**Example 261.** The most simplest case is when $M = S = k[x_0, \cdots, x_N]$, and collecting all the degrees at once, we have a complex of graded $S$-modules

$$0 \to F_{N+1} = \wedge^{N+1} V \otimes S(-N - 1) \to \cdots \to F_1 = \wedge^1 V \otimes S(-1) \to F_0 = S \to 0$$

which is exact at every $F_i$ except for $F_0$. Indeed, it is a minimal graded $S$-free resolution of the quotient module $\text{coker}[F_1 \to F_0] = S/(x_0, \cdots, x_N) \cong k$.

**Definition 262.** The Koszul cohomology group $K_{p,q}(M,V)$ is the cohomology of the above complex at $\wedge^p V \otimes M_q$.

**Proposition 263.** Let $M$ be a finitely generated graded $S$-module, and let $V = S_1$. Then $\dim K_{p,q}(M,V) = \beta_{p,p+q}(M)$.

**Proof.** Thanks to the symmetry property of Tor functors, the Tor group $\text{Tor}^S_p(M,k)$ can be computed from a free resolution of $k$. Since the Koszul complex

$$0 \to \wedge^{N+1} V \otimes S(-N - 1) \to \cdots \to \wedge^2 V \otimes S(-2) \to \wedge^1 V \otimes S(-1) \to S \to k \to 0$$

which provides a (minimal) free resolution of $k$. Hence, the $p$-th Tor group $\text{Tor}^S_p(M,k)$ can be computed from the (co)homology at the $p$-th step. In particular, $\dim_k \text{Tor}^S_p(M,k)_{p,q} = \dim K_{p,q}(M,V)$ since the Koszul complex for $M$ is obtained by taking a tensor product $(- \otimes_S M)$ to the Koszul complex for $k$.

The Koszul cohomology groups can be easily defined in the geometric context as follows.

**Definition 264.** Let $X$ be a projective variety, and let $L$ be a globally generated line bundle on $X$. Let $V = H^0(X,L)$.

The Koszul cohomology group $K_{p,q}(X,L)$ is the Koszul cohomology of the graded $\text{Sym}(V)$-module

$$R(L) = \bigoplus_q H^0(X,L^q).$$

Precisely, it is the cohomology at the middle of the complex

$$\wedge^{p+1} V \otimes H^0(X,L^{p-1}) \to \wedge^p V \otimes H^0(X,L^q) \to \wedge^{p-1} V \otimes H^0(X,L^{q+1}).$$

Let $\mathcal{F}$ be a coherent sheaf on $X$. We define $R(\mathcal{F},L) := \bigoplus_q H^0(X,\mathcal{F} \otimes L^q)$, and the Koszul cohomology group $K_{p,q}(X,\mathcal{F},L)$ be the cohomology of $R(\mathcal{F},L)$ at $\wedge^p V \otimes H^0(X,\mathcal{F} \otimes L^q)$.

When $L = \mathcal{O}_X(1)$ induces an embedding $X \hookrightarrow \mathbb{P}^N$ so that $X$ is projectively normal, then two rings $S(X)$ and $R(L) = \bigoplus_q H^0(X,L^q)$ coincides; in particular, $K_{p,q}(S(X),H^0(L)) = K_{p,q}(X,L)$ as desired. In general, there is an inclusion of rings $S(X) \hookrightarrow R(L)$, whose difference is measured by the Hartshorne-Rao module $\bigoplus_q H^1(X,\mathcal{I}_X(q))$. In particular, $X$ is $q$-normal if and only if the natural map $\text{Sym}^q H^0(L) \to H^0(L^q)$ is surjective.
Lemma 265. Let $X$ be a projective variety, and let $L$ be a very ample line bundle on $X$. Consider the embedding $X \hookrightarrow \mathbb{P}^N$ given by the complete linear series $|L|$. Then $X$ is projective normal if and only if

$$K_{0,q}(X, L) = 0$$

for every $q \geq 1$.

Proof. The projective normality is equivalent to the surjectivity of the multiplication map $H^0(L) \otimes H^0(L^{q-1}) \to H^0(L^q)$ for every $q \geq 1$. By definition, the cokernel of this multiplication map coincides with the Koszul cohomology group $K_{0,q}(X, L)$.

