Still the dualizing sheaf $\omega_X^\circ = \mathcal{E}xt^r_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$ is not familiar enough. We already know that the canonical sheaf ω_X is the dualizing sheaf when $X = \mathbb{P}^n$ is a projective space. Hence, in the general case $X \subseteq \mathbb{P}^N$, it is natural to compare the canonical sheaves $(\omega_X, \omega_{\mathbb{P}^N})$ and the dualizing sheaf ω_X. We assume that X is Cohen-Macaulay and equidimensional, since the natural maps θ^i become isomorphisms in this case. First, we will see what happens locally.

Lemma 196. *Let R be a local Cohen-Macaulay ring with canonical module ω_R. If A is a local R-algebra which is finitely generated as an R-module (for instance, $A = R/I$), and A is also Cohen-Macaulay, then A has a canonical module and*

$$\omega_A \simeq \mathrm{Ext}^r_R(A, R)$$

where $r = \dim R - \dim A$.

In particular, we may expect that the canonical sheaf ω_X will be a strong candidate for the dualizing sheaf ω_X° – if the local behavior of X is good enough. When $X \subseteq \mathbb{P}^N$ is locally a complete intersection, then X is cut out by $r = \mathrm{codim}(X, \mathbb{P}^N) = N - n$ equations in the local ring at each point $x \in X \subseteq \mathbb{P}^N$, and such c generators form a regular sequence.

Let us describe this phenomenon in a general algebraic setting. Let A be a ring, and let $f_1, \cdots, f_r \in A$. The *Koszul complex* $K_\bullet(f_1, \cdots, f_r)$ is a complex such that K_1 is a free A-module of rank r with basis $e_1, \cdots, e_r$, and $K_p := \wedge^p K_1$. The boundary map $d : K_p \to K_{p-1}$ is defined as

$$d(e_{i_1} \wedge \cdots \wedge e_{i_p}) := \sum (-1)^{j-1} f_{i_j} (e_{i_1} \wedge \cdots \wedge \hat{e_{i_j}} \wedge \cdots \wedge e_{i_p}).$$

One can show that $d^2 = 0$, and hence $K_\bullet(f_1, \cdots, f_r)$ is a homological complex of A-modules. If M is an A-module, we write $K_\bullet(f_1, \cdots, f_r; M) := K_\bullet(f_1, \cdots, f_r) \otimes_A M$.

Lemma 197. *If $f_1, \cdots, f_r$ form a regular sequence for M, then*

$$h_i(K_\bullet(f_1, \cdots, f_r; M)) = \begin{cases} M/(f_1, \cdots, f_r)M & i = 0; \\ 0 & i > 0, \end{cases}$$

in particular, the Koszul complex gives a free resolution of $M/(f_1, \cdots, f_r)M$.

This leads to the following description of the dualizing sheaf for a local complete intersection $X \subseteq \mathbb{P}^N$.

Theorem 198. *Let X be a closed subscheme of $\mathbb{P}^N$ which is a local complete intersection of codimension r. Let $\mathscr{I}$ be the ideal sheaf of X. Then $\omega_X^\circ \simeq \omega_{\mathbb{P}^N} \otimes \wedge^r(\mathscr{I}/\mathscr{I}^2)^\vee$, in particular, ω_X° is an invertible sheaf on X.*

Proof. We need to compute $\omega_X^\circ = \mathcal{E}xt^r_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$. Let U be an affine open subset which is small enough, so that $\mathscr{I}$ can be generated by r elements $f_1, \cdots, f_r \in A = \Gamma(U, \mathcal{O}_{\mathbb{P}^N})$. Let $x \in X \cap U$ be a point corresponding to an ideal $\mathfrak{m} \subseteq A$. Being a local complete intersection, $f_1, \cdots, f_r$ form a regular sequence for $A_{\mathfrak{m}}$. In particular, the localized Koszul complex

