Topic 1 : Sheaf

1 Presheaves and sheaves

> "The concept of a *sheaf* provides a systematic way of keeping track of local algebraic data on a topological space" – Hartshorne, 'Algebraic Geometry'

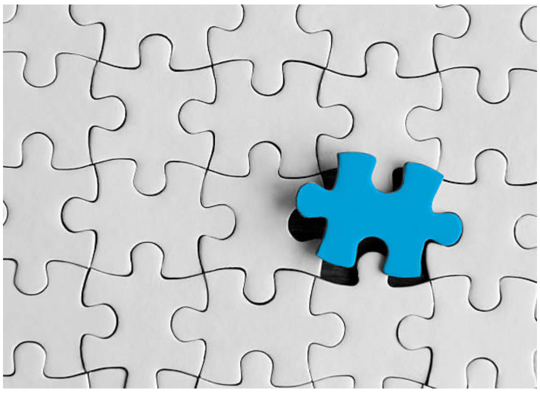

Figure 1: A jig-saw puzzle is consisted of small pieces and rules for patching them together

Let us recall our old memory. Let $X \subset \mathbb{R}^N$ be an n-dimensional topological manifold. One effective way to study X is not only considering X itself, but also observing continuous functions to $\mathbb{R}$. Since an atlas $\{(U_\alpha, \varphi_\alpha)\}$ defines X, a continuous function f can be seen as a collection of continuous functions on smaller open subsets $\{(U_\alpha, f|_{U_\alpha})\}$. Indeed, we may assume that each U_α is an n-dimensional Euclidean ball, which is a fundamental object in classical topology, and $f|_{U_\alpha}$ is a real-valued continuous function. When we are interested in smooth manifolds, one may substitute X by a smooth manifold and observe smooth functions. It is straightforward to assign local data from additional structures such as: continuous functions, smooth functions, or holomorphic functions, etc. In every cases, such data can be restricted to smaller open sets in a natural way.
Conversely, suppose that we have a collection of local data which comes from a single $(X, f : X \to \mathbb{R})$. In order to make it sense, local pieces should recover the original $(X, f : X \to \mathbb{R})$ by gluing them together. Hence, a right notion for the 'playground' should contain the following data:

- the topology of X, that is, the set of all open subsets of X;

- the collection of a local structure we want to observe, for instance, the set of local continuous functions on each subset $U \subseteq X$;
- a notion of "restriction" when $V \subseteq U$ is a further open subset, for instance, sending $f_U : U \to \mathbb{R}$ to $f_U|_V : V \to \mathbb{R}$;
- a notion of "gluing" when we have two local data compatible on their intersection.

It is hard to pin down who invented the sheaf theory. It seems reasonable to the connection between the extension problem (such as analytic continuations). Such an idea was mostly developed by several people working in algebraic topology during 30's and 40's, including Jean Leray. The work when he was in a prisoner camp is essential to bring out the modern sheaf theory and the development of spectral sequences. Serre brought the sheaf theory to algebraic geometry in 50's, and then it becomes a fundamental language in this world.
Sheaves have various applications in algebraic topology and in algebraic geometry. First, geometric structures like vector bundles on a smooth manifold (or an algebraic variety) can be expressed in terms of sheaves of modules on the given space. Also, sheaves provide (a hint of) a cohomology theory generalizing the usual cohomology such as singular cohomology. Sheaves are designed in a much more general and abstract way, however, in this course, we only deal with a few simpler cases.

Definition 1. Let X be a topological space. A *presheaf* $\mathcal{F}$ of abelian groups on X is a contravariant functor from $\mathfrak{Top}(X)$ to $\mathfrak{Ab}$.

Note that the category $\mathfrak{Top}(X)$ is consisted of the objects as the open subsets of X and the morphisms are the inclusion maps between open subsets. Therefore, $\mathcal{F}$ is a collection of the data

(1) abelian group $\mathcal{F}(U)$ for each open subset $U \subseteq X$;
(2) a (homo)morphism of abelian groups $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for every inclusion $V \subseteq U$ of open subsets of X

satisfying the following conditions

(a) $\mathcal{F}(\emptyset) = 0$, the empty set assigns a terminal object;
(b) $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map;
(c) $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ where $W \subseteq V \subseteq U$ are three open subsets.

