It is better to introduce a classical notation for our convenience.

Notation 240. We say that a curve C has a $\mathfrak{g}_d^r$ if there is a morphism $f : C \to \mathbb{P}^r$ of degree d, equivalently, there is a linear system of dimension r and degree d.

In particular, C is hyperelliptic if and only if C has a $\mathfrak{g}_2^1$.
Now we are ready to prove M. Noether's theorem:

Theorem 241 (M. Noether)**.** *Let C be a curve of genus $g \geq 2$. Then K_C is very ample if and only if C is not hyperelliptic.*

Proof. Suppose first that K_C is very ample, that is, for any effective divisor D of degree 2, we have

$$h^0(C, K_C - D) = h^0(C, K_C) - 2 = g - 2.$$

By Serre duality, $h^0(C, D) = h^1(C, K_C - D) = 1$ for any effective divisor D of degree 2. In particular, the linear system $|D|$ cannot give a morphism to $\mathbb{P}^1$, and hence, C cannot be hyperelliptic.
Conversely, if K_C is not very ample, then there are two points $P, Q \in C$ (possibly coincide) such that $h^0(C, K_C - P - Q) \neq h^0(C, K_C) - 2$. Since K_C is base-point-free, the only possibility is that $h^0(C, K_C - P - Q) = h^0(C, K_C) - 1 = g - 1$. In this case, $h^0(C, P + Q) = h^1(C, K_C - P - Q) = 2$, that is, the complete linear system $|P + Q|$ induces a morphism $f : C \to \mathbb{P}^1$ of degree 2. $\square$

Remark 242. Indeed, when C is not hyperelliptic, the natural map $\mathrm{Sym}^n(H^0(K_C)) \to H^0(nK_C)$ is surjective for every n. In other words, the image of C under the morphism induced by $|K_C|$ is projectively normal.

Note that every curve of genus 2 is hyperelliptic, since the linear system $|K_C|$ is a $\mathfrak{g}_2^1$ on C. Hence, it is worthwhile to consider the curves of genus at least 3. In the case, $|K_C|$ gives a morphism $C \to \mathbb{P}^{g-1}$. When C is not hyperelliptic, this morphism is indeed an embedding.

Definition 243. Let C be a non-hyperelliptic curve of genus $g \geq 3$. The embedding $C \hookrightarrow \mathbb{P}^{g-1}$ determined by the canonical divisor K_C is called the *canonical embedding* of C (determined up to an automorphism of $\mathbb{P}^{g-1}$, which is a linear change of coordinates). Its image, a curve of degree $2g - 2$, is called a *canonical curve*.

Example 244. Let C be a non-hyperelliptic curve of genus 3. Its canonical curve is a quartic curve in $\mathbb{P}^2$. Conversely, any plane quartic curve C has $\omega_C \simeq \mathcal{O}_C(1)$, so it is a canonical curve. In particular, there is a non-hyperelliptic curve of genus 3.

Example 245. Let C be a non-hyperelliptic curve of genus 4. Its canonical curve is a sextic curve in $\mathbb{P}^3$. Let $\mathscr{I}$ be the ideal sheaf. We have the following short exact sequence

$$0 \to \mathscr{I}(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_C(2) \to 0.$$

Since $\mathcal{O}_C(2)$ has degree $12 = 2 \cdot 6$, it is nonspecial. In particular, $h^0(C, \mathcal{O}_C(2)) = \chi(C, \mathcal{O}_C(2)) = 1 - g + \deg \mathcal{O}_C(2) = 9$. From the left exact sequence

$$0 \to H^0(\mathbb{P}^3, \mathscr{I}(2)) \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(C, \mathcal{O}_C(2)),$$

we have

$$h^0(\mathbb{P}^3, \mathscr{I}(2)) \geq 1.$$

In particular, there is at least one quadric surface $Q \subset \mathbb{P}^3$ containing C. Since C is not contained in a plane $\mathbb{P}^2$ (if then, its (arithmetic) genus must be of the form $\binom{d-1}{2}$ for some integer d), such a Q must be irreducible and reduced. If there are two different quadrics Q, Q' both contain C, then $\deg C \leq \deg Q \cap Q' = 4$ contradicts to the assumption, hence there is a unique quadric surface Q containing C. In particular, $h^0(\mathbb{P}^3, \mathscr{I}(2)) = 1$.
Consider the short exact sequence

$$0 \to \mathscr{I}(3) \to \mathcal{O}_{\mathbb{P}^3}(3) \to \mathcal{O}_C(3) \to 0,$$