$$K_\bullet(f_1, \cdots, f_r; A_{\mathfrak{m}})$$

gives a free resolution of $A_{\mathfrak{m}}/(f_1, \cdots, f_r)A_{\mathfrak{m}}$ over $A_{\mathfrak{m}}$. Hence, replacing U by a smaller affine open neighborhood if necessary, we may assume that $K_\bullet(f_1, \cdots, f_r)$ gives a free resolution of $A/(f_1, \cdots, f_r)A$ over $U = \operatorname{Spec} A$. Sheafifying it gives a free resolution $K_\bullet(f_1, \cdots, f_r; \mathcal{O}_U)$ of $\mathcal{O}_X$ over U. We may compute the sheaf Ext by using this Koszul resolution; we have

$$\begin{aligned} \mathcal{E}xt^r_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N})|_U &= h^r(\mathcal{H}om(K_\bullet(f_1, \cdots, f_r; \mathcal{O}_U), \omega_{\mathbb{P}^N}|_U)) \\ &\simeq \omega_{\mathbb{P}^N}|_U/(f_1, \cdots, f_r)\omega_{\mathbb{P}^N}|_U \\ &= (\omega_{\mathbb{P}^N} \otimes \mathcal{O}_X)|_U. \end{aligned}$$

Note that this isomorphism depends on the choice of a regular sequence $f_1, \cdots, f_r$ generating $\mathscr{I}$. If $g_i := \sum_{j=1}^r c_{ij} f_j$, $1 \le i \le r$, they give another basis, and the exterior powers of the matrix (c_{ij}) will give an isomorphism of Koszul complexes. In particular, we have a factor $\det(c_{ij})$ on $K_r = \wedge^r K_1$, and hence, our isomorphism of $\mathcal{E}xt^r$ changes by $\det(c_{ij})$.
Hence, we have to twist it to make it intrinsic, in other words, we have to multiply $\det(c_{ij})^{-1}$ on the transition functions. Note that the sheaf $\mathscr{I}/\mathscr{I}^2$ is a locally free sheaf of rank r, free over U, with basis $f_1, \cdots, f_r$. Hence, its determinant $\wedge^r(\mathscr{I}/\mathscr{I}^2)$ is free of rank 1, with basis $f_1 \wedge \cdots \wedge f_r$. Note that if we substitute the basis by g_i as above, the element $g_1 \wedge \cdots \wedge g_r = \det(c_{ij})(f_1 \wedge \cdots \wedge f_r)$. Therefore, the sheaf $\wedge^r(\mathscr{I}/\mathscr{I}^2)^\vee$ has the desired transition functions, and hence we have an intrinsic isomorphism (over U)

$$\mathcal{E}xt^r_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N}) \simeq \omega_{\mathbb{P}^N} \otimes \mathcal{O}_X \otimes \wedge^r(\mathscr{I}/\mathscr{I}^2)^\vee$$

as desired. This isomorphism is independent of the choice of basis. In particular, we may cover $\mathbb{P}^N$ by such open subsets, and hence the local isomorphisms will glue together and form the required isomorphism. □

Thanks to the adjunction formula, we immediately have the following description of the dualizing sheaf when X is smooth.

Corollary 199. *If X is a nonsingular projective variety over an algebraically closed field k, then the dualizing sheaf ω_X° is isomorphic to the canonical sheaf ω_X.*

The following theorem will be very useful in advance.

Theorem 200 (Kodaira, Akizuki-Nakano, Deligne-Illusie)**.** *Let X be a smooth projective variety of dimension n over a field k of characteristic 0. Let $\mathcal{L}$ be an ample line bundle on X. Then,*

$$H^q(X, \Omega^p_{X/k} \otimes \mathcal{L}) = 0 \text{ for all } p+q > n.$$

Equivalently, by Serre duality, we have

$$H^q(X, \Omega^p_{X/k} \otimes \mathcal{L}^{-1}) = 0 \text{ for all } p+q < n.$$

Corollary 201 (Kodaira vanishing theorem)**.** *Let X be a smooth projective variety of dimension n over a field k of characteristic 0. Let $\mathcal{L}$ be an ample line bundle on X. Then*

$$H^i(X, \omega_X \otimes \mathcal{L}) = 0 \text{ for all } i > 0,$$

and equivalently,

$$H^i(X, \mathcal{L}^{-1}) = 0 \text{ for all } i < n.$$

Topic 4 – Algebraic curves

An algebraic curve is an algebraic variety of dimension 1. We will mainly study irreducible algebraic curves over an algebraically field k of characteristic 0. Up to birational equivalence, the irreducible curves are equivalent to the algebraic function fields over k of transcendence degree 1. Indeed, a smooth projective curve can be completely determined by its function field. When $k = \mathbb{C}$, projective curves over k coincide with compact Riemann surfaces – connected, complex analytic manifold complex dimension 1. Indeed, there are equivalent "trichotomy":

(i) the category of smooth irreducible projective curves (algebraic manifolds);

(ii) the category of compact Riemann surfaces (analytic manifolds);

(iii) the opposite category of function fields of transcendence degree 1.

