# Exercise Sheet 4 (17. 01. 2019)

**Exercise 1.** Let $D$ be an effective divisor on $C$. Show that $\dim |D| \leq \deg D$, and the equality holds if and only if either $D = 0$ or $g = 0$.

**Exercise 2.** A curve $C$ is called *hyperelliptic* if $g \geq 2$ and there exists a finite morphism $f : C \to \mathbb{P}^1$ of degree 2.

(1) Let $C$ be a curve of genus 2. Show that the canonical divisor $K_C$ is base-point-free, $\deg K_C = 2$, and $h^0(C, K_C) = 2$. Conclude that $C$ is hyperelliptic.

(2) Show that there is a hyperelliptic curve of genus $g$ for every $g \geq 2$.

**Exercise 3** (Automorphisms of a curve)**.** Let $C$ be a curve over $k = \mathbb{C}$ of genus $g$.

(1) Let $g = 0$. Show that the automorphism group $\mathrm{Aut}(\mathbb{P}^1)$ is the group of Möbius transformations

$$\left\{ f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0 \right\}.$$

(2) Let $g = 1$. In the case, $C$ is an elliptic curve, and hence it has a group structure. Show that the translation by a point of $C$ is an automorphism. In particular, $C$ is a subgroup of $\mathrm{Aut}(C)$.

(3) Let $g \geq 2$. In the case, it is known that $G = \mathrm{Aut}(C)$ is finite. So let $n = |G|$ be its order. Since $G$ acts on the function field $K(C)$ of $C$, we have a field extension $L := K(C)^G \hookrightarrow K(C)$. This gives a finite morphism of curves $f : C \to C'$ of degree $n$.

   If $P \in C$ is a ramification point of index $e_P = r$, show that $f^{-1}(f(P))$ consists of $n/r$ points, each having ramification index $r$.

(4) Note that $f$ is branched over finite number of points on $C'$. Let $P_1, \cdots, P_s$ be a maximal set of ramification points of $C$ lying over distinct points of $Y$. In particular, $s$ is the number of branch points of $C'$. Let $e_{P_i} = r_i$. Show that

$$\frac{1}{n}(2g-2) = 2g(C') - 2 + \sum_{i=1}^{s} \left(1 - \frac{1}{r_i}\right).$$

(5) Since $g \geq 2$, the value appearing in the above equality must be $> 0$. Under the assumption $g(C') \geq 0$, $s \geq 0$, $r_i \geq 2$ for $1 \leq i \leq s$ which makes the above value to be strictly greater than 0, show that the minimum value of the expression

$$2g(C') - 2 + \sum_{i=1}^{s} \left(1 - \frac{1}{r_i}\right)$$

   is $\frac{1}{42}$ (occurs when $g(C') = 0, s = 3, r_1 = 2, r_2 = 3, r_3 = 7$). Conclude that $n \leq 84(g-1)$.

**Exercise 4.** Let $C$ be a plane curve of degree 4. Show that any effective divisor linearly equivalent to the canonical divisor is the hyperplane divisor $C \cap L$ for some line $L \subseteq \mathbb{P}^2$. Show that there is no linear system of divisors of degree 2 of dimension 1, in other words, $h^0(C, D) < 2$ for any (effective) divisor $D$ of degree 2. Conclude that $C$ cannot be hyperelliptic.