Of course, we may define a presheaf of rings, or of sets just by replacing the words "abelian groups" by "rings", or "sets". The abelian groups $\mathcal{F}(U)$, also denoted by $\Gamma(U, \mathcal{F})$, is called the *sections* of the presheaf $\mathcal{F}$ over the open set U. The morphisms ρ_{UV} are called *restriction maps*. We write $s|_V := \rho_{UV}(s)$ where $s \in \mathcal{F}(U)$.
Roughly speaking, a sheaf is a presheaf whose sections are determined by further local data, as we discussed above.

Definition 2. A presheaf $\mathcal{F}$ is a *sheaf* if it satisfies the following further conditions:

(3) if $\{U_i\}$ is an open covering of an open subset $U \subseteq X$, and if $s, t \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$;

(4) if $\{U_i\}$ is an open covering of an open subset $U \subseteq X$, and if we have sections $s_i \in \mathcal{F}(U_i)$ for each i such that they agree on the intersections: $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for every pair i, j, then there is a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for each i.

Example 3. Let X be a variety over a field k. For each open subset $U \subseteq X$, let $\mathcal{O}_X(U)$ be the ring of regular functions from U to k, and for each $V \subseteq U$, let $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ be the usual restriction map. Then $\mathcal{O}$ is a sheaf of rings on X, which is called the *sheaf of regular functions* on X, or the *structure sheaf* of X (when we regard X as a locally ringed space, or a scheme).
Let $Y \subseteq X$ be a closed subset of X. For each open set $U \subseteq X$, we assign $U \mapsto \mathscr{I}_Y(U)$ where $\mathscr{I}_Y(U) \subseteq \mathcal{O}_X(U)$ be the ideal of regular functions vanish at every point of $Y \cap U$. This forms a sheaf $\mathscr{I}_Y$, and called the *sheaf of ideals* of Y, or *the ideal sheaf* of Y.

Example 4. One can define the sheaf of continuous functions on any topological space, or the sheaf of differentiable functions on a differentiable manifold, or the sheaf of holomorphic functions on a complex manifold.

Example 5. Let A be an abelian group. We define the *constant sheaf* $\mathscr{A}$ on X as follows. Give A the discrete topology, and for each $U \subseteq X$, let $\mathscr{A}(U)$ be the group of all continuous maps from U to A. We have $\mathscr{A}(U) \simeq A$ for every connected open subset U.

Example 6. Let A be an abelian group, and $P \in X$ be a point. We define a sheaf $i_P(A)$ on X as follows: $i_P(A)(U) = A$ if $P \in U$ and $i_P(A)(U) = 0$ otherwise. $i_P(A)$ is called the *skyscraper sheaf.*

Now we will see more definitions to play with sheaves. They are not quite different from the basic notions in commutative algebra and homological algebra.

Definition 7. Let $\mathcal{F}$ be a presheaf on X, and let $P \in X$ be a point. We define the *stalk* $\mathcal{F}_P$ of $\mathcal{F}$ at P be the direct limit of the groups $\mathcal{F}(U)$ for every open subset U containing P, via the restriction maps.

An element of $\mathcal{F}_P$ is represented by a pair $\langle U, s\rangle$, where U is an open neighborhood of P, and $s \in \mathcal{F}(U)$. Two pairs $\langle U, s\rangle$ and $\langle V, t\rangle$ defines the same element in $\mathcal{F}_P$ if and only if there is a smaller open subset $W \subseteq U \cap V$ such that $s|_W = t|_W$. Hence, the elements of the stalk $\mathcal{F}_P$ are germs of sections at the point P.

Exercise 8. What are the stalks of the previous examples?

Definition 9. Let $\mathcal{F}$ and $\mathcal{G}$ are presheaves on X. A *morphism* $\varphi : \mathcal{F} \to \mathcal{G}$ is a collection of morphisms of abelian groups $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set $U \subseteq X$ such that the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ {\scriptstyle \rho^{\mathcal{F}}_{UV}}\downarrow & & \downarrow{\scriptstyle \rho^{\mathcal{G}}_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

commutes for any inclusion of open sets $V \subseteq U$, where $\rho^{\mathcal{F}}$ and $\rho^{\mathcal{G}}$ denotes the restriction maps in $\mathcal{F}$ and $\mathcal{G}$. A morphism of sheaves follows the same definition. An *isomorphism* is a morphism which admits a two-sided inverse.