There are also technical merits to compute the syzygies by Koszul cohomology groups. We will observe just a few basic tools.

Definition 266. Let $C$ be a curve, and $L$ be a globally generated line bundle. The kernel $\mathcal{M}_L$ of the evaluation map $ev : H^0(C, L) \otimes \mathcal{O}_C \to L$ is called the kernel bundle. Note that the short exact sequence

$$0 \to \mathcal{M}_L \to H^0(C, L) \otimes \mathcal{O}_C \xrightarrow{ev} L \to 0$$

is the pullback of the Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^N}(1) \to H^0(C, L) \otimes \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_{\mathbb{P}^N}(1) \to 0$$

on the projective space $\mathbb{P}^N = \mathbb{P}H^0(C, L)^\vee$ via the morphism defined by $L$.

Example 267 (Castelnuovo base-point-free pencil trick). Let $C$ be a curve, and let $L$ be a globally generated line bundle such that the linear series $P^V \subseteq |L|$ is a base-point-free $g^1_d$ on $C$, where $d = \deg L$. Then $L$ is globally generated by its two independent sections in $V \subseteq H^0(C, L)$. In particular, we have the following ladder of short exact sequences

$$
\begin{array}{cccccccc}
0 & \to & L' & \to & V \otimes \mathcal{O}_C & \to & L & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{M}_L & \to & H^0(C, L) \otimes \mathcal{O}_C & \to & L & \to & 0. \\
\end{array}
$$

The above diagram is very useful when we check the normality of $C$ embedded by $L$, together with a certain twist by a power of $L$.

Proposition 268. Let $C, L$ be as above, and let $V = H^0(C, L)$. We have

$$K_{p,q}(C, L) \simeq \ker[\wedge^{p+1} V \otimes H^0(L^{q-1}) \to H^0(\wedge^p \mathcal{M}_L \otimes L^q)]$$

$$\simeq \ker[H^1(\wedge^{p+1} \mathcal{M}_L \otimes L^{q-1}) \to \wedge^{p+1} V \otimes H^1(L^{q-1})].$$

Similarly,

$$K_{p,q}(C, F, L) \simeq \ker[\wedge^{p+1} V \otimes H^0(F \otimes L^{q-1}) \to H^0(\wedge^p \mathcal{M}_L \otimes F \otimes L^q)]$$

$$\simeq \ker[H^1(\wedge^{p+1} \mathcal{M}_L \otimes F \otimes L^{q-1}) \to \wedge^{p+1} V \otimes H^1(F \otimes L^{q-1})].$$
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Proof. Consider the sequence of vector bundles
\[ \cdots \to \wedge^{p+1} V \otimes L^{q-1} \to \wedge^p V \otimes L^q \to \wedge^{p-1} V \otimes L^{q+1} \to \cdots. \]

Note that the Koszul cohomology group $K_{p,q}(C,L)$ is the cohomology (at the middle) of the sequence obtained by taking global sections of the above sequence. On the other hand, by definition, we have a short exact sequence
\[ 0 \to \mathcal{M}_L \to V \otimes \mathcal{O}_C \to L \to 0. \]
Taking the $p$-th exterior powers, we have
\[ 0 \to \wedge^p \mathcal{M}_L \to \wedge^p V \otimes \mathcal{O}_C \to \wedge^{p-1} \mathcal{M}_L \otimes L \to 0. \]
Twisting by $L^q$, we have a short exact sequence
\[ 0 \to \wedge^p \mathcal{M}_L \otimes L^q \to \wedge^p V \otimes L^q \to \wedge^{p-1} \mathcal{M}_L \otimes L^{q+1} \to 0. \]
Hence, the first sequence factors through those vector bundles appearing in the left and right:

In particular, the kernel and the image of the Koszul differential
\[ \delta : \wedge^p V \otimes H^0(L^q) \to \wedge^{p-1} V \otimes H^0(L^{q+1}) \]
are given by
\[ \ker \delta \simeq H^0(\wedge^p \mathcal{M}_L \otimes L^q), \quad \text{im} \delta \simeq H^0(\wedge^{p-1} \mathcal{M}_L \otimes L^{q+1}) \]
which gives the formula.

