2 Riemann–Hurwitz theorem

Suppose that we have two curves X, Y and a nonconstant morphism $f : X \to Y$. Note that f is dominant, hence, it is equivalent to give a field extension of function fields of X and Y, and hence, a finite extension. In particular, the morphism $f : X \to Y$ must be a finite morphism of degree $[K(X) : K(Y)]$. It is natural to compare the canonical divisors of X and Y via f, in other words, to measure the difference between two divisors K_X and f^*K_Y. Thanks to the cotangent sequence, we have a right exact sequence

$$f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0.$$

Hence, it sounds natural to describe the difference by the relative sheaf of differentials $\Omega_{X/Y}$.
On the other hand, there is a geometric way to understand the morphism f, by observing the fibers. When $Q \in Y$ is a point, then the fiber $f^{-1}(Q) \subseteq X$ is a set of points of length $d = \deg f$. In particular, this gives a family of (effective) divisors on X parametrized by points of Y. By the upper-semicontunity, a "general" fiber is composed of d distinct points, and a "special" fiber has a multiplicity > 1 at some points.
This leads to the notion of ramification points as follows.

Definition 216. Let $P \in X$ be a point. The *ramification index* e_P is defined as:

> Let $Q = f(P)$, and let $t \in \mathcal{O}_{Y,Q}$ be the local parameter at Q. Via the natural map $f^\# : \mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$, define $e_P := v_P(f^\#(t))$ where v_P is the valuation of the discrete valuation ring $\mathcal{O}_{X,P}$.

If $e_P = 1$, we say f is *unramified at* P. If $e_P > 1$, we say f is *ramified at* P, and that Q is a *branch point* of f. A morphism of varieties (needs not to be curves) φ is called *étale* if it is locally finite and unramified.

Following the above notion, the pullback of divisors $f^* : \operatorname{Div} Y \to \operatorname{Div} X$ is given by

$$f^*(Q) = \sum_{P \in f^{-1}(Q)} e_P \cdot P.$$

Since $\mathcal{O}_X(f^*D) \simeq f^*\mathcal{O}_Y(D)$ for any divisor D on Y, it is better to take the homomorphism of the Picard groups. Note that any morphism $f : X \to Y$ of curves is *separable*, that is, $K(X)$ is a separable extension of $K(Y)$, since the base field k has characteristic 0. In particular,

Proposition 217. *Let $f : X \to Y$ be a finite morphism of curves. Then the relative cotangent sequence $0 \to f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$ is exact also on the left.*

Proof. It is sufficient to show that the first map is injective. Since both are locally free of rank 1, it will be sufficient to show that the map is nonzero at the generic point. Since $K(X)$ is separable over $K(Y)$, the sheaf $\Omega_{X/Y}$ is zero at the generic point of X. In particular, $f^*\Omega_Y \to \Omega_X$ is surjective at the generic point, which cannot be zero. $\square$

Since $f^*\Omega_Y$ and Ω_X are line bundles on X, it sounds like that the sheaf $\Omega_{X/Y}$ corresponds to the difference of two divisors $K_X - f^*K_Y$. However, the sheaf $\Omega_{X/Y}$ is not an invertible sheaf, and it is zero at the generic point of X. Indeed, it is a torsion sheaf, and hence, supported on a proper closed subset of X.
Let us study a little more on the sheaf $\Omega_{X/Y}$ of relative differentials. Let $P \in X$ be a point, and let $Q = f(P)$ be its image. Let t be a local parameter of Y at Q, and let u be a local parameter of X at P. Then, dt is a generator of a free module $\Omega_{Y,Q}$, and du is a generator of a free module $\Omega_{X,P}$. In particular, there is a unique element $g \in \mathcal{O}_{X,P}$ such that $f^*dt = g \cdot du$, and we denote it by $\frac{dt}{du}$.

Proposition 218. *Let $f : X \to Y$ be a finite morphism of curves. Then:*

(1) $\Omega_{X/Y}$ is a torsion sheaf on X, with support equal to the set of ramification points of f. In particular, f is ramified at finitely many points;

(2) for each $P \in X$, the stalk $(\Omega_{X/Y})_P$ is a principal $\mathcal{O}_{X,P}$-module of finite length equal to $v_P\left(\frac{dt}{du}\right)$;

(3) length$(\Omega_{X/Y})_P = e_P - 1$.

