Example 261. The most simplest case is when $M = S = k[x_0, \cdots, x_N]$, and collecting all the degrees at once, we have a complex of graded S-modules

$$0 \to F_{N+1} = \wedge^{N+1} V \otimes S(-N-1) \to \cdots \to F_1 = \wedge^1 V \otimes S(-1) \to F_0 = S \to 0$$

which is exact at every F_i except for F_0. Indeed, it is a minimal graded S-free resolution of the quotient module $\operatorname{coker}[F_1 \to F_0] = S/(x_0, \cdots, x_N) \simeq k$.

Definition 262. The Koszul cohomology group $K_{p,q}(M, V)$ is the cohomology of the above complex at $\wedge^p V \otimes M_q$.

Proposition 263. *Let M be a finitely generated graded S-module, and let $V = S_1$. Then* $\dim K_{p,q}(M, V) = \beta_{p,p+q}(M)$.

Proof. Thanks to the symmetry property of Tor functors, the Tor group $\operatorname{Tor}_p^S(M, k)$ can be computed from a free resolution of k. Since the Koszul complex

$$0 \to \wedge^{N+1} V \otimes S(-N-1) \to \cdots \to \wedge^2 V \otimes S(-2) \to \wedge^1 V \otimes S(-1) \to S \to k \to 0$$

which provides a (minimal) free resolution of k. Hence, the p-th Tor group $\operatorname{Tor}_p^S(M, k)$ can be computed from the (co)homology at the p-th step. In particular, $\dim_k \operatorname{Tor}_p^S(M, k)_{p,q} = \dim K_{p,q}(M, V)$ since the Koszul complex for M is obtained by taking a tensor product $(- \otimes_S M)$ to the Koszul complex for k.

□

The Koszul cohomology groups can be easily defined in the geometric context as follows.

Definition 264. Let X be a projective variety, and let L be a globally generated line bundle on X. Let $V = H^0(X, L)$.
The *Koszul cohomology group* $K_{p,q}(X, L)$ is the Koszul cohomology of the graded $\operatorname{Sym}(V)$-module

$$R(L) = \bigoplus_q H^0(X, L^q).$$

Precisely, it is the cohomology at the middle of the complex

$$\wedge^{p+1} V \otimes H^0(X, L^{q-1}) \to \wedge^p V \otimes H^0(X, L^q) \to \wedge^{p-1} V \otimes H^0(X, L^{q+1}).$$

Let $\mathcal{F}$ be a coherent sheaf on X. We define $R(\mathcal{F}, L) := \bigoplus_q H^0(X, \mathcal{F} \otimes L^q)$, and the Koszul cohomology group $K_{p,q}(X, \mathcal{F}, L)$ be the cohomology of $R(\mathcal{F}, L)$ at $\wedge^p V \otimes H^0(X, \mathcal{F} \otimes L^q)$.

When $L = \mathcal{O}_X(1)$ induces an embedding $X \hookrightarrow \mathbb{P}^N$ so that X is projectively normal, then two rings $S(X)$ and $R(L) = \bigoplus_q H^0(X, L^q)$ coincides; in particular, $K_{p,q}(S(X), H^0(L)) = K_{p,q}(X, L)$ as desired. In general, there is an inclusion of rings $S(X) \hookrightarrow R(L)$, whose difference is measured by the Hartshorne-Rao module $\bigoplus_q H^1(X, \mathscr{I}_X(q))$. In particular, X is q-normal if and only if the natural map $\operatorname{Sym}^q H^0(L) \to H^0(L^q)$ is surjective.

Lemma 265. *Let X be a projective variety, and let L be a very ample line bundle on X. Consider the embedding $X \hookrightarrow \mathbb{P}^N$ given by the complete linear series $|L|$. Then X is projective normal if and only if*

$$K_{0,q}(X, L) = 0$$

for every $q \geq 1$.

Proof. The projective normality is equivalent to the surjectivity of the multiplication map $H^0(L) \otimes H^0(L^{q-1}) \to H^0(L^q)$ for every $q \geq 1$. By definition, the cokernel of this multiplication map coincides with the Koszul cohomology group $K_{0,q}(X, L)$. □

There are also technical merits to compute the syzygies by Koszul cohomology groups. We will observe just a few basic tools.