we have $h^0(\mathbb{P}^3, \mathscr{I}(3)) \geq 5$ by a similar argument. Since the image of the multiplication map $H^0(\mathbb{P}^3, \mathscr{I}(2)) \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathbb{P}^3, \mathscr{I}(3))$ has dimension at most 4, there is a cubic surface $F \subset \mathbb{P}^3$, not containing Q, such that $C \subset F$. Since $\deg Q \cap F = 6$, it must coincide with C. Therefore, we conclude that a canonical curve of genus 4 is a (smooth) complete intersection of quadric and cubic surfaces in $\mathbb{P}^3$.
Conversely, if we have a smooth complete intersection C of quadric and cubic surfaces in $\mathbb{P}^3$, then the adjunction formula implies that $\omega_C \simeq \mathcal{O}_C(1)$, so C is a canonical curve (of genus 4).

Before to proceed, let us have a short tour on the canonical divisor on a hyperelliptic curve C of genus $g \geq 2$. Since K_C is base-point-free, it still induces a morphism $C \to \mathbb{P}^{g-1}$, however, it is not an embedding since K_C is not very ample.

Proposition 246. *Let C be a hyperelliptic curve of genus $g \geq 2$. Then C has a unique $\mathfrak{g}^1_2$. If $\varphi_0 : C \to \mathbb{P}^1$ is the corresponding $2-1$ morphism, then the canonical morphism $\varphi : C \to \mathbb{P}^{g-1}$ factors through φ_0 followed by $(g-1)$-uple embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$. In particular, the image $\varphi(C)$ is a rational normal curve of degree $g-1$ and φ is a morphism of degree 2 onto $\varphi(C)$. Furthermore, $K_C \sim (g-1)\mathfrak{g}^1_2$, in other words, every effective divisor in $|K_C|$ is a sum of $(g-1)$ divisors in the (unique) $\mathfrak{g}^1_2$.*

Proof. Since C is hyperelliptic, it has a $\mathfrak{g}^1_2$. Choose and fix one. For any divisor $P_1 + P_2 \in \mathfrak{g}^1_2$, we already seen that P_1‘ is a base point of $|K_C - P_2|$. In particular, $|K_C|$ does not separate P_1 and P_2, in other words, $\varphi(P_1) = \varphi(P_2)$. Since $\mathfrak{g}^1_2$ has infinitely many effective divisors in it (parametrized by $\mathbb{P}^1$), the morphism φ cannot be birational. Let the degree of the surjection $\varphi : C \to \varphi(C)$ be $\mu \geq 2$, and let $d = \deg \varphi(C)$. Since $\deg K_C = 2g-2$, we have $d\mu = 2g-2$, hence $d \leq g-1$.
To resolve (potential) singular points of $\varphi(C)$, we take its normalization $\widetilde{\varphi(C)} \to \varphi(C) \subseteq \mathbb{P}^{g-1}$. In particular, this normalization morphism comes from a linear system of degree $d \leq g-1$ and dimension $g-1$ on $\widetilde{\varphi(C)}$. The only possibility is $d = g-1$ and the

genus of $\widetilde{\varphi(C)}$ is 0. Hence, it is isomorphic to $\mathbb{P}^1$ and the linear system must be the unique complete linear system of degree $g-1$, namely, $|(g-1)Q|$. In particular, the morphism coincides with the $(g-1)$-uple embedding of $\mathbb{P}^1$, and hence, the image $\varphi(C)$ is a (nonsingular) rational normal curve of degree $g-1$.
Note that $\mu = 2$, and $\varphi : C \to \varphi(C)$ collapses the pairs of the $\mathfrak{g}^1_2$ as above. Thus, it must be equal to the composition of the map $\varphi_0 : C \to \mathbb{P}^1$ determined by our $\mathfrak{g}^1_2$ with the $(g-1)$-uple embedding of $\mathbb{P}^1$. The $\mathfrak{g}^1_2$ is determined by φ, and so is uniquely determined. Finally, any effective divisor linearly equivalent to K_C is the pull-back of some hyperplane section of $\varphi(C)$, hence, it is a sum of $g-1$ divisors in the unique $\mathfrak{g}^1_2$. Conversely, any sum of $g-1$ points on $\varphi(C)$ is a hyperplane section, so we may identify the linear system $|K_C|$ with the set of sums of $g-1$ divisors in $\mathfrak{g}^1_2$. □

To provide a finer classification of curves, we need other invariants than the genus. Among them, the most important invariant is the *gonality*, which is geometrically defined:

Definition 247. Let C be a curve of genus g. The *gonality* of C is the minimal possible degree of a finite morphism $C \to \mathbb{P}^1$. C has the gonality $k = \mathrm{gon}(C)$ implies that C has $\mathfrak{g}^1_k$ but no $\mathfrak{g}^1_\ell$ with $\ell < k$.