Proof. Note that $\Omega_{X/Y} = \Omega_X / f^*\Omega_Y$ is a quotient of two invertible sheaves. The stalk $(\Omega_{X/Y})_P$ can be written as $\Omega_{X,P}/f^*\Omega_{Y,Q}$, which is isomorphic to $\mathcal{O}_{X,P}/(\frac{dt}{du})$ as an $\mathcal{O}_{X,P}$-module. This implies the second statement. Furthermore, $(\Omega_{X/Y})_P = 0$ if and only if f^*dt generates $\Omega_{X,P}$, in other words, t is also a local parameter for $\mathcal{O}_{X,P}$. Hence, f is unramified at P which implies the first statement.
Finally, if f is ramified at P of index e_P, then we may write $t = au^{e_P}$ for some unit $a \in \mathcal{O}_{X,P}$. Therefore,

$$dt = e_P \cdot au^{e_P - 1}du + u^{e_P}da = u^{e_P-1}(e_P \cdot adu + uda).$$

Since we are working over a field of characteristic 0, we have $e_P \cdot adu \neq 0$ and thus $v_P(dt/du) = e - 1$. $\square$

Definition 219. Let $f : X \to Y$ be a finite morphism of curves. We define the *ramification divisor* R of f to be

$$R = \sum_{P \in X} length(\Omega_{X/Y})_P \cdot P.$$

Immediately, we have the following proposition which describes the difference of two canonical sheaves:

Proposition 220 (Ramification formula)**.** *Let $f : X \to Y$ be a finite morphism of curves. Then $K_X \sim f^*K_Y + R$.*

Proof. The structure sheaf $\mathcal{O}_R$ of R is isomorphic to the sheaf $\Omega_{X/Y}$. Hence, we have a short exact sequence

$$0 \to f^*\Omega_Y \to \Omega_X \to \mathcal{O}_R \to 0.$$

Twisting by $\Omega_X^\vee$, we have

$$0 \to f^*\Omega_Y \otimes \Omega_X^\vee \to \mathcal{O}_X \to \mathcal{O}_R \to 0,$$

which has to coincide with the exact sequence

$$0 \to \mathscr{I}_R = \mathcal{O}_X(-R) \to \mathcal{O}_X \to \mathcal{O}_R \to 0.$$

□

Corollary 221 (Riemann-Hurwitz formula)**.** *Let $f : X \to Y$ be a finite morphism of curves of degree n. Then*

$$2g(X) - 2 = n \cdot (2g(Y) - 2) + \deg R = n \cdot (2g(Y) - 2) + \sum_{P \in X} (e_P - 1).$$

Example 222. By the Riemann-Hurwitz formula, we can show easily that $\mathbb{P}^1$ does not admit an unramified finite cover by another curve X. Let $f : X \to \mathbb{P}^1$ be a finite étale cover of $\mathbb{P}^1$ of degree n. Then,

$$2g(X) - 2 = n(2g(\mathbb{P}^1) - 2) + \deg R = -2n + \deg R = -2n \leq -2$$

since $n \geq 1$. The only possibility is that $g(X) = 0$ and $n = 1$, that is, $X = \mathbb{P}^1$ and f is the identity map.

Example 223. Suppose that there is a dominant, finite morphism $f : X \to Y$ of curves. Then $g(X) \geq g(Y)$ since $\deg f \geq 1$ and $\deg R \geq 0$. The equality occurs only if $n = 1$, or $g(X) = g(Y) = 0$, or $g(Y) = 1$ and f is unramified.
As a consequence, we have the following "Lüroth's theorem":

> Let X be a unirational curve, that is, there is a finite, dominant rational map $f : \mathbb{P}^1 \dashrightarrow X$. Then X is rational.

If there is such a rational map, it extends to a finite, dominant morphism from $\mathbb{P}^1$ to X. Hence, $g(X) = 0$ and we conclude that $X \simeq \mathbb{P}^1$.

Exercise 224 (Hyperelliptic revisited)**.** Let C be a curve of genus 2 over the field $k = \mathbb{C}$.

(1) Show that the canonical divisor K_C induces a finite morphism $f : C \to \mathbb{P}^1$ of degree 2 ramified at 6 points.