Proposition 269 (Green duality for curves). Let $C, L$ be as above, and let $r = \dim |L| = h^0(L) - 1$. Then $K_{p,q}(C,L) \simeq K_{r-p-1,2-q}(C,K_C,L)^\vee$. 

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Proof. Note that

$$K_{p,q}(C, L) \simeq \text{coker } [\wedge^{p+1} V \otimes H^0(L^{q-1}) \to H^0(\wedge^p \mathcal{M}_L \otimes L^q)]$$

which is dual to

$$\ker [H^1(K_C \otimes \wedge^p \mathcal{M}_L^\vee \otimes L^{-q}) \to \wedge^{p+1} V^\vee \otimes H^1(K_C \otimes L^{1-q})]$$

by the above formula and Serre duality. Since the rank of $\mathcal{M}_L$ is $r$ and $\det \mathcal{M}_L \simeq L^{-1}$, we have the following perfect pairing

$$\wedge^p \mathcal{M}_L \otimes \wedge^{r-p} \mathcal{M}_L \to \wedge^r \mathcal{M}_L \simeq L^{-1}.$$  

In particular, $\wedge^{r-p} \mathcal{M}_L \simeq \wedge^p \mathcal{M}_L^\vee \otimes L^{-1}$. Also note that $\wedge^{p+1} V^\vee \simeq \wedge^{(r+1)-(p+1)} V$ by the same reason. Hence,

$$K_{p,q}(C, L)^\vee \simeq \ker [H^1(\wedge^{r-p} \mathcal{M}_L \otimes K_C \otimes L^{1-q}) \to \wedge^{r-p} V \otimes H^1(K_C \otimes L^{1-q})]$$

which is an expression of the Koszul cohomology group $K_{r-p-1,2-q}(C, K_C, L)$. □

**Corollary 270.** Let $C$ be a nonhyperelliptic curve. Then $K_{p,q}(C, K_C) \simeq K_{g-p-2,3-q}(C, K_C)^\vee$. In particular, the Betti table of a canonical curve is symmetric.

**Proof.** Since $K_C$ is very ample, and hence $K_{p,q}(C, K_C) \simeq K_{g-p-2,3-q}(C, K_C, K_C)^\vee = K_{g-p-2,3-1}(C, K_C)^\vee$. □

**Example 271.** Let $C$ be a nonhyperelliptic curve of genus $g = 3$. Then $|K_C|$ embeds $C$ as a degree 2 curve in $\mathbb{P}^{g-1} = \mathbb{P}^2$. Hence, the Betti table of the canonical curve will be

$$
\begin{array}{ccc}
1 & - & - \\
- & - & - \\
- & 1 & - \\
- & - & - \\
- & - & 1 \\
\end{array}
$$

Let $C$ be a nonhyperelliptic curve of genus $g = 4$. Then $|K_C|$ embeds $C$ as a degree 6 curve in $\mathbb{P}^3$, and the image is projectively normal. We already observed that $C$ is a complete intersection of these quadric and cubic hypersurfaces, and hence, the Betti table of the canonical curve will be

$$
\begin{array}{ccc}
1 & - & - \\
- & 1 & - \\
- & 1 & - \\
- & - & - \\
\end{array}
$$

When $C$ is a nonhyperelliptic curve of genus $g = 5$, there are two possibilities for the Betti table of the canonical curve: either

$$
\begin{array}{ccc}
1 & - & - \\
- & 3 & - \\
- & - & 3 \\
- & - & - \\
\end{array}
$$

or

$$
\begin{array}{ccc}
1 & - & - \\
- & - & 1 \\
- & - & - \\
\end{array}
$$

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, or

\[
\begin{array}{cccc}
1 & - & - & - \\
- & 3 & 2 & - \\
- & 2 & 3 & - \\
- & - & - & 1 \\
\end{array}
\]

which depends on the existence of \( g_3^1 \) on \( C \).