A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves on X induces a morphism $\varphi_P : \mathcal{F}_P \to \mathcal{G}_P$ on the stalks, for any $P \in X$. The following proposition shows the local nature of a sheaf.

Proposition 10. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. φ is an isomorphism if and only if the induced map φ_P is an isomorphism for every $P \in X$.*

Proof. It is enough to show that φ is an isomorphism when all the induced maps on the stalks are isomorphisms. To do this, we claim that each $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism where $U \subseteq X$ is an open subset. Suppose the claim is true. Let ψ be the inverse defined by $\psi(U) = \varphi(U)^{-1}$. Since

$$\varphi(\psi(\rho^{\mathcal{G}}_{UV}(t))) = \rho^{\mathcal{G}}_{UV}(t) = \rho^{\mathcal{G}}_{UV}(\varphi(\psi(t))) = \varphi(\rho^{\mathcal{F}}_{UV}(\psi(t)))$$

for any $t \in \mathcal{G}(U)$ and an open subset $V \subseteq U$, we conclude that $\psi(\rho^{\mathcal{G}}_{UV}(t)) = \rho^{\mathcal{F}}_{UV}(\psi(t))$ commutes.
First we show that $\varphi(U)$ is injective. Let $s \in \mathcal{F}(U)$ be an element such that $\varphi(s) = 0$ in $\mathcal{G}(U)$. For every point $P \in U$, the image $\varphi(s)_P$ of $\varphi(s)$ in the stalk $\mathcal{G}_P$ is zero. Since φ_P is injective, we have $s_P = 0$ in $\mathcal{F}_P$ for every $P \in X$. It means, there exist an open neighborhood $P \in W_P \subseteq U$ such that $s|_{W_P} = 0$. Since $\{W_P\}$ gives a covering of U, we conclude that $s = 0$ in U from the sheaf property.
Next, we show that $\varphi(U)$ is surjective. Let $t \in \mathcal{G}(U)$ be a section, and t_P be its germ at $P \in U$. Since each φ_P is an isomorphism, we have $s_P \in \mathcal{F}_P$ such that $\varphi_P(s_P) = t_P$. Let s_P is represented by a section $s(P)$ on an open neighborhood $P \in W_P \subseteq U$. Then $\varphi(s(P))$ and $t|_{W_P}$ are two sections in W_P whose germs at P coincide. Hence, replacing W_P by a smaller neighborhood if necessary, we may assume that $\varphi(s(P)) = t|_{W_P}$ in $\mathcal{G}(W_P)$. For any $P, Q \in X$, $\varphi(s(P))|_{W_P \cap W_Q} = t|_{W_P \cap W_Q} = \varphi(s(Q))|_{W_P \cap W_Q}$ coincide on the intersection. Since φ is injective, we have $s(P)|_{W_P \cap W_Q} = s(Q)|_{W_P \cap W_Q}$. Hence, by the sheaf property, there is a section $s \in \mathcal{F}(U)$ such that $s|_{W_P} = s(P)$ for each $P \in X$. Again by the sheaf property for $\mathcal{G}$, we conclude that $\varphi(s) = t$. □

Caution.

(1) The proposition only holds for a morphism of sheaves, not for presheaves.

(2) It is not true that φ is an isomorphism when $\mathcal{F}_P$ and $\mathcal{G}_P$ are isomorphic for each $P \in X$. Isomorphisms on the stalks must be induced from a single morphism φ.

For instance, let $X = \{P, Q\}$ with the discrete topology, and let A be an abelian group. We take $\mathcal{F}$ as a presheaf defined by $\mathcal{F}(U) = A$ for any nonempty open subset $U \subset X$, and let $\mathcal{G}$ as the constant sheaf. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves defined by

$$\varphi(\{P\}) = id_A, \varphi(\{Q\}) = id_A, \varphi(X) = \Delta_A$$

where $\Delta_A(a) = (a, a) \in A \oplus A$. Clearly, they are not isomorphic, but the induced maps on all the stalks are isomorphisms.