(2) Let $\alpha_1, \cdots, \alpha_6 \in k$ be six distinct elements, and let K be the field extension of $k(x)$ by the equation $y^2 - \prod_{i=1}^6 (x - \alpha_i) = 0$. Let $f : C \to \mathbb{P}^1$ be the corresponding morphism of curves. Show that $g(C) = 2$, and the map f is the same as the morphism induced by the canonical divisor K_C. In particular, f is ramified exactly over the six points $(x = \alpha_i)$ of $\mathbb{P}^1$.

(3) (Möbius transformation) Let P_1, P_2, P_3 be three distinct points of $\mathbb{P}^1$, and let Q_1, Q_2, Q_3 be another three distinct points of $\mathbb{P}^1$. Show that there is an automorphism of $\mathbb{P}^1$ sending P_i to Q_i. In particular, the above six points $\alpha_1, \cdots, \alpha_6$ can be normalized by $0, 1, \infty, \beta_1, \beta_2, \beta_3$ where β_i's are three distinct complex numbers different from $0, 1$.

(4) Show that the permutation group $\mathfrak{S}_6$ of six letters acts on the set of triples

$$\{(\beta_1, \beta_2, \beta_3) \mid \beta_i \in \mathbb{C} \setminus \{0, 1\}\}.$$

(5) Conclude that there is an 1-1 correspondence between the set of isomorphism classes of curves of genus 2 and the set of orbits described in above.

(6) Similarly, describe the parameter space of hyperelliptic curve of genus g as the set of $(2g-1)$-tuples of distinct elements in $\mathbb{C} \setminus \{0, 1\}$ modulo a finite group action.

Exercise 225 (Automorphisms of a curve)**.** Let C be a curve over $k = \mathbb{C}$ of genus g.

(1) Let $g = 0$. Show that the automorphism group $\mathrm{Aut}(\mathbb{P}^1)$ is the group of Möbius transformations

$$\left\{ f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0 \right\}.$$

(2) Let $g = 1$. In the case, C is an elliptic curve, and hence it has a group structure. Show that the translation by a point of C is an automorphism. In particular, C is a subgroup of $\mathrm{Aut}(C)$.

(3) Let $g \geq 2$. In the case, it is known that $G = \mathrm{Aut}(C)$ is finite. So let $n = |G|$ be its order. Since G acts on the function field $K(C)$ of C, we have a field extension $L := K(C)^G \hookrightarrow K(C)$. This gives a finite morphism of curves $f : C \to C'$ of degree n.

If $P \in C$ is a ramification point of index $e_P = r$, show that $f^{-1}(f(P))$ consists of n/r points, each having ramification index r.

(4) Note that f is branched over finite number of points on C'. Let $P_1, \cdots, P_s$ be a maximal set of ramification points of C lying over distinct points of Y. In particular, s is the number of branch points of C'. Let $e_{P_i} = r_i$. Show that

$$\frac{1}{n}(2g-2) = 2g(C') - 2 + \sum_{i=1}^{s} \left(1 - \frac{1}{r_i}\right).$$

(5) Since $g \geq 2$, the value appearing in the above equality must be > 0. Under the assumption $g(C') \geq 0$, $s \geq 0$, $r_i \geq 2$ for $1 \leq i \leq s$ which makes the above value to be strictly greater than 0, show that the minimum value of the expression

$$2g(C') - 2 + \sum_{i=1}^{s} \left(1 - \frac{1}{r_i}\right)$$

is $\frac{1}{42}$ (occurs when $g(C') = 0, s = 3, r_1 = 2, r_2 = 3, r_3 = 7$). Conclude that $n \leq 84(g-1)$.

