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Some of the numbers can be read off from the Hilbert function of $C$, in particular:

**Proposition 277.** $\kappa_{1,1} = \left(\frac{d-g-1}{2}\right)$ and $\kappa_{r-1,2} = g$.

**Proof.** Since there is no linear form in the ideal $I_C$, we have $\kappa_{1,1} = \dim(I_C)_2 = \dim S_2 - h^0(O_C(2)) = \binom{r+2}{2} - (1 - g + 2d)$. Since $r = d - g$, we have the desired result. Moreover, by Green’s duality, $K_{r-1,2}(C, O_C(1)) = K_{0,0}(C, K_C, O_C(1)) = H^0(K_C)$ is $g$-dimensional as stated. □

**Proposition 278.** Let $C$ be a curve of high degree as above. If $\kappa_{1,1} = 0$, then $\kappa_{j,1} = 0$ for every $j \geq i$. If $\kappa_{i,2} = 0$, then $\kappa_{j,2} = 0$ for every $j \leq i$.

**Proof.** Note that $\kappa_{i+1,1}$ counts the number of independent linear relations among minimal generators of $S(-i-1)^{\oplus \kappa_{i,1}} \subseteq F_1$ appearing in $i$-th term of the minimal free resolution of $C$. Hence, if $\kappa_{1,1} = 0$, no (linear) relations can occur, which forces $\kappa_{i+1,1} = 0$. The second statement comes from the same argument, applied on the “dual resolution” $\text{Hom}(F_\ast, S(-r - 1))$ which is a free resolution of the graded module $\bigoplus_j H^0(\omega_C(j))$. □

To sum up, the Betti table of a curve of high degree has the following shape:

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>⋯</th>
<th>$a$</th>
<th>$a + 1$</th>
<th>⋯</th>
<th>$b - 1$</th>
<th>$b$</th>
<th>⋯</th>
<th>$r - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>–</td>
<td>–</td>
<td>⋯</td>
<td>–</td>
<td>–</td>
<td>⋯</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>1</td>
<td>–</td>
<td>$\kappa_{1,1}$</td>
<td>$\kappa_{2,1}$</td>
<td>⋯</td>
<td>$\kappa_{a,1}$</td>
<td>$\kappa_{a+1,1}$</td>
<td>⋯</td>
<td>$\kappa_{b-1,1}$</td>
<td>–</td>
<td>⋯</td>
</tr>
<tr>
<td>2</td>
<td>–</td>
<td>–</td>
<td>⋯</td>
<td>–</td>
<td>$\kappa_{a+1,2}$</td>
<td>⋯</td>
<td>$\kappa_{b-1,2}$</td>
<td>$\kappa_{b,2}$</td>
<td>⋯</td>
<td>$g$</td>
</tr>
</tbody>
</table>

**Question 279.** It is natural to ask the following questions for a curve of high degree $C \subset \mathbb{P}^r$ of degree $d = 2g + 1 + p$.

1. What is the number $a = a(C)$ so that $\kappa_{a,2} = 0$ but $\kappa_{a+1,2} \neq 0$? (such a number is called the Green-Lazarsfeld index)

2. What is the number $b$ so that $\kappa_{b,1} = 0$ but $\kappa_{b-1,1} \neq 0$?

As we seen above, Green’s $(2g + 1 + p)$-theorem implies that $a(C) \geq p$. An upper bound of $a(C)$ comes from the presence of special secants:

**Definition 280.** A degenerate $q$-secant plane of $C \subset \mathbb{P}^r$ is a linear subspace $\Lambda \subseteq \mathbb{P}^r$ such that $\text{length}(C \cap \Lambda) \geq q$, and $\dim \Lambda \leq q - 2$.

If we choose general $q$ points $P_1, \cdots, P_q$ of $C$ with $q < r$, then their linear span $\Lambda = \langle P_1, \cdots, P_q \rangle$ is a linear subspace of dimension $(q - 1)$ which intersects $C$ exactly at $P_1, \cdots, P_q$. Hence, a degenerate $q$-secant plane implies that there are $q$-points on $C$ which form a special configuration in this manner. We address a known result without proofs:

**Proposition 281.** Let $C$ be as above. If $C$ has a degenerate $q$-secant plane, then $a(C) \leq q - 3$. Furthermore, $C$ always has a degenerate $q$-secant plane for the value $q = p + 3 + \max \left(0, \left\lceil \frac{p^2 - 3}{2} \right\rceil \right)$
For the values of $b(C)$, one has an upper bound, called $K_{p,1}$-theorem:

**Theorem 282.** $b(C) \leq r$, and the equality holds (i.e., $\kappa_{r-1,1} \neq 0$) if and only if $C$ is a rational normal curve.

A well-known proof of the $K_{p,1}$-theorem uses a notion of “syzygy scheme” and Castelnuovo theory, so we will skip in this lecture.

A lower bound comes from the following nonvanishing theorem by Green and Lazarsfeld.

**Theorem 283** (Green-Lazarsfeld nonvanishing). Let $X$ be a smooth projective variety, $L$ be a very ample line bundle. Let $M_1, M_2$ be two line bundles such that $L \cong M_1 \otimes M_2$ and

$$r_i = h^0(X, M_i) - 1 \geq 1$$

for $i = 1, 2$. Then $K_{r_1+r_2-1,1}(X, L) \neq 0$.

If we are able to find certain $M_1, M_2$ for our curve of high degree, then $b(C) \geq r_1 + r_2$.

A nonzero cohomology class in $K_{r_1+r_2-1,1}(X, L)$ provided from the above nonvanishing theorem is called a Green-Lazarsfeld class.

**Question 284.** Let $C$ be a curve (of high degree).

(1) Do the degenerate secant planes completely determine the value $a(C)$?

