Section 0 – Goal of the lecture

The first and the main purpose of the lecture is to understand the notion of sheaves on topological spaces and algebraic varieties, and to be familiar with the sheaf cohomology theory. And then, as applications, we will cover a small part of the theory of algebraic curves and Ulrich sheaves.

One might say that a very naive concept of the sheaf is: a formal and abstract notion which contains an additional structure on the given space. When we are trying to understand an object (not only for a topological space, or a geometric object, but also one may consider a virtual object), we observe their "characters" including its shape, color, smell, taste, etc. Such additional information makes it possible to characterize and classify; one may compare with those properties of another object, and measure how they are close or different.

Roughly speaking, a sheaf is a collection of local data which are compatible with the topology of the underlying space. Following a very naive local-to-global principle, one may obtain global information from a collection of local information. For example, a topological manifold itself is consisted of local charts and an atlas. Hence, a manifold itself is an object which can be completely recovered from small pieces like balls in Euclidean spaces. There is an interesting geometric structure on a manifold, called a vector bundle. Locally, it looks like a trivial fibration with the projection map $U \times \mathbb{R}^n \to U$. What is more: there are local isomorphisms on the intersection of local pieces which enable to glue them together and to form a global structure. We will see how a vector bundle can be understood as a sheaf. In fact, vector bundles appear as central objects during the whole lecture.

Cohomology is a kind of general tools which assign an algebraic objects (e.g., groups or modules) from a geometric object. Mostly, it is defined from a cochain complex, and measures how the given complex is far from exact. There are several different cohomology theories, however, we will focus on the sheaf cohomology theory in this lecture. One might imagine a "good" cohomology theory for algebraic varieties, for instance, we are happier with following properties:

1. functoriality; a theory should translate geometric questions into certain algebraic questions.
2. able to compute; of course, a theory should equip with homological algebra, and hence one can use computational tools freely.
3. local simplicity; on a sufficiently small part, a theory can be described in a very easy way.
4. sometimes a theory provides a more fluent and concrete explanation,

in particular, coincides with another (well-known) theory under certain conditions.

An algebraic curve is an algebraic variety of dimension 1. We are mostly interested in the case of complex projective curves. In this case, a curve has dimension 2 as a real manifold, and it is compact, connected, and orientable. It is often called a compact Riemann surface. We will apply the techniques to study algebraic curves. In fact, the sheaf cohomology for curves is quite simple, and we do not need these huge machineries. Nevertheless, curves are enough good examples which lead us to a better understanding. We will also discuss the canonical embedding of a curve, related results, and a famous conjecture by Green.

If time permits, we will discuss Ulrich bundles on algebraic varieties. Ulrich bundles are vector bundles satisfying a number of cohomology vanishing conditions. On its algebraic side, such a notion appears naturally in the classification problem of Cohen-Macaulay modules. Very recently, people realized that Ulrich bundles appear in several different places of mathematics. We will see some examples of Ulrich bundles and construct Ulrich bundles on algebraic curves, or hypersurfaces.

For the simplicity, we will mostly work over an algebraically closed field k of characteristic 0. If there is no comment for the base ring or the base field, we assumed that $k = \mathbb{C}$ denotes the field of complex numbers. This makes us possible to use several notions and theorems in differential/complex geometry without spending a lot of time.