Definition 266. Let C be a curve, and L be a globally generated line bundle. The kernel $\mathcal{M}_L$ of the evaluation map $ev : H^0(C, L) \otimes \mathcal{O}_C \to L$ is called the *kernel bundle*. Note that the short exact sequence

$$0 \to \mathcal{M}_L \to H^0(C, L) \otimes \mathcal{O}_C \stackrel{ev}{\to} L \to 0$$

is the pullback of the Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^N}(1) \to H^0(C, L) \otimes \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_{\mathbb{P}^N}(1) \to 0$$

on the projective space $\mathbb{P}^N = \mathbb{P}H^0(C, L)^\vee$ via the morphism defined by L.

Example 267 (Castelnuovo base-point-free pencil trick)**.** Let C be a curve, and let L be a globally generated line bundle such that the linear series $\mathbb{P}V \subseteq |L|$ is a base-point-free $\mathfrak{g}^1_d$ on C, where $d = \deg L$. Then L is globally generated by its two independent sections in $V \subseteq H^0(C, L)$. In particular, we have the following ladder of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^\vee & \longrightarrow & V \otimes \mathcal{O}_C & \longrightarrow & L & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{M}_L & \longrightarrow & H^0(C, L) \otimes \mathcal{O}_C & \longrightarrow & L & \longrightarrow & 0.
\end{array}$$

The above diagram is very useful when we check the normality of C embedded by L, together with a certain twist by a power of L.

Proposition 268. *Let C, L be as above, and let $V = H^0(C, L)$. We have*

$$\begin{aligned}
K_{p,q}(C, L) &\simeq \operatorname{coker}[\wedge^{p+1} V \otimes H^0(L^{q-1}) \to H^0(\wedge^p \mathcal{M}_L \otimes L^q)] \\
&\simeq \ker[H^1(\wedge^{p+1} \mathcal{M}_L \otimes L^{q-1}) \to \wedge^{p+1} V \otimes H^1(L^{q-1})].
\end{aligned}$$

Similarly,

$$\begin{aligned}
K_{p,q}(C, \mathcal{F}, L) &\simeq \operatorname{coker}[\wedge^{p+1} V \otimes H^0(\mathcal{F} \otimes L^{q-1}) \to H^0(\wedge^p \mathcal{M}_L \otimes \mathcal{F} \otimes L^q)] \\
&\simeq \ker[H^1(\wedge^{p+1} \mathcal{M}_L \otimes \mathcal{F} \otimes L^{q-1}) \to \wedge^{p+1} V \otimes H^1(\mathcal{F} \otimes L^{q-1})].
\end{aligned}$$

Proof. Consider the sequence of vector bundles

$$\cdots \to \wedge^{p+1} V \otimes L^{q-1} \to \wedge^p V \otimes L^q \to \wedge^{p-1} V \otimes L^{q+1} \to \cdots .$$

Note that the Koszul cohomology group $K_{p,q}(C, L)$ is the cohomology (at the middle) of the sequence obtained by taking global sections of the above sequence.
On the other hand, by definition, we have a short exact sequence

$$0 \to \mathcal{M}_L \to V \otimes \mathcal{O}_C \to L \to 0.$$

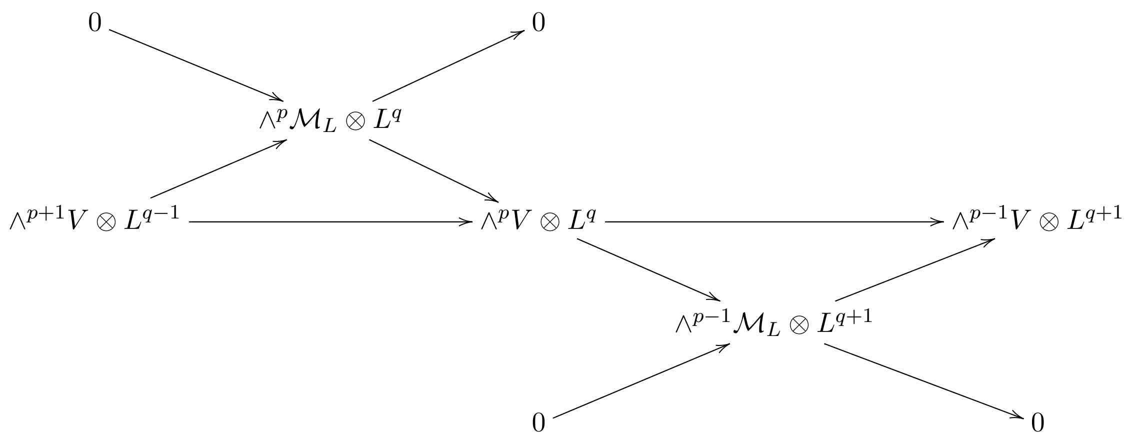
Taking the p-th exterior powers, we have