Definition 272. A very ample line bundle \( L \) on a smooth projective variety \( X \) satisfies the property \((N_p)\) if \(|L|\) embeds \( X \) into a projective space projective normally, and certain Koszul cohomology groups vanish:

\[ K_{i,j}(X, L) = 0 \text{ for all } i \leq p, j \geq 2. \]

Example 273. In the above cases of canonical curves, all of them satisfies the property \((N_0)\). When \( C \) is the canonical curve of a nonhyperelliptic curve of genus 5 which is not trigonal, it also satisfies the property \((N_1)\). In other words,

\[
\begin{array}{|c|}
\hline
(N_0) & \text{the embedded variety is projectively normal;} \\
(N_1) & \text{the variety is projectively normal and defined by quadric equations;} \\
(N_2) & \text{the property \((N_1)\) and the syzygies are generated by linear relations;} \\
(N_3) & \text{\((N_2)\) and the further relations among the syzygies are generated by linear relations;} \\
\vdots & \vdots \\
(N_p) & \text{\((N_1)\) and the minimal free resolution is linear between the 1st and the \( p \)-th term.} \\
\hline
\end{array}
\]

5 Curves of high degree

Let \( C \) be a curve of genus \( g \). We learned that any line bundle of degree at least \( 2g + 1 \) is very ample. What happens if we take a line bundle of degree higher than \( 2g + 1 \)? The image of such a curve embedded by the complete linear series is called a curve of high degree.

First of all, to make sure that the syzygy of a curve of high degree can be read off from the Koszul cohomology groups \( K_{p,q}(C, L) \) where \( L = \mathcal{O}_C(1) \), we need the projective normality.

Proposition 274. Let \( C \) be a curve of high degree in \( \mathbb{P}^r \), where \( r = \deg(C) - g \). Then it is projectively normal, in other words, the natural inclusion

\[ S_C \hookrightarrow \bigoplus_n H^0(C, \mathcal{O}_C(n)) \]

is an isomorphism.

The statement claims that \( C \subset \mathbb{P}^r \) is projectively normal. We already saw that the projective normality is equivalent to the vanishing of Koszul cohomology groups

\[ K_{0,q}(C, L) = 0 \]
for every $q \geq 1$. When $q = 1$, it is trivial, which is obtained from taking the global sections of the evaluation map $H^0(L) \otimes \mathcal{O}_C \to L$. For $q \geq 2$, the following theorem by Green will imply the above proposition. We need a small lemma first:

**Lemma 275.** Let $C$ be a curve of genus $g$. Then,

1. a general line bundle $L$ on $C$ of degree $\geq g - 1$ is nonspecial, that is, $h^1(L) = 0$;

2. a general line bundle $L$ on $C$ of degree $\geq g + 1$ is base-point-free.

**Proof.** Let $L$ be a line bundle on $C$ of degree $g - 1$. By Riemann-Roch, the assumption $h^1(L) = 0$ is equivalent to $h^0(L) = 0$. Suppose not, namely, $H^0(L) \neq 0$. Then there is an effective divisor contained in $|L|$, and hence, $L$ is in the image of the following map

$$
C^{g-1} \to \text{Pic}^{g-1}(C);
$$

$$(P_1, \cdots, P_{g-1}) \mapsto \mathcal{O}_C(P_1 + \cdots + P_{g-1}).$$

Since the image can have at most dimension $(g - 1)$, it cannot fill the $g$-dimensional space $\text{Pic}^{g-1}(C)$. We conclude that $h^0(L) = h^1(L) = 0$ for a general $L \in \text{Pic}^{g-1}(C)$. When $L$ is a general line bundle of degree $d \geq g - 1$, choose $(d - g + 1)$ general points $P_1, \cdots, P_{d-g+1} \in C$. Then the line bundle $L(-P_1\cdots-P_{d-g+1})$ is a general line bundle of degree $g - 1$, which is nonspecial; in particular, $L$ itself must be nonspecial.