We now define the kernel, cokernel, and image of a morphism of (pre)sheaves.

Definition 11. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. We define the *presheaf kernel*, *presheaf cokernel*, *presheaf image* of φ to be the presheaves given by $U \mapsto \ker \varphi(U)$, $U \mapsto \operatorname{coker} \varphi(U)$, and $U \mapsto \operatorname{im} \varphi(U)$, respectively.

Caution. Let $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. The presheaf kernel of φ is a sheaf, but the presheaf cokernel and image are not sheaves in general.

Definition–Theorem 12. Given a presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$, with the universal property that for any sheaf $\mathcal{G}$ and a morphism $\varphi : \mathcal{F} \to \mathcal{G}$, there is a unique morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \psi \circ \theta$. The pair $(\mathcal{F}^+, \theta)$ is unique up to unique isomorphism. $\mathcal{F}^+$ is called the *sheaf associated* to the presheaf $\mathcal{F}$, or the *sheafification* of $\mathcal{F}$.

Proof. For any open set $U \subseteq X$, let $\mathcal{F}^+(U)$ be the set of functions $s^+ : U \to \bigcup_{P \in U} \mathcal{F}_P$ such that

(i) for each $P \in U$, $s^+(P) \in \mathcal{F}_P$, and

(ii) for each $P \in U$, there is an open neighborhood $P \in V \subseteq U$ and an element $t \in \mathcal{F}(V)$ such that for every $Q \in V$, the germ t_Q of t at Q is equal to $s^+(Q)$.

Then $\mathcal{F}^+$ with the natural restriction maps is a sheaf. All the other conditions are clear, so let us only check the gluing property. Let $\{U_i\}$ be an open covering of an open set $U \subseteq X$. Suppose that we have functions $s_i^+ : U_i \to \bigcup_{P \in U_i} \mathcal{F}_P$ compatible on the intersections. It is straightforward that there is a function $s^+ : U \to \bigcup_{P \in U} \mathcal{F}_P$ such that $s^+|_{U_i} = s_i^+$ and $s^+(P) \in \mathcal{F}_P$ for each $P \in U$. Let $P \in U_i \subseteq U$. By the condition (ii), there is an open neighborhood $P \in V_i \subseteq U_i$ and an element $t_i \in \mathcal{F}(V_i)$ such that the germ of t_i at $Q \in V_i$ is equal to $s_i^+(Q) = s^+|_{U_i}(Q) = s^+(Q)$. Thus we have $s^+ \in \mathcal{F}^+(U)$. It is also clear that the natural map

$$s \in \mathcal{F}(U) \mapsto \left(s^+ : U \to \bigcup_{P \in U} \mathcal{F}_P \right)$$

by $s^+(P) = s_P$ defines a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$.
Now let $\mathcal{G}$ be a sheaf and $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism (of presheaves). We define $\psi : \mathcal{F}^+ \to \mathcal{G}$ as follows. Let $U \subseteq X$ be an open set, and let $s^+ \in \mathcal{F}^+(U)$. For each $P \in U$, there is an open neighborhood $P \in V_P \subseteq U$ and an element $t(P) \in \mathcal{F}(V_P)$ whose germ at Q is equal to $s^+(Q)$ for every $Q \in V_P$. We have sections $\varphi(t(P)) \in \mathcal{G}(V_P)$ for each $P \in X$. Since two germs $\varphi_Q(s^+(Q))$ and $\varphi(t(P))_Q$ coincide in $\mathcal{G}_Q$ for every $Q \in V_P$, we replace V_P by a smaller open subset if necessary, and may assume that $\{\varphi(t(P))\}$ are compatible on the intersections. Since $\mathcal{G}$ is a sheaf, they glue together and form a section $\psi(s^+)$. This defines a morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$. It is straightforward that this equals to $\varphi(s)$ when $s^+ = \theta(s)$.
The uniqueness of the pair $(\mathcal{F}^+, \theta)$ follows from the universal property. □

Remark 13. Note that the induced map on stalks $\theta_P : \mathcal{F}_P \to \mathcal{F}_P^+$ is an isomorphism for each $P \in X$. When $\mathcal{F}$ itself is a sheaf, then $\mathcal{F}$ is isomorphic to $\mathcal{F}^+$.