$$0 \to \wedge^p \mathcal{M}_L \to \wedge^p V \otimes \mathcal{O}_C \to \wedge^{p-1} \mathcal{M}_L \otimes L \to 0.$$

Twisting by L^q, we have a short exact sequence

$$0 \to \wedge^p \mathcal{M}_L \otimes L^q \to \wedge^p V \otimes L^q \to \wedge^{p-1} \mathcal{M}_L \otimes L^{q+1} \to 0.$$

Hence, the first sequence factors through those vector bundles appearing in the left and right:



In particular, the kernel and the image of the Koszul differential

$$\delta : \wedge^p V \otimes H^0(L^q) \to \wedge^{p-1} V \otimes H^0(L^{q+1})$$

are given by

$$\begin{aligned} \ker \delta &\simeq H^0(\wedge^p \mathcal{M}_L \otimes L^q), \\ \operatorname{im} \delta &\simeq H^0(\wedge^{p-1} \mathcal{M}_L \otimes L^{q+1}) \end{aligned}$$

which gives the formula. □

Proposition 269 (Green duality for curves)**.** *Let C, L be as above, and let $r = \dim |L| = h^0(L) - 1$. Then $K_{p,q}(C, L) \simeq K_{r-p-1,2-q}(C, K_C, L)^\vee$.*

Proof. Note that

$$K_{p,q}(C,L) \simeq \operatorname{coker}\left[\wedge^{p+1}V \otimes H^0(L^{q-1}) \to H^0(\wedge^p \mathcal{M}_L \otimes L^q)\right]$$

which is dual to

$$\ker\left[H^1(K_C \otimes \wedge^p \mathcal{M}_L^\vee \otimes L^{-q}) \to \wedge^{p+1}V^\vee \otimes H^1(K_C \otimes L^{1-q})\right]$$

by the above formula and Serre duality. Since the rank of $\mathcal{M}_L$ is r and $\det \mathcal{M}_L \simeq L^{-1}$, we have the following perfect pairing

$$\wedge^p \mathcal{M}_L \otimes \wedge^{r-p}\mathcal{M}_L \to \wedge^r \mathcal{M}_L \simeq L^{-1}.$$

In particular, $\wedge^{r-p}\mathcal{M}_L \simeq \wedge^p \mathcal{M}_L^\vee \otimes L^{-1}$. Also note that $\wedge^{p+1}V^\vee \simeq \wedge^{(r+1)-(p+1)}V$ by the same reason. Hence,

$$K_{p,q}(C,L)^\vee \simeq \ker\left[H^1(\wedge^{r-p}\mathcal{M}_L \otimes K_C \otimes L^{1-q}) \to \wedge^{r-p}V \otimes H^1(K_C \otimes L^{1-q})\right]$$

which is an expression of the Koszul cohomology group $K_{r-p-1,2-q}(C, K_C, L)$. □

Corollary 270. *Let C be a nonhyperelliptic curve. Then $K_{p,q}(C,K_C) \simeq K_{g-p-2,3-q}(C,K_C)^\vee$. In particular, the Betti table of a canonical curve is symmetric.*

Proof. Since K_C is very ample, and hence $K_{p,q}(C,K_C) \simeq K_{g-p-2,2-q}(C,K_C,K_C)^\vee = K_{g-p-2,3-1}(C,K_C)^\vee$. □

Example 271. Let C be a nonhyperelliptic curve of genus $g = 3$. Then $|K_C|$ embeds C as a degree $2g-2 = 4$ curve in $\mathbb{P}^{g-1} = \mathbb{P}^2$. Hence, the Betti table of the canonical curve will be