The only 1-gonal curve is $\mathbb{P}^1$, so it is not interesting. When $g(C) \geq 2$, the smallest possible gonality it can have is 2, and hence $\mathrm{gon}(C) = 2$ if and only if C is hyperelliptic. There is another important invariant called the *Clifford index*. For a nonspecial divisor D on C, we can compute the dimension of the vector space $H^0(C, D)$, or of the complete linear system $|D|$ by the Riemann-Roch formula. However, if D is special, $h^0(C, D)$ does not depend only on its degree. There are some useful theorems on $h^0(C, D)$, including the following Clifford's theorem.

Theorem 248 (Clifford)**.** *Let D be an effective special divisor on the curve C, that is, $h^1(C, D) > 1$. Then*

$$h^0(C, D) - 1 = \dim |D| \leq \frac{1}{2} \deg D.$$

Furthermore, the equality holds if and only if either $D = 0$, or $D = K$, or C is hyperelliptic and D is a multiple of the unique $\mathfrak{g}^1_2$.

The above theorem leads to the following definition of the Clifford index:

Definition 249. The *Clifford index* of a curve C is the minimum value of $(\deg D - 2\dim |D|)$, taken over all the effective special divisors D on C different from 0 or K_C.

Clifford's theorem implies that the Clifford index of C is nonnegative, and it is 0 if and only if C has a $\mathfrak{g}^1_2$, *i.e.*, C is hyperelliptic. Note that both the gonality and the Clifford index measures how the given curve C is apart from hyperelliptic curves.

Remark 250. If we have a $\mathfrak{g}^1_k$ for some $2 \leq k$ on a curve C of genus $g > 2$, then the divisor D appearing in a linear series in $\mathfrak{g}^1_k$ contributes to the computation of the Clifford index, in particular, $\mathrm{Cliff}(C) \leq k-2$. In general, if a curve C is k-gonal, then

$$k - 3 \leq \mathrm{Cliff}(C) \leq k - 2$$

holds, and the case $\mathrm{Cliff}(C) = k - 3$ happens rarely.

The following theorem, which is a special case of the theorem called "geometric Riemann-Roch formula", is also helpful in many places:

Theorem 251 (Geometric Riemann-Roch formula)**.** *Let $D = P_1 + \cdots + P_d$ be an effective divisor, which consists of d distinct points on a nonhyperelliptic curve C of genus $g \geq 2$. Let $\varphi : C \hookrightarrow \mathbb{P}^{g-1}$ be the canonical embedding. Then*

$$h^0(C, D) = d - \dim \overline{\varphi(D)},$$

where $\overline{\varphi(D)}$ denotes the linear span of d points $\varphi(P_1), \cdots, \varphi(P_d) \in \mathbb{P}^{g-1}$.

Proof. Only for a sketch of the proof. The Riemann-Roch formula and the Serre duality imply: $h^0(C, D) = 1 - g + d + h^1(C, D) = 1 - g + d + h^0(C, K_C - D)$. Since $|K_C|$ defines an embedding of C into $\mathbb{P}^{g-1}$, the linear system $|K_C - D|$ admits the following geometric interpretation:

> $|K_C - D|$ is consisted of hyperplane sections of C (= effective divisors in $|K_C|$) which contain $D = \varphi(P_1) + \varphi(P_2) + \cdots + \varphi(P_d)$.

Note that a hyperplane $H \subset \mathbb{P}^{g-1}$ contains $\varphi(P_1), \cdots, \varphi(P_d)$ if and only if H contains their linear span $\overline{\varphi(D)}$. The dimension of the family of such hyperplanes of $\mathbb{P}^{g-1}$ is $(g-1) - \dim \overline{\varphi(D)} - 1 = g - 2 - \dim \overline{\varphi(D)}$, which gives the value $\dim |K_C - D|$. In particular, $h^0(C, K_C - D) = g - 1 - \dim \overline{\varphi(D)}$. □

Exercise 252. Let $C \subseteq \mathbb{P}^{g-1}$ be a nonhyperelliptic canonical curve of genus $g \geq 3$. Show that the number of independent quadratic generators for the ideal of C is $\binom{g-2}{2}$.