2 More definitions and basic properties

In several literatures, one can find the same process via the notion of *espace étalé*. Let $\mathcal{F}$ be a presheaf on X. We define the *espace étalé* as the topological space $\text{Spé}(\mathcal{F})$ together with a projection $\pi : \text{Spé}(\mathcal{F}) \to X$ in a following way. As a set, $\text{Spé}(\mathcal{F}) = \coprod_{P \in X} \mathcal{F}_P$ be the disjoint union of all the stalks. The projection is defined in a trivial way: $\pi(s) := P$ when $s \in \mathcal{F}_P$. Let $s \in \mathcal{F}(U)$ be a section over an open set U. Then we have an element $\bar{s} : U \to \text{Spé}(\mathcal{F})$ by sending $P \mapsto s_P$, its germ at P. Since $\pi \circ \bar{s} = id_U$, one may consider $\bar{s}$ as a "(topological) section" of π over U. The topology of $\text{Spé}(\mathcal{F})$ is given by the strongest topology such that all the maps $\bar{s} : U \to \text{Spé}(\mathcal{F})$ are continuous for any choice of open sets $U \subseteq X$ and $s \in \mathcal{F}(U)$. Via this notion, $\mathcal{F}^+(U)$ is identified with the set of continuous sections of $\text{Spé}(\mathcal{F})$ over U.

Definition 14. Let $\mathcal{F}$ be a sheaf on X, and let $s \in \mathcal{F}(U)$ be a section on an open set $U \subseteq X$. The *support* of s is defined to be the set $\{P \in U \mid s_P \neq 0\}$. We immediately see that the set $Supp(s) \subseteq U$ is closed; if we take a point $P \in U \setminus Supp(s)$ in the complement, then there is an open neighborhood $P \in V \subseteq U$ such that $s|_V = 0$. The *support* of $\mathcal{F}$ is defined to be the set $\{P \in X | \mathcal{F}_P \neq 0\}$.

Definition 15. Let $\mathcal{F}, \mathcal{G}$ be sheaves, and $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. A *subsheaf* $\mathcal{F}'$ of a sheaf $\mathcal{F}$ is a sheaf such that for every open set $U \subseteq X$, $\mathcal{F}'(U) \subseteq \mathcal{F}(U)$ is a subgroup, and the restriction maps are induced by those of $\mathcal{F}$. The *kernel* of φ, denoted by $\ker\varphi$, is the presheaf kernel of φ (which is a sheaf). A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is *injective* if $\ker\varphi = 0$. The *image* of φ, denoted by $\operatorname{im}\varphi$, is the sheaf associated to the presheaf image of φ. By the universal property, there is a natural injective map $\operatorname{im}\varphi \to \mathcal{G}$. A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is *surjective* if $\operatorname{im}\varphi = \mathcal{G}$. The *cokernel* of φ, denoted by $\operatorname{coker}\varphi$, is the sheaf associated to the presheaf cokernel of φ. The *sheaf of local morphisms* of $\mathcal{F}$ into $\mathcal{G}$ (or *sheaf hom* for short), denoted by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, is defined by $U \mapsto \text{Hom}(\mathcal{F}|_U \to \mathcal{G}|_U)$.
A sequence $\cdots \to \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \to \cdots$ of sheaves is *exact* if $\ker\varphi^i = \operatorname{im}\varphi^{i-1}$ at each stage.
Let $\mathcal{F}'$ be a subsheaf of $\mathcal{F}$. The *quotient sheaf* $\mathcal{F}/\mathcal{F}'$ is the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{F}'(U)$.