1	–
–	–
–	–
–	1

Let C be a nonhyperelliptic curve of genus $g = 4$. Then $|K_C|$ embeds C as a degree 6 curve in $\mathbb{P}^3$, and the image is projectively normal. We already observed that C is a complete intersection of these quadric and cubic hypersurfaces, and hence, the Betti table of the canonical curve will be

1	–	–
–	1	–
–	1	–
–	–	1

When C is a nonhyperelliptic curve of genus $g = 5$, there are two possibilities for the Betti table of the canonical curve: either

1	–	–	–
–	3	–	–
–	–	3	–
–	–	–	1

, or

1	–	–	–
–	3	2	–
–	2	3	–
–	–	–	1

which depends on the existence of $\mathfrak{g}^1_3$ on C.

Definition 272. A very ample line bundle L on a smooth projective variety X satisfies the *property* (N_p) if $|L|$ embeds X into a projective space projective normally, and certain Koszul cohomology groups vanish:

$$K_{i,j}(X,L) = 0 \text{ for all } i \leq p, j \geq 2.$$

Example 273. In the above cases of canonical curves, all of them satisfies the property (N_0). When C is the canonical curve of a nonhyperelliptic curve of genus 5 which is not trigonal, it also satisfies the property (N_1). In other words,

(N_0)	the embedded variety is projectively normal;
(N_1)	the variety is projectively normal and defined by quadric equations;
(N_2)	the property (N_1) and the syzygies are generated by linear relations;
(N_3)	(N_2) and the further relations among the syzygies are generated by linear relations;
⋮	⋮
(N_p)	(N_1) and the minimal free resolution is linear between the 1st and the p-th term.

5 Curves of high degree

Let C be a curve of genus g. We learned that any line bundle of degree at least $2g+1$ is very ample. What happens if we take a line bundle of degree higher than $2g+1$? The image of such a curve embedded by the complete linear series is called a *curve of high degree.*
First of all, to make sure that the syzygy of a curve of high degree can be read off from the Koszul cohomology groups $K_{p,q}(C,L)$ where $L = \mathcal{O}_C(1)$, we need the projective normality.

Proposition 274. *Let C be a curve of high degree in $\mathbb{P}^r$, where $r = \deg(C) - g$. Then it is projectively normal, in other words, the natural inclusion*

$$S_C \hookrightarrow \bigoplus_n H^0(C, \mathcal{O}_C(n))$$

is an isomorphism.

The statement claims that $C \subset \mathbb{P}^r$ is projectively normal. We already saw that the projective normality is equivalent to the vanishing of Koszul cohomology groups

$$K_{0,q}(C,L) = 0$$

for every $q \geq 1$. When $q = 1$, it is trivial, which is obtained from taking the global sections of the evaluation map $H^0(L) \otimes \mathcal{O}_C \to L$. For $q \geq 2$, the following theorem by Green will imply the above proposition. We need a small lemma first:

Lemma 275. *Let C be a curve of genus g. Then,*

(1) a general line bundle L on C of degree $\geq g-1$ is nonspecial, that is, $h^1(L) = 0$;

(2) a general line bundle L on C of degree $\geq g+1$ is base-point-free.

Proof. Let L be a line bundle on C of degree $g-1$. By Riemann-Roch, the assumption $h^1(L) = 0$ is equivalent to $h^0(L) = 0$. Suppose not, namely, $H^0(L) \neq 0$. Then there is an effective divisor contained in $|L|$, and hence, L is in the image of the following map

$$\begin{array}{rcl} C^{g-1} & \to & \mathrm{Pic}^{g-1}(C) \\ (P_1, \cdots, P_{g-1}) & \mapsto & \mathcal{O}_C(P_1 + \cdots + P_{g-1}). \end{array}$$