Now let $L$ be a line bundle of degree $d \geq g + 1$. Consider the incidence set of nonspecial line bundles of degree $d$ with base points:

$$
\mathcal{U} = \{(L, P) \mid h^0(L) = h^0(L - P) \text{ and } h^1(L) = 0\}
$$

It is enough to show that $\dim \mathcal{U} < g = \dim \text{Pic}^d(C)$. Let $\pi : \mathcal{U} \to \text{Pic}^{d-1}(C)$ be the morphism sending $(L, P)$ to $L(-P)$. Note that a fiber $\pi^{-1}(L')$ of $\pi$ over a line bundle $L' \in \text{Pic}^{d-1}(C)$ is contained in the set

$$
\{(L'(P), P) \mid P \in C\} \simeq C
$$

which is parametrized by $C$. In particular, $\dim \pi^{-1}(L')$ is at most 1 for any $L' \in \text{Pic}^{d-1}(C)$. Also note that if $(L, P) \in \mathcal{U}$, then $h^0(L(-P)) = h^0(L) = 1 - g + d$ and $h^1(L(-P)) = 1$ by Riemann-Roch. In particular, $L(-P)$ is a special line bundle of degree $d - 1$. On the other hand, via the Serre dual map

$$
\text{Pic}^{d-1}(C) \to \text{Pic}^{2g-2-(d-1)}(C);
$$

$L'$ isomorphic to $\omega_C \otimes (L')^\vee$, the locus of special line bundles $\{L' \in \text{Pic}^{d-1}(C) \mid h^1(L') > 0\}$ is isomorphic to the locus $\{L'' \in \text{Pic}^{2g-2-(d-1)}(C) \mid h^0(L'') > 0\}$, which is the image of $C^{2g-2-(d-1)} \to 109$
Pic^{2g-2-(d-1)} as above. Hence, the locus has dimension at most \((g-2) + 2\) \((d-1) \leq g-2\).

We conclude that
\[
\dim U \leq (g-2) + \dim \pi^{-1}(L') \leq g-1
\]
which is strictly smaller than \(\dim \text{Pic}^d(C)\).

**Theorem 276** (Green). Let \(C\) be a smooth curve of genus \(g\), and let \(L\) be a very ample line bundle of degree \(d\). Then:

1. \(K_{p,q}(C, L) = 0\) for \(q \geq 3\) if \(H^1(L) = 0\).
2. \(K_{p,2}(C, L) = 0\) if \(d \geq 2g+1+p\), that is, a high degree curve of degree \(d \geq 2g+1+p\), \(p \geq 0\) satisfies the property \((N_p)\).

**Proof.** By Green’s duality, we have \(K_{p,q}(C, L)^! \simeq K_{r-1-p,2-q}(C, K_C, L)\) where \(r = h^0(C, L) - 1\). If \(h^1(L) = 0\), we have \(h^0(K_C - L) = 0\) by Serre duality, and in particular, \(h^0(K_C + (2-q)L) = 0\) for any \(q \geq 3\). The term \(\wedge^{r-1-p} V \otimes H^0(K_C + (2-q)L) = 0\), and hence the cohomology group \(K_{r-1-p,2-q}(C, K_C, L)\) also vanishes.

Now consider the case \(q = 2\) and \(d = \deg L \geq 2g+1+p\) with \(p \geq 0\). Since \(H^1(L) = 0\), we have \(K_{p,2}(C, L) \simeq H^1(C, \wedge^{p+1} M_L \otimes L)\). By Serre duality, the vanishing of \(K_{p,2}(C, L)\) is equivalent to show that
\[
H^0(C, \wedge^{p+1} M_L^! \otimes \omega_C \otimes L^!) = 0.
\]

Let \(P \in C\) be a (general) point. Since \(L - P\) is still base-point-free, one has the following ladder with exact rows and columns.

\[
\begin{array}{ccccccccc}
\node{0} & \node{0} & \node{0} \\
\node{0} & \node{\mathcal{M}_L(-P)} & \node{H^0(L(-P)) \otimes \mathcal{O}_C} & \node{L(-P)} & \node{0} \\
\node{0} & \node{\mathcal{M}_L} & \node{H^0(L) \otimes \mathcal{O}_C} & \node{L} & \node{0} \\
\node{0} & \node{\mathcal{J}_P = \mathcal{O}_C(-P)} & \node{\mathcal{O}_C} & \node{\mathcal{O}_P} & \node{0} \\
\node{0} & \node{0} & \node{0} & \node{0} & \node{0} \\
\end{array}
\]