Remark 16. A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves is injective if and only if the map on sections $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for each $U \subseteq X$. However, the corresponding statement for the surjective morphisms is not true.
For instance, let $X = S^1 \subset \mathbb{R}^2$ with the usual topology, $\mathcal{F}$ be the sheaf of all $\mathbb{R}$-valued continuous functions, and let $\mathcal{G}$ be the sheaf of all $S^1 = \mathbb{R}/\mathbb{Z}$-valued continuous functions. A natural morphism $\mathcal{F} \to \mathcal{G}$ is surjective on any small open neighborhood, however, there is no global $\mathbb{R}$-valued continuous function from S^1 which maps to id_{S^1}.
We give another example here. Let $X = \mathbb{C} \setminus \{0\}$ be the punctured complex plane, $\mathcal{F}$ be the sheaf of holomorphic functions, and let $\mathcal{G}$ be the sheaf of nowhere-zero holomorphic functions. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be the exponential map $\varphi(f) = e^f$. On a small open neighborhood, we may take the log branch, and hence φ is a surjective morphism of

sheaves. However, it is impossible to take a log branch on the whole punctured complex plane, in particular, there is no global holomorphic function $f : X \to \mathbb{C}$ such that $e^f(z) = z$ for every $z \in X$.

Definition 17. A sheaf $\mathcal{F}$ is called a *flasque sheaf* if the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective for every inclusion $V \subseteq U$ of open subsets.

Let us briefly observe just one major property of flasque sheaves, which will be useful in the remaining course. Let $0 \to \mathcal{F}' \overset{\varphi}{\to} \mathcal{F} \overset{\psi}{\to} \mathcal{F}'' \to 0$ be a short exact sequence of sheaves. In general, the (global) section functor $\Gamma(U, -)$ is left exact but not exact: the map $\mathcal{F}(U) \to \mathcal{F}''(U)$ needs not to be surjective. On the other hand, when $\mathcal{F}'$ is flasque, then we have a short exact sequence of abelian groups $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ for any open set $U \subseteq X$. A sketch of proof is as follows:

> Let $s'' \in \mathcal{F}''(U)$. Since the induced map on stalks $\psi_P : \mathcal{F}_P \to \mathcal{F}''_P$ is surjective, we may take $s_P \in \psi_P^{-1}(s''_P)$ for each $P \in X$. Let s_P is represented by a pair $(V_P, s(P))$ where $P \in V_P \subseteq U$ be an open neighborhood and $s(P) \in \mathcal{F}(V_P)$ such that $\psi(s(P)) = s''|_{V_P}$. The problem is that $s(P)$ and $s(Q)$ needs not to be the same on the intersection $V_P \cap V_Q$. On the other hand, followed from the left exactness, there is a section $s'(PQ) \in \mathcal{F}'(V_P \cap V_Q)$ such that $\varphi(s'(PQ)) = s(P)|_{V_P \cap V_Q} - s(Q)|_{V_P \cap V_Q}$. Since $\mathcal{F}'$ is flasque, there is a section $s'(Q) \in \mathcal{F}'(V_Q)$ such that $s'(Q)|_{V_P \cap V_Q} = s'(PQ)$ for each $Q \in U$. We may replace $s(Q) \in \mathcal{F}(V_Q)$ by $s(Q) + \varphi(s'(Q))$ for every $Q \in U$ to correct the difference.

So far, we discussed sheaves on a single topological space X. We also need to define some operations on sheaves with two topological spaces X and Y, together with a continuous map $f : X \to Y$.

Definition 18. Let $f : X \to Y$ be a continuous map between two topological spaces, and let $\mathcal{F}$ be a sheaf on X, $\mathcal{G}$ be a sheaf on Y. We define the *direct image sheaf* $f_*\mathcal{F}$ on Y by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ for each open set $V \subseteq Y$. We define the *inverse image sheaf* $f^{-1}\mathcal{G}$ on X to be the sheaf associated to the presheaf $U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$. Note that the stalk is easy to compute: $(f^{-1}\mathcal{G})_P = \mathcal{G}_{f(P)}$. Also note that the functor $f_*(-)$ is left exact, and the functor $f^{-1}(-)$ is exact.
When $Z \subset X$ is a subset, and $i : Z \to X$ be the inclusion map, then $\mathcal{F}|_Z := i^{-1}\mathcal{F}$ is called the *restriction* of $\mathcal{F}$ onto Z.

Remark 19. When we deal with a morphism $f : X \to Y$ of locally ringed spaces, we often play with sheaves of $\mathcal{O}_Y$-modules. Since $f^{-1}\mathcal{G}$ does not have an $\mathcal{O}_X$-module structure in general, one defines another functor f^* and the sheaf $f^*\mathcal{G}$ which is different from $f^{-1}\mathcal{G}$.

We finish by two fundamental propositions.