Since the image can have at most dimension $(g-1)$, it cannot fill the g-dimensional space $\mathrm{Pic}^{g-1}(C)$. We conclude that $h^0(L) = h^1(L) = 0$ for a general $L \in \mathrm{Pic}^{g-1}(C)$. When L is a general line bundle of degree $d \geq g-1$, choose $(d-g-1)$ general points $P_1, \cdots, P_{d-g-1} \in C$. Then the line bundle $L(-P_1 \cdots - P_{d-g-1})$ is a general line bundle of degree $g-1$, which is nonspecial; in particular, L itself must be nonspecial.
Now let L be a line bundle of degree $d \geq g+1$. Consider the incidence set of nonspecial line bundles of degree d with base points:

$$\mathcal{U} = \{(L, P) \mid h^0(L) = h^0(L-P) \text{ and } h^1(L) = 0\}$$

$$\mathrm{Pic}^d(C) \xleftarrow{\ pr_1\ } \mathcal{U} \xrightarrow{\ pr_2\ } C$$

It is enough to show that $\dim \mathcal{U} < g = \dim \ \mathrm{Pic}^d(C)$. Let $\pi : \mathcal{U} \to \mathrm{Pic}^{d-1}(C)$ be the morphism sending (L, P) to $L(-P)$. Note that a fiber $\pi^{-1}(L')$ of π over a line bundle $L' \in \mathrm{Pic}^{d-1}(C)$ is contained in the set

$$\{(L'(P), P) \mid P \in C\} \simeq C$$

which is parametrized by C. In particular, $\dim \ \pi^{-1}(L')$ is at most 1 for any $L' \in \mathrm{Pic}^{d-1}(C)$. Also note that if $(L, P) \in \mathcal{U}$, then $h^0(L(-P)) = h^0(L) = 1 - g + d$ and $h^1(L(-P)) = 1$ by Riemann-Roch. In particular, $L(-P)$ is a special line bundle of degree $d-1$. On the other hand, via the Serre dual map

$$\begin{array}{rcl} \mathrm{Pic}^{d-1}(C) & \to & \mathrm{Pic}^{2g-2-(d-1)}(C) \\ L' & \mapsto & \omega_C \otimes (L')^{\vee}, \end{array}$$

the locus of special line bundles $\{L' \in \mathrm{Pic}^{d-1}(C) \mid h^1(L') > 0\}$ is isomorphic to the locus $\{L'' \in \mathrm{Pic}^{2g-2-(d-1)}(C) \mid h^0(L'') > 0\}$, which is the image of $C^{2g-2-(d-1)} \to$

$\mathrm{Pic}^{2g-2-(d-1)}$ as above. Hence, the locus has dimension at most $2g-2-(d-1) \leq g-2$. We conclude that

$$\dim\ \mathcal{U} \leq (g-2) + \dim\ \pi^{-1}(L') \leq g-1$$

which is strictly smaller than $\dim\ \mathrm{Pic}^d(C)$.

□

Theorem 276 (Green)**.** *Let C be a smooth curve of genus g, and let L be a very ample line bundle of degree d. Then:*

(1) $K_{p,q}(C, L) = 0$ for $q \geq 3$ if $H^1(L) = 0$.

(2) $K_{p,2}(C, L) = 0$ if $d \geq 2g+1+p$, that is, a high degree curve of degree $d \geq 2g+1+p$, $p \geq 0$ satisfies the property (N_p).

Proof. By Green's duality, we have $K_{p,q}(C, L)^\vee \simeq K_{r-1-p,2-q}(C, K_C, L)$ where $r = h^0(C, L) - 1$. If $h^1(L) = 0$, we have $h^0(K_C - L) = 0$ by Serre duality, and in particular, $h^0(K_C + (2-q)L) = 0$ for any $q \geq 3$. The term $\wedge^{r-1-p} V \otimes H^0(K_C + (2-q)L) = 0$, and hence the cohomology group $K_{r-1-p,2-q}(C, K_C, L)$ also vanishes.
Now consider the case $q = 2$ and $d = \deg L \geq 2g+1+p$ with $p \geq 0$. Since $H^1(L) = 0$, we have $K_{p,2}(C, L) \simeq H^1(C, \wedge^{p+1}\mathcal{M}_L \otimes L)$. By Serre duality, the vanishing of $K_{p,2}(C, L)$ is equivalent to show that

$$H^0(C, \wedge^{p+1} M_L^\vee \otimes \omega_C \otimes L^\vee) = 0.$$

Let $P \in C$ be a (general) point. Since $L - P$ is still base-point-free, one has the following ladder with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{M}_{L(-P)} & \to & H^0(L(-P)) \otimes \mathcal{O}_C & \to & L(-P) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{M}_L & \to & H^0(L) \otimes \mathcal{O}_C & \to & L & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{I}_P = \mathcal{O}_C(-P) & \to & \mathcal{O}_C & \to & \mathcal{O}_P & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