Topologically, compact Riemann surfaces are completely determined by their genera. Since the topological genus g coincides with the geometric genus p_g, one may expect that the canonical sheaf and their global sections will play significant roles in the curve theory. Being smooth, the genus also coincide with the arithmetic genus p_a. In particular, the Hilbert function/polynomial will also behave nicely.
There are several ways to give "different algebraic (or, complex) structure" on a curve of given genus. Nevertheless, it is quite natural to classify algebraic curves up to its genus, namely, into 3 subcases $g = 0$ (rational), $g = 1$ (elliptic), and $g \geq 2$ (of general type). We will not focus on the moduli problem, however, focus on their canonical sheaves.
The goal of this topic is to understand:

1. basic theorems including Riemann-Roch and Riemann-Hurwitz theorem;
2. embeddings to a projective space, mostly canonical embeddings;
3. canonical models and syzygies of canonical curves.

In particular, we will address Green's canonical syzygy conjecture, and review some related problems and results as the goal.
During the whole lectures on algebraic curves, a *curve* C denotes an irreducible, smooth, projective variety of dimension 1 over an algebraically closed field k of characteristic 0, unless there is a specific comment. A *point* of C means a closed point of C.

1 Riemann–Roch theorem

Proposition 202. *Let C be a curve. We have $p_a(C) = p_g(C) = h^1(C, \mathcal{O}_C)$.*

Proof. The arithmetic genus is given by $p_a(C) = 1 - \chi(C, \mathcal{O}_C) = 1 - h^0(C, \mathcal{O}_C) + h^1(C, \mathcal{O}_C) = h^1(C, \mathcal{O}_C)$ since C is connected. The geometric genus $p_g(C) = h^0(C, \omega_C) = h^1(C, \mathcal{O}_C)$ by Serre duality. □

We will call this number simply the *genus* of C, and denote it by g, or $g(C)$.

Remark 203. The genus g of a curve C is always nonnegative. Conversely, given any nonnegative integer g, there is a curve C of genus g. For instance, let $X = \mathbb{P}^1 \times \mathbb{P}^1 = V(xy - zw) \subseteq \mathbb{P}^3$ be the quadric hypersurface. Note that $\omega_X = \mathcal{O}_X(-2) = \mathcal{O}_{\mathbb{P}^1}(-2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-2)$. Consider a divisor of type $(g+1, 2)$ on X. Such a divisor is very ample, and hence, there is a smooth irreducible divisor of type $(g+1, 2)$ thanks to Bertini's theorem. Such a divisor, considered as a variety of dimension 1, is a curve of genus g.

We want to take the sheaf cohomology theory as a key tool of approaches. Our objects of interests are vector bundles, which are locally free sheaves of finite rank, and in particular, line bundles to keep a track together with Serre duality and Kodaira vanishing theorem. Since the set of the equivalence classes of Weil divisors $\mathrm{Cl}(C)$ identify with the set of the isomorphism classes of line bundles $\mathrm{Pic}(C)$, via a correspondence $D \mapsto \mathcal{O}_C(D)$, the statement should be described in terms of divisors – a formal sum of points in integer coefficients. Hence, we may consider the cohomology of a divisor, and write as $H^i(C, D) = H^i(C, \mathcal{O}_C(D))$. If there is no confusion, we may skip C and denote just by $H^i(D)$.