3 Embeddings in projective spaces and the canonical embedding

From this section, we are interested in the extrinsic behavior of a projective curve $C \subseteq \mathbb{P}^N$, in other words, C as a closed subvariety of $\mathbb{P}^N$. Since any divisor D on a curve C of positive degree is ample, there is always a very ample line bundle on C, of sufficiently high degree. Among them, the most interesting case is the canonical embedding, which is the case when the canonical divisor K_C is very ample. In fact, it can be described in geometric way as follows:

Theorem 226 (M. Noether)**.** *Let C be a curve of genus $g \geq 3$. Then K_C is very ample if and only if C is not hyperelliptic.*

In this section, we will focus on basic and general properties of curves, together with embeddings to projective spaces.
First of all, we study basic properties of line bundles (= divisors) on a curve C. Since an embedding to a projective space is determined by a linear system determined by a very ample divisor, it is important to analyze the positivity of line bundles. Note that a divisor D on C is ample if and only if $\deg D > 0$, and D is very ample if it is isomorphic to $\mathcal{O}(1)$ for some embedding into a projective space. Also note that D is base-point-free if the set $\bigcap_{D' \sim D} \operatorname{Supp} D'$ is empty, equivalently, the line bundle $\mathcal{O}_C(D)$ is globally generated.

Proposition 227. *Let D be a divisor on a curve C. Then:*

(1) D is base-point-free if and only if $h^0(C, D-P) = h^0(C, D) - 1$ for every point $P \in C$;

(2) D is very ample if and only if $h^0(C, D-P-Q) = h^0(C, D) - 2$ for every two points $P, Q \in C$, including the case $P = Q$.

Proof. Consider the short exact sequence

$$0 \to \mathcal{O}_C(D-P) \to \mathcal{O}_C(D) \to k(P) \to 0.$$

Taking the global section, we have a left exact sequence $0 \to H^0(C, \mathcal{O}_C(D-P)) \to H^0(C, \mathcal{O}_C(D)) \to k$ of k-vector spaces. In particular, $h^0(C, D)$ is either $h^0(C, D-P)+1$ or $h^0(C, D-P)$. Consider the addition map φ_P defined as $D' \mapsto D'+P$ which induces an injection $H^0(C, D-P) \to H^0(C, D)$, which will denote also by φ_P. Hence, $h^0(C, D) = h^0(C, D-P)$ if and only if the addition map $\varphi_P : H^0(C, D-P) \to H^0(C, D)$ is surjective. In particular, any effective divisor D' linearly equivalent to D can be expressed as a sum of two effective divisor $D' = E + P$ for some $E \in \operatorname{Div}(C)$. In particular, $P \in \operatorname{Supp} D'$ for any $D' \sim D$, that is, P is a base point of D.

For the very ampleness, we may assume that D is base-point-free. Clearly, a very ample divisor is base-point-free. On the other hand, if we have $h^0(C, D-P-Q) = h^0(C,D)-2$, then $h^0(C, D-P) = h^0(C,D)-1$ since $h^0(C, D-P)$, obtained by subtracting a single point, is either $h^0(C,D)$ or $h^0(C,D)-1$, in particular, can differ at most 1. Hence, D is base-point-free.
Note that D is very ample if and only if the complete linear system $|D|$ separates the points and tangent vectors. Separating points means that for any distinct two points $P, Q \in C$, the point Q is not a base point of $|D-P|$, which we already proved. Separating tangent vectors means that for any point $P \in C$, there is an effective divisor $D' \sim D$ such that P occurs with multiplicity 1 in D'. Since P is a smooth point of C, this is equivalent to say that $\dim T_P(D') < \dim T_P(C) = 1$, that is, $\dim T_P(D') = \dim D' = 0$. But this says that P is not a base point of $|D-P|$, in particular,

$$h^0(D-2P) = h^0(D) - 2.$$

□

Corollary 228. *Let D be a divisor on a curve C of genus g. Then:*

(1) if $\deg D \geq 2g$, then D is base-point-free;

(2) if $\deg D \geq 2g+1$, then D is very ample.

Proof. When D is nonspecial, the degree of D completely determines the value $h^0(C,D)$ by Riemann-Roch formula. Since any divisor of degree at least $2g-1$ is nonspecial by Kodaira vanishing, it is easy to verify the above statements. □

Example 229. When $g = 0$, *i.e.*, $C \simeq \mathbb{P}^1$, then a divisor D is ample $\Leftrightarrow$ very ample $\Leftrightarrow \deg D > 0$. On the other hand, when $g > 0$, the ample divisor $\mathcal{O}_C(P)$, $P \in C$ is never very ample. Indeed, it is not even base-point-free:

> $h^0(C, \mathcal{O}_C(P)) \geq h^0(C, \mathcal{O}_C) + 1 = 2$ implies that there is an effective divisor (of degree 1) which is linearly equivalent to P but different from P. In particular, it must be another point $Q \in C$. Hence, there is a degree 1 morphism $f : C \to \mathbb{P}^1$ sending P to $0 \in \mathbb{P}^1$ and Q to $\infty \in \mathbb{P}^1$, which is possible only when $C \simeq \mathbb{P}^1$. This contradicts to the assumption $g > 0$.