This process can be repeated: choose general points $P_1, \cdots, P_{r-1} \in C$, where $r = h^0(L) - 1 = d - g$. By induction on r, one has the following short exact sequence

$$0 \to \mathcal{M}_{L(-P_1-\cdots-P_{r-1})} \to \mathcal{M}_L \to \bigoplus_{i=1}^{r-1} \mathcal{O}_C(-P_i) \to 0.$$

Note that the line bundle $L(-P_1-\cdots-P_{r-1})$ is a base-point-free pencil, since $h^0(L(-P_1-\cdots-P_{r-1}))=h^0(L)-(r-1)=2$. Therefore, Castelnuovo pencil trick gives an isomorphism $M_{L(-P_1-\cdots-P_{r-1})}\simeq L^\vee(P_1+\cdots+P_{r-1})$. Taking the dual of the above short exact sequence, we have

$$0\to\bigoplus_{i=1}^{r-1}\mathcal{O}_C(P_i)\to\mathcal{M}_L^\vee\to L(-P_1-\cdots-P_{r-1})\to 0.$$

Taking its $(p+1)$-th exterior power, we have

$$0\to\wedge^{p+1}\left(\bigoplus_{i=1}^{r-1}\mathcal{O}_C(P_i)\right)\to\wedge^{p+1}\mathcal{M}_L^\vee\to\wedge^p\left(\bigoplus_{i=1}^{r-1}\mathcal{O}_C(P_i)\right)\otimes L(-P_1-\cdots-P_{r-1})\to 0.$$

Twisting by $\omega_C\otimes L^\vee$, we see that the terms on the left is a direct sum of line bundles of degree $(p+1)+(2g-2)-d=2g+p-1-d<0$, hence have no global sections. The terms on the right is a direct sum of line bundles of the form $\omega_C(-P_{i_1}-\cdots-P_{i_{r-p-1}})$, subtracting $(r-p-1)$ points among general points $P_1,\cdots,P_{r-1}\in C$. It has no global section since the rank of the zeroth cohomology group drops by 1 when we subtract a general point, and $r-p-1=d-g-p-1\geq g$ in our case. In particular, $K_{p,2}(C,L)\simeq H^0(C,\wedge^{p+1}\mathcal{M}_L^\vee\otimes\omega_C\otimes L^\vee)^\vee=0$. □

Hence, the Betti table of a curve of high degree $C\subset\mathbb{P}^r$ of degree $d=2g+1+p$ has the following shape:

	0	1	2	$\cdots$	$p-1$	p	$p+1$	$\cdots$	$r-1$
0	$\kappa_{0,0}=1$	$-$	$-$	$\cdots$	$-$	$-$	$-$	$\cdots$	$-$
1	$-$	$\kappa_{1,1}$	$\kappa_{2,1}$	$\cdots$	$\kappa_{p-1,1}$	$\kappa_{p,1}$	$\kappa_{p+1,1}$	$\cdots$	$\kappa_{r-1,1}$
2	$-$	$-$	$-$	$\cdots$	$-$	$-$	$\kappa_{p+1,2}$	$\cdots$	$\kappa_{r-1,2}$

where $\kappa_{p,q}=\dim K_{p,q}(C,\mathcal{O}_C(1))=\beta_{p,p+q}(C)$. The horizontal strip $(\kappa_{1,1}\quad\kappa_{2,1}\quad\cdots\quad\kappa_{r,1})$ is called the *quadratic strand*, since it denotes the quadric generators and their linear relations. The next strip $(\kappa_{1,2}\quad\kappa_{2,2}\quad\cdots\quad\kappa_{r,2})$ is called the *cubic strand*. It is composed of the cubic generators and their linear relations, and quadratic relations among the terms lie on the quadratic strand. Green's $(2g+p+1)$-theorem says that the first p-terms in the quadratic strand of a curve of degree $\geq 2g+p+1$ becomes zero.