4 Syzygies and Koszul cohomology

In the last section, we studied basic properties of a curve as a closed subvariety of a projective space. In particular, we learned two important theorems:

(1) a divisor D is very ample if $\deg D \geq 2g + 1$;

(2) the canonical divisor K_C is very ample if and only if C is not hyperelliptic.

We may ask for the next case as generalizations. For instance, if D is a divisor of degree $\geq 2g + 2$, then what can we say about D additionally? What can we say about the canonical divisor if C gets much further from being hyperelliptic?
The answer for the first question is that the image of C is defined by quadratic equations. Suppose we have a projective variety $X \subset \mathbb{P}^N$. Composing with a further d-uple embedding for some $d \gg 0$, one immediately has that the image of X tends to be cut out only by quadratic equations, since Veronese varieties do. Hence, we may expect that if the divisor D gets more positive, then the image of C by a map defined by $|D|$ tends

to be cut out by quadratic equations. Green's $(2g+1+p)$-theorem generalizes this idea in a beautiful statement.
The second question is much tricky. First, we have to find out a good notion which measures a difference between C and hyperelliptic curves. The notion of gonality is very intuitive and geometric, and works very nice in many places. However, in some cases, there are some weird curves which makes the problem hard to characterize and generalize. One good answer is the following Enriques-Petri theorem:

Theorem 253 (Enriques-Petri)**.** *Let $C \subseteq \mathbb{P}^{g-1}$ be a smooth, non-hyperelliptic, canonical curve of genus $g \geq 3$. Then C is projectively normal. Furthermore, C is defined by quadratic equations unless C is trigonal or C is isomorphic to a plane quintic curve.*

A further generalization of this theorem is known as Green's canonical syzygy conjecture. To understand those theorems, we need more information on the ideal of C. An approach to the (extrinsic) geometry of a projective variety $X \subseteq \mathbb{P}^N$ is observing the defining ideal I_X of X. Following Hilbert's philosophy, it can be achieved by taking the generators of I_X (which are homogeneous polynomials), and then observing their relations and higher relations. The celebrating syzygy theorem says that such a process terminates in a finite number of steps; we will have a free resolution which has more fluent information than the original setup.
Let $S = k[x_0, \cdots, x_N]$ be the homogeneous coordinate ring of X. Any finitely generated graded S-module M (for instance, $M = S_X = S/I_X$, the homogeneous coordinate ring of X) has a free S-resolution of finite length. Some resolutions might have different lengths, or might contain some reducible factors which can be cancelled. Hence, it is natural to consider a minimal resolution among those free resolutions.

Definition 254. A complex of graded S-modules $\cdots \to F_i \stackrel{d_i}{\to} F_{i-1} \to \cdots$ is called *minimal* if for each i the image of d_i is contained in $\mathfrak{m}F_{i-1}$, where $\mathfrak{m} = (x_0, \cdots, x_N)$ is the irrelevant maximal ideal. If we consider d_i as a matrix with entries in homogeneous polynomials, then a minimal resolution cannot have a nonzero constant as an entry of d_i.

Proposition 255. *Let $F_\bullet : \cdots \to F_i \stackrel{d_i}{\to} F_{i-1} \to \cdots$ be a graded free resolution. Then $F_\bullet$ is minimal if and only if for each i, the map d_i takes a basis of F_i to a minimal set of generators of the image of d_i.*

Proof. The complex is minimal if and only if the induced map

$$\overline{d_{i+1}} : F_{i+1}/\mathfrak{m}F_{i+1} \to F_i/\mathfrak{m}F_i$$

is zero. In the case, the induced map $F_i/\mathfrak{m}F_i \to (\operatorname{im} d_i)/\mathfrak{m}(\operatorname{im} d_i)$ becomes an isomorphism. By Nakayama's lemma, the image of a basis of F_i form a minimal set of generators of $(\operatorname{im} d_i)$. □

One can prove the following uniqueness theorem of a minimal free resolution.

Theorem 256. *Let M be a finitely generated graded S-module. If $F_\bullet$ and $G_\bullet$ are minimal graded free resolutions of M, then there is a graded isomorphism of complexes $F_\bullet \to G_\bullet$ inducing the identity map on M.*

In particular, the rank of the terms in a minimal free resolution gives an invariant of M.

Definition 257. Let M be a finitely generated graded S-module, and let $F_\bullet$ be its minimal free resolution. The *graded Betti number* $\beta_{i,j}(M)$ is defined as the number of summands $S(-j)$ appearing in the i-th term F_i.

