k+1, V) \mid p \in \Lambda\} & \\ \swarrow^{p} & & \searrow^{q} \\ \mathbb{P}(V^\vee) = \mathbb{P}^n & & Gr(k+1, V) \end{array}$$

Let $\mathcal{U} := q_* p^*(\Omega_{\mathbb{P}^n}(1))$, and let $\mathcal{Q} := q_* p^*(\mathcal{O}_{\mathbb{P}^n}(1))$. Show that there is an exact sequence

$$0 \to \mathcal{U} \to V \otimes \mathcal{O}_{Gr(k+1,V)} \to \mathcal{Q} \to 0.$$

Exercise 5. Let $(X, \mathcal{O}_X)$ be a ringed space. We denote $\operatorname{Pic} X$ by the group of isomorphism classes of invertible sheaves. Show that $\operatorname{Pic} X \simeq H^1(X, \mathcal{O}_X^\times)$, where $\mathcal{O}_X^\times$ denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U, \mathcal{O}_X)$, with the multiplication as the group operation.

[Hint : Let $\mathcal{L}$ be an invertible sheaf. Cover X by open subsets U_i on which $\mathcal{L}$ is free. Fix local isomorphisms $\varphi_i : \mathcal{O}_{U_i} \xrightarrow{\sim} \mathcal{L}|_{U_i}$. On $U_i \cap U_j$, we have an automorphism $\varphi_i^{-1} \circ \varphi_j$ of $\mathcal{O}_{U_{i,j}}$. These automorphisms give an element of $\check{\mathrm{H}}^1(\mathfrak{U}, \mathcal{O}_X^\times)$. Now use the fact that $\varinjlim \check{\mathrm{H}}^1(\mathfrak{U}, \mathcal{F}) = H^1(X, \mathcal{F})$.]

Exercise 6. Let C be a projective plane curve, defined by a single homogeneous equation $f(x, y, z) = 0$ of degree d. Assume that $(1 : 0 : 0) \notin C$, equivalently, f does not contains a x^d term.

(1) Show that C is covered by two affine open subsets $U = C \cap U_y = \{y \neq 0\}$ and $V = C \cap U_z = \{z \neq 0\}$.
(2) Compute the Čech complex explicitly.
(3) Verify that $h^0(C, \mathcal{O}_C) = 1$ and $h^1(C, \mathcal{O}_C) = \binom{d-1}{2}$.

Exercise 7. Compute the cohomology groups $H^i(X, \mathcal{O}_X)$ where $X = \mathbb{A}^2_k \setminus \{(0,0)\}$ is a punctured affine plane. [Hint : compute Čech cohomology groups with respect to an affine open cover $\{U_x, U_y\}$, where $U_x \subseteq \{(x,y) \mid x \neq 0\} \subseteq \mathbb{A}^2_k$ and $U_y \subseteq \{(x,y) \mid y \neq 0\} \subseteq \mathbb{A}^2_k$.] Conclude that $\mathbb{A}^2_k \setminus \{(0,0)\}$ is not affine.