Note that P is the only base point in this case.

Example 230. When D is a very ample divisor of degree d on a curve C, which induces an embedding of C into a projective space $\mathbb{P}^N$. The image of C, as a projective variety, has degree d.

Example 231. Let C be an elliptic curve. Since $g = 1$, any divisor D of degree at least $3 = 2g+1$ is very ample. However, there is no very ample divisor D of degree 2. Indeed, such D must be base-point-free, hence, $|D|$ gives a morphism to $\mathbb{P}^1$ since $\dim |D| = h^0(C,D) - 1 = 1$, which cannot be an embedding of C. When $g \geq 2$, the smallest degree among very ample divisors on C does not coincide with the bound $2g+1$ we found, and provides an (intrinsic) geometric information of C.

We give a short remark on linear projections. Note that a linear projection at a linear subspace $\Lambda \subseteq \mathbb{P}^N$ of dimension r is a rational map $\pi_\Lambda : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-r-1}$, corresponding to the quotient $V \to V/W$ where $\mathbb{P}(V) = \mathbb{P}^N$ and $\mathbb{P}(W) = \Lambda$. When we have a (quasi-) projective variety $X \subseteq \mathbb{P}^N$ which does not intersect with Λ, then it gives a morphism $\pi_\Lambda : X \to \mathbb{P}^{N-r-1}$, which is also called a (outer) projection. Since any linear projection can be decomposed into a sequence of linear projections at a point, we only consider the case $\Lambda = \{P\}$ the projection center is a single point of $\mathbb{P}^N$.
A secant line L of X is a line in $\mathbb{P}^N$ which equals to the line $\overline{Q_1Q_2}$ for some distinct two points $Q_1, Q_2 \in X$. A tangent line L to X at a point $Q \in X$ is a line such that the (embedded) tangent space $T_Q(L)$ at Q is a subspace of $T_Q(X)$ as subspaces of $T_Q(\mathbb{P}^N)$. When X is a curve, and Q is a smooth point of X, then there is a unique tangent line to X at the point Q.

Proposition 232. *Let $X \subseteq \mathbb{P}^N$ be a projective variety, and let $P \in \mathbb{P}^N \setminus X$ be a point not on X. Let $\pi_P : X \to \mathbb{P}^{N-1}$ be the linear projection at the point P. Then π_P is an embedding if and only if*

(i) no secant line of X passes through P;

(ii) no tangent line of X (at any point $Q \in X$) passes through P.

Proof. If P lies on a secant line $\overline{Q_1Q_2}$ of X, then π_P fails to separate $\pi_P(Q_1)$ and $\pi_P(Q_2)$. Similarly, if P lies on a tangent line L of X at $Q \in X$, then π_P fails to separate the tangent vectors of $\pi_P(X)$ at $\pi_P(Q)$. □

Definition 233. Let $X \subseteq \mathbb{P}^N$ be a projective variety of dimension n. We define the tangent variety $\mathrm{Tan}(X)$ as

$$\mathrm{Tan}(X) := \overline{\bigcup L}, \text{ where } L \text{ is a tangent line of } X \text{ at some point } Q \in X \ ,$$

and the secant variety $\mathrm{Sec}(X)$ as

$$\mathrm{Sec}(X) := \overline{\bigcup L}, \text{ where } L \text{ is a secant line of } X \ .$$

Note that both $\mathrm{Tan}(X)$ and $\mathrm{Sec}(X)$ are projective varieties in $\mathbb{P}^N$, and $\mathrm{Tan}(X) \subseteq \mathrm{Sec}(X)$. When X is smooth, $\dim \mathrm{Tan}(X) \leq 2n$ and $\dim \mathrm{Sec}(X) \leq 2n+1$.