Recall that a divisor D can be expressed as a finite sum $D = \sum n_i P_i$ for some $P_i \in C$, $n_i \in \mathbb{Z}$. Its degree is $\deg(D) = \sum n_i$. A divisor is effective if all the coefficients $n_i \geq 0$ are nonnegative. The set of the effective divisors linearly equivalent to a given divisor D is called the *complete linear system* and denoted by $|D|$. From the construction, the elements of $|D|$ are in $1-1$ correspondence with the projectivized space

$$((H^0(C, \mathcal{O}_C(D)) \setminus \{0\})/k^\times,$$

so $|D|$ carries the structure of the set of closed points of a projective space. Indeed, the global sections $H^0(C, \mathcal{O}_C(D))$ define a map to the projective space $|D|$, which is well-defined outside the base points

$$Bs(D) := \bigcap_{D' \in |D|} \mathrm{Supp}(D').$$

The following statement is quite elementary, but useful.

Lemma 204. *Let D be a divisor on a curve C. If $\deg D < 0$, then $h^0(D) = 0$. If $\deg D = 0$ and $h^0(D) \neq 0$, then $\mathcal{O}_C(D) \simeq \mathcal{O}_C$, and hence $h^0(D) = 1$.*

Proof. If $h^0(D) \neq 0$, then D is linearly equivalent to an effective divisor $D' \in |D|$. Since $\deg D = \deg D'$, it must be nonnegative. When $\deg D = 0$, such D' is an effective divisor of degree 0. There is only one such divisor, namely $D' = 0$. □

Since the dimension of a curve C is 1, the sheaf of differentials $\Omega_{C/k}$ coincides with the canonical sheaf ω_C. We call any divisor in the corresponding linear equivalence class a *canonical divisor*, and denote by K_C. Note that $H^i(C, D) \simeq H^{1-i}(C, K_C - D)^\vee$ thanks to Serre duality.

Theorem 205 (Riemann-Roch)**.** *Let D be a divisor on a curve C of genus g. Then,*

$$\chi(C, D) = h^0(C, D) - h^1(C, D) = \deg D + 1 - g.$$

Proof. First of all, consider the case $D = 0$. The formula says:

$$h^0(C, \mathcal{O}_C) - h^1(C, \mathcal{O}_C) = 1 - g.$$

This is straightforward: $h^0(C, \mathcal{O}_C) = 1$ since it is connected, and $h^1(C, \mathcal{O}_C) = h^0(C, \omega_C) = g$ by Serre duality.
Now, let D be any divisor, and let P be any point of C. We claim that the formula is true for D if and only if it is true for $D + P$. If the claim holds, then the whole statement becomes true since any divisor can be reached from 0 in a finite number of steps by adding/subtracting a point each step.
Consider P as a closed subscheme of C. The structure sheaf $\mathcal{O}_P$ is a skyscraper sheaf $k(P)$, assigning k at the point P. Also note that the ideal sheaf $\mathscr{I}_P = \mathcal{O}_C(-P)$. We have an exact sequence

$$0 \to \mathcal{O}_C(-P) \to \mathcal{O}_C \to k(P) \to 0.$$

Twist by a line bundle $\mathcal{O}_C(D + P)$, which preserves the exactness, we have:

$$0 \to \mathcal{O}_C(D) \to \mathcal{O}_C(D + P) \to k(P) \to 0.$$

The Euler characteristic is additive on the short exact sequence, in other words,

$$\chi(C, O_C(D + P)) = \chi(C, \mathcal{O}_C(D)) + 1.$$

On the other hand, $\deg(D + P) = \deg(D) + 1$, so the formula is true for D if and only if it is true for $D + P$. $\square$

Remark 206. Let C be a curve in $\mathbb{P}^N$ of genus g and degree d, and let D be a hyperplane section. In particular, $\mathcal{O}_C(D) = \mathcal{O}_C(1)$. The Hilbert polynomial gives us that

$$\chi(C, \mathcal{O}_C(nD)) = nd + 1 - p_a = nd + 1 - g,$$

which is a special case of the Riemann-Roch formula.

Remark 207. This is a special case of the following Riemann-Roch formula for vector bundles:

> Let C be a curve of genus g, and let $\mathcal{E}$ be a locally free sheaf on C of rank r. Then,
>
> $$\chi(C, \mathcal{E}) = r(1 - g) + \deg \mathcal{E}$$
>
> where $\deg \mathcal{E} := \deg(\det \mathcal{E})$.