This process can be repeated: choose general points \(P_1, \ldots, P_{r-1} \in C\), where \(r = h^0(L) - 1 = d - g\). By induction on \(r\), one has the following short exact sequence
\[
0 \to \mathcal{M}_L(-P_1, \ldots, -P_{r-1}) \to \mathcal{M}_L \to \bigoplus_{i=1}^{r-1} \mathcal{O}_C(-P_i) \to 0.
\]
Note that the line bundle \( L(-P_1 - \cdots - P_{r-1}) \) is a base-point-free pencil, since \( h^0(L(-P_1 - \cdots - P_{r-1})) = h^0(L) - (r - 1) = 2 \). Therefore, Castelnuovo pencil trick gives an isomorphism \( M_L(-P_1 - \cdots - P_{r-1}) \cong L^r(P_1 + \cdots + P_{r-1}) \). Taking the dual of the above short exact sequence, we have

\[
0 \to \bigoplus_{i=1}^{r-1} \mathcal{O}_C(P_i) \to \mathcal{M}_L^r \to L(-P_1 - \cdots - P_{r-1}) \to 0.
\]

Taking its \((p+1)\)-th exterior power, we have

\[
0 \to \wedge^{p+1} \left( \bigoplus_{i=1}^{r-1} \mathcal{O}_C(P_i) \right) \to \wedge^{p+1} \mathcal{M}_L^r \to \wedge^p \left( \bigoplus_{i=1}^{r-1} \mathcal{O}_C(P_i) \right) \otimes L(-P_1 - \cdots - P_{r-1}) \to 0.
\]

Twisting by \( \omega_C \otimes L^r \), we see that the terms on the left is a direct sum of line bundles of degree \((p+1) + (2g - 2) - d = 2g + p - 1 - d < 0\), hence have no global sections. The terms on the right is a direct sum of line bundles of the form \( \omega_C(-P_1 - \cdots - P_{r-p-1}) \), subtracting \((r-p-1)\) points among general points \( P_1, \cdots, P_{r-1} \in C \). It has no global section since the rank of the zeroth cohomology group drops by 1 when we subtract a general point, and \( r - p - 1 = d - g - p - 1 \geq g \) in our case. In particular, \( K_{p,2}(C,L) \cong H^0(C, \wedge^{p+1} \mathcal{M}_L^r \otimes \omega_C \otimes L^r)^r = 0 \).

Hence, the Betti table of a curve of high degree \( C \subset \mathbb{P}^r \) of degree \( d = 2g + 1 + p \) has the following shape:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>\cdots</th>
<th>p-1</th>
<th>p</th>
<th>p+1</th>
<th>\cdots</th>
<th>r-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>\kappa_{0,0} = 1</td>
<td>-</td>
<td>-</td>
<td>\cdots</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>\cdots</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>-</td>
<td>\kappa_{1,1}</td>
<td>\kappa_{2,1}</td>
<td>\cdots</td>
<td>\kappa_{p-1,1}</td>
<td>\kappa_{p,1}</td>
<td>\kappa_{p+1,1}</td>
<td>\cdots</td>
<td>\kappa_{r-1,1}</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>\cdots</td>
<td>-</td>
<td>\kappa_{p+1,2}</td>
<td>\cdots</td>
<td>\kappa_{r-1,2}</td>
<td></td>
</tr>
</tbody>
</table>

where \( \kappa_{p,q} = \dim K_{p,q}(C, \mathcal{O}_C(1)) = \beta_{p+p+q}(C) \). The horizontal strip \((\kappa_{1,1} \ \kappa_{2,1} \ \cdots \ \kappa_{r,1})\) is called the quadratic strand, since it denotes the quadric generators and their linear relations. The next strip \((\kappa_{1,2} \ \kappa_{2,2} \ \cdots \ \kappa_{r,2})\) is called the cubic strand. It is composed of the cubic generators and their linear relations, and quadratic relations among the terms lie on the quadratic strand. Green’s \((2g + p + 1)\)-theorem says that the first \(p\)-terms in the quadratic strand of a curve of degree \( \geq 2g + p + 1 \) becomes zero.