Corollary 234. *Let X be a smooth projective variety of dimension n. Then X can be embedded into $\mathbb{P}^{2n+1}$.*

Proof. First, take any embedding $X \hookrightarrow \mathbb{P}^N$. If $N > 2n+1$, then there is a point $P \in \mathbb{P}^N \setminus \mathrm{Sec}(X)$. The linear projection π_P at a point P gives an embedding $X \hookrightarrow \mathbb{P}^{N-1}$. By repeating the process, we have the desired result. □

Remark 235. The smallest embedding dimension also provides a geometric information of X. Of course, $2n+1$ is not always the smallest possible, for instance, a plane curve has embedding dimension $2 < 3 = 2n+1$.

Remark 236. Suppose that the projection center $P \in \mathbb{P}^N$ is a point on X. In this case, we have a rational map $\pi_P : X \dashrightarrow \mathbb{P}^{N-1}$, well-defined on $X \setminus \{P\}$. This can be completed by taking its projective closure (or, consider the strict transform of X inside the blowing up of $\mathbb{P}^N$ at P and extend it into a morphism). Indeed, we have a morphism $X \to \mathbb{P}^{N-1}$, again denoted by π_P, which is called an *inner projection* of X at P. This morphism is determined by the linear subsystem whose support contains P. In particular, when $X = C$ is a smooth curve embedded by the complete linear system $|D|$ for some divisor D on C, then the inner projection of C at P is the morphism induced by the complete linear system $|D - P|$ (note that $D - P$ is base-point-free since D is very ample).
In the case, one can show that the induced morphism $\pi_P : C \to \mathbb{P}^{N-1}$ is very ample if and only if there is no trisecant lines of C containing P, where a trisecant line of C is a line L with $length(L \cap C) \geq 3$.
For example, a twisted cubic C in $\mathbb{P}^3$ has no trisecant lines, and hence, any inner projection at a point $P \in C$ gives an embedding $C \to \mathbb{P}^2$ as a degree 2 curve, in other words, a plane conic.
Another example: let C be a complete intersection of two quadrics in $\mathbb{P}^3$. One can show that C has no trisecant lines. Hence, an inner projection at a point $P \in C$ gives an embedding $C \to \mathbb{P}^2$ as a degree 3 curve, in other words, a plane cubic. One can show that the (arithmetic) genus of C is 1, thus C is an elliptic curve.

Exercise 237. Let C be a plane curve of degree 4. Show that any effective divisor linearly equivalent to the canonical divisor is the hyperplane divisor $C \cap L$ for some line $L \subseteq \mathbb{P}^2$. Show that there is no linear system of divisors of degree 2 of dimension 1, in other words, $h^0(C, D) < 2$ for any (effective) divisor D of degree 2. Conclude that C cannot be hyperelliptic.

Exercise 238 (Curves of degree 4)**.** Let $C \subset \mathbb{P}^N$ be a curve of degree 4 of genus g. Show that either

1. $g = 3$, in which case X is a plane quartic; or
2. $g = 1$, in which case X is a complete intersection of two quadric surfaces in $\mathbb{P}^3$; or
3. $g = 0$, in which case X is either a rational normal quartic in $\mathbb{P}^4$, or X is a rational quartic in $\mathbb{P}^3$.

Now we want to focus on the canonical divisor K_C of degree $2g - 2$. If $g = 0$, then $K_C = \mathcal{O}(-2)$ does not have any nonzero global section, and hence the linear system $|K_C|$ is empty. If $g = 1$, then $K_C = \mathcal{O}_C$, and hence the linear system $|K_C|$ gives a constant map from C to a point. Hence, considering the linear system $|K_C|$ is meaningful when $g \geq 2$, that is, C is a curve of general type.

Lemma 239. *Let C be a curve of genus $g \geq 1$. Then the canonical divisor K_C is base-point-free.*

Proof. Note that $h^0(C, \mathcal{O}_C(P)) = h^0(C, \mathcal{O}_C) = 1$ for any $P \in C$ unless C is rational. Also note that a point P is a base point of K_C if and only if $h^0(C, K_C) = h^0(K_C - P)$. Since $h^0(C, K_C - P) = h^1(P)$ by Serre duality, we have

$$h^0(C, K_C - P) = h^0(C, P) - (1 - g + \deg P) = g - 1$$

by Riemann-Roch, where the value equals to $h^0(C, K_C) - 1 = g - 1$. Therefore, no point P can be a base point of K_C. $\square$