Example 208. Let C be a curve of genus g, and let K_C be its canonical divisor. Since $h^0(C, K_C) = g$ and $h^1(C, K_C) = h^0(C, \mathcal{O}_C) = 1$, we have

$$\chi(C, K_C) = g - 1 = \deg K_C + 1 - g,$$

hence $\deg K_C = 2g - 2$.

Definition 209. A divisor D on a curve C is *special* if $h^1(C, D) \neq 0$. Otherwise, it is *nonspecial*. If D is a divisor of degree $> 2g - 2$, then $K_C - D$ has negative degree, so $h^0(K_C - D) = h^1(D) = 0$. Thus any divisor of degree at least $2g - 1$ is nonspecial.

Remark 210. Let D be a divisor on a curve C. Then it is *ample* if and only if $\deg D > 0$. If it is ample, then there is a positive integer n such that nD is very ample, hence, the linear system $|nD|$ provides an embedding of C into a projective space. In particular, $h^0(C, nD) \geq 2$. Conversely, suppose that $\deg D$ is positive. Choose and fix any embedding $C \subseteq \mathbb{P}^N$. Any coherent sheaf $\mathcal{F}$ on C is a quotient of a direct sum of line bundles $\bigoplus \mathcal{O}_C(-q)$ for some $q \in \mathbb{Z}$. Twist by $\mathcal{O}_C(nD)$ for a given integer n, we have a surjection by reading off the cohomology long exact sequence

$$\bigoplus H^1(C, \mathcal{O}_C(-q) \otimes \mathcal{O}_C(nD)) \to H^1(C, \mathcal{F} \otimes \mathcal{O}_C(nD)) \to 0.$$

Since any line bundle of degree $> 2g-2$ is nonspecial, the term on left vanishes when $n \gg 0$. Hence, $H^1(C, \mathcal{F} \otimes \mathcal{O}_C(nD)) = 0$ for sufficiently large n. Thanks to the cohomological criterion of the ampleness, this concludes that D is indeed ample.

Definition 211. A curve C is called *rational* if $g(C) = 0$. This is compatible with the definition by a birational equivalence:

> Let P, Q be any two points of C. Then the degree of the divisor $P - Q$ is $0 > 2g - 1 = -1$, so $P - Q$ is nonspecial. Hence $h^0(C, P - Q) = 1$, in other words, $P - Q \sim 0$. In particular, there is a rational function f in the function field $K(C)$ such that $(f) = P - Q$. The inclusion of fields $k(f) \hookrightarrow K(C)$ gives a finite morphism $\varphi : C \to \mathbb{P}^1$, where $\varphi^*(0_{\mathbb{P}^1}) = P$ and $\varphi^*(\infty_{\mathbb{P}^1}) = Q$. In particular, the degree of φ must be 1, hence, φ is birational.

A curve C is called *elliptic* if $g(C) = 1$. In the case, the canonical divisor K_C has degree $2g - 2 = 0$, with a nontrivial global section $h^0(K_C) = 1$. In particular, $K_C = 0$. Otherwise, a curve C is called *of general type*. The canonical divisor K_C has positive degree in this case.

Exercise 212. Let C be an elliptic curve, and let P_0 be any point. Fix P_0. Show that the map $P \mapsto \mathcal{O}_C(P - P_0)$ gives a bijection from C to $\mathrm{Pic}^0(C) \subseteq \mathrm{Pic}(C)$, the subgroup of isomorphism classes of degree 0 line bundles. This gives a group structure on the set of points on C.

Exercise 213. Let D be an effective divisor on C. Show that $\dim |D| \leq \deg D$, and the equality holds if and only if either $D = 0$ or $g = 0$.

Exercise 214. Let C be a curve of genus g. Show that there is a finite morphism $f : C \to \mathbb{P}^1$ of degree $\leq g + 1$.

Exercise 215. A curve C is called *hyperelliptic* if $g \geq 2$ and there exists a finite morphism $f : C \to \mathbb{P}^1$ of degree 2.

(1) Let C be a curve of genus 2. Show that the canonical divisor K_C is base-point-free, $\deg K_C = 2$, and $h^0(C, K_C) = 2$. Conclude that C is hyperelliptic.

(2) Show that there is a hyperelliptic curve of genus g for every $g \geq 2$.