Proposition 258. *Let $M, F_\bullet$ be as above. Any minimal set of homogeneous generators of F_i contains precisely* $\dim_k \operatorname{Tor}_i^S(M,k)_j$ *of degree j, that is,* $\beta_{i,j}(M) = \dim_k \operatorname{Tor}_i^S(M,k)_j$.

Proof. The $k = S/\mathfrak{m}$-vector space $\operatorname{Tor}_i^S(M,k)_j$ is the degree j component of the graded vector space which is the i-th homology of the complex $F_\bullet \otimes_S k$. Since $F_\bullet$ is minimal, all the maps in $F_\bullet \otimes_S k$ are zero. In particular, $\operatorname{Tor}_i^S(M,k) = F_i \otimes_S k$. By Nakayama's lemma, $\dim \operatorname{Tor}_i^S(M,k)_j$ coincides with the number of degree j generators which F_i needed. □

Definition 259. The table consisted of graded Betti numbers

$$(\beta_{i,i+j}(M)) = \begin{pmatrix} \vdots & \vdots & \vdots & \\ \beta_{0,0}(M) & \beta_{1,1}(M) & \beta_{2,2}(M) & \cdots \\ \beta_{0,1}(M) & \beta_{1,2}(M) & \beta_{2,3}(M) & \cdots \\ \beta_{0,2}(M) & \beta_{1,3}(M) & \beta_{2,4}(M) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

is called the Betti table of M.

Example 260. Let $C \subset \mathbb{P}^3$ be the twisted cubic $\{[s^3 : s^2t : st^2 : t^3] \mid [s:t] \in \mathbb{P}^1\}$. The defining ideal of C is generated by three quadric equations $x_1x_3 - x_2^2, -x_0x_3 + x_1x_2, x_0x_2 - x_1^2$. One may compute the minimal free resolution of the homogeneous coordinate ring S_C:

$$0 \to F_2 = S(-3)^2 \xrightarrow{d_2} F_1 = S(-2)^3 \xrightarrow{d_1} F_0 = S \to S_C \to 0$$

where the boundary maps are given by

$$d_1 = \begin{pmatrix} x_1x_3 - x_2^2 & -x_0x_3 + x_1x_2 & x_0x_2 - x_1^2 \end{pmatrix}$$

and

$$d_2 = \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}.$$

Hence, $\beta_{0,0}(S_C) = 1, \beta_{1,2}(S_C) = 3$, and $\beta_{2,3}(S_C) = 2$ and all the others are zero. We have the following Betti table

1	–	–
–	3	2

Still the computation of graded Betti numbers looks weird, since we need very explicit information on the graded module M. Even in the case $M = S_X$, we have no computational idea unless we know the generators of the defining ideal. One geometric idea is to associate the Koszul cohomology groups $K_{p,q}(X, L)$ associated to a very ample line bundle L on X which corresponds to the embedding $X \hookrightarrow \mathbb{P}^N$. The idea was suggested by M. Green in 1984. In so many cases, classical results concerning the generators/relations of the ideal I_X can be rephrased in terms of Koszul cohomology groups.
Since S is the symmetric algebra of the vector space V with basis $x_0, \cdots, x_N$, we have the following sequence of vector spaces:

$$0 \to (\wedge^{N+1} V) \to \cdots \to \wedge^2 V \to \wedge^1 V = V \to k \to 0,$$

where the $(p+1)$-th map is defined to be a natural extension of

$$x_{i_0} \wedge \cdots \wedge x_{i_p} \mapsto \sum_j (-1)^{j+1} x_{i_0} \wedge \cdots \wedge \widehat{x_{i_j}} \wedge \cdots \wedge x_{i_p}.$$

When we have a graded S-module $M = \bigoplus_q M_q$, we may plug in the grading structure as follows. The map

$$\delta : \wedge^p V \otimes M_q \to \wedge^{p-1} V \otimes M_{q+1}$$

defined as

$$\delta(x_{i_0} \wedge \cdots \wedge x_{i_{p-1}} \otimes m) := \sum_j (-1)^{j+1} x_{i_0} \wedge \cdots \wedge \widehat{x_{i_j}} \wedge \cdots \wedge x_{i_{p-1}} \otimes x_{i_j} m$$

induces a complex

$$\cdots \to \wedge^{p+1} V \otimes M_{q-1} \to \wedge^p V \otimes M_q \to \wedge^{p-1} V \otimes M_{q+1}$$

of total degree $p + q$, which is called the *Koszul complex.*