(2) Do the Green-Lazarsfeld classes completely determine the value $b(C)$?

Both of the questions seem to be extremely difficult in general. There is an answer to the second question when $C$ is a curve of sufficiently high degree. We first begin with a consequence of the Green-Lazarsfeld nonvanishing.

**Corollary 285.** Let $C$ be a $k$-gonal curve, and let $L$ be a very ample line bundle on $C$ of degree $\deg L = 2g+1+p$ for $p \geq 0$, so that $|L|$ embeds $C$ into $\mathbb{P}^N$ where $N = g+p+1$. Then $K_{N-k,1}(C, L) \neq 0$.

**Proof.** Apply the Green-Lazarsfeld nonvanishing theorem for a pair $M$ and $L \otimes M^\vee$, where $M$ is a line bundle which gives a $g_1^k$ on $C$. Since $h^0(C, L \otimes M^\vee) \geq g+p+2-k = N+1-k$, the Koszul cohomology groups $K_i(C, L)$ cannot be zero for $1 \leq i \leq h^0(L-M)-1$, where the range covers $i = N-k$.

When $\deg L$ is sufficiently large, then the divisor $L-M$ becomes nonspecial, and hence the number $h^0(L-M) - 1$ coincides with $N-k$. Therefore, we may ask a natural question whether this result is sharp:

**Question 286.** Let $L$ be a very ample divisor on $C$ with $\deg L \gg 0$, so that $|L|$ embeds $C$ into $\mathbb{P}^N$ where $N = \deg L - g$. Does $K_{p,1}(C, L) = 0$ when $p > N-k$?

The problem, once known as Green-Lazarsfeld’s gonality conjecture, is turned out to be true by Ein and Lazarsfeld.
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**Theorem 287** (Ein-Lazarsfeld, Rathmann). Let $L$ be any very ample divisor on $C$ with $\deg L > 4g - 4$, so that $|L|$ embeds $C$ into $\mathbb{P}^N$ where $N = \deg L - g$. Then $K_{p,1}(C, L) = 0$ for $p > N - k$.

In particular, we are able to read off the gonality of $C$ from the shape of Betti table for $C \subset \mathbb{P}^N$ a curve of sufficiently high degree.

**Remark 288.** For a generic $k$-gonal curve $C$ of genus $g$, Farkas and Kemeny showed that $\deg(L) \geq 2g - 1 + k$ is enough for the degree condition for $L$. Their bound is sharp; every $k$-gonal curve $C$ of genus $g$ has a line bundle of degree $2g - 2 + k$ which fails to verify the statement of the gonality conjecture. However, their result cannot cover every $k$-gonal curve of genus $g$; plane curves do not satisfy the statement (of course, they are NOT general in the moduli).
6 Enriques-Petri theorem and Green’s conjecture

We begin with another consequence of Green-Lazarsfeld nonvanishing theorem, asking the Clifford index:

**Question 289.** Is it possible to read off the Clifford index of $C$ from a certain Betti table?

Going back to the nonvanishing theorem, we have the following consequence:

**Corollary 290.** Let $C$ be a nonhyperelliptic curve of genus $g \geq 3$ such that $\text{Cliff}(C) = p$. Then the Koszul cohomology group $K_{g-2-p,1}(C, K_C) \neq 0$ does not vanish.

**Proof.** Let $A$ be an effective divisor which computes the Clifford index of $C$. In particular, if we denote by $d = \deg A$, $r = h^0(A) - 1 \geq 1$, then $p = d - 2r$. By Riemann-Roch, the divisor $K_C - A$ is also a special divisor such that $h^0(K_C - A) - 1 = g - d + r - 1$. Hence, the nonvanishing theorem implies that $K_{g-d+2r-2,1}(C, K_C) = K_{g-2-p,1}(C, K_C)$ does not vanish.

We may also ask that the above nonvanishing result is sharp, which is a famous Green’s conjecture:

**Conjecture 291 (Green’s canonical syzygy conjecture).** Let $C$ be as above. Then $K_{i,1}(C, K_C) = 0$ for $i > g - 2 - \text{Cliff}(C)$.

Passing by Green’s duality theorem, we have an equivalent statement for $K_{p,2}(C, K_C)$:

**Conjecture 292.** Let $C$ be as above. Then the canonical curve satisfies the property $(N_p)$ for $p < \text{Cliff}(C)$.

The zeroth case $p = 0$ corresponds to M. Noether’s theorem. The case $p = 1$ corresponds to Enriques-Petri theorem:

**Theorem 293 (Enriques-Petri).** Let $C$ be a nonhyperelliptic curve of genus $g \geq 4$. The canonical curve of $C$ is defined only by quadric equations if and only if neither $C$ is trigonal and nor $C$ is isomorphic to a plane quintic.

**Proof.** ($\Rightarrow$) When $C$ is trigonal, then the divisor $D$ associated to the $3-1$ morphism $C \to \mathbb{P}^1$ contributes to the Clifford index of $C$; in particular, $\text{Cliff}(C) \leq \deg D - h^0(D) + 2 = 1$. Since $C$ is not hyperelliptic, the Clifford index cannot be zero. Similarly, when $C$ is isomorphic to a plane quintic, then the hyperplane divisor $D$ satisfies $h^0(D) = 3$ and $\deg D = 5$, which also contribute to the Clifford index of $C$. In particular, $\text{Cliff}(C) \leq 1$, and we conclude that $\text{Cliff}(C) = 1$ by the same reason. In any cases, $\text{Cliff}(C) = 1$, and hence, the canonical curve fails to satisfy the property $(N_1)$ since the Koszul cohomology group $K_{g-3,1}(C, K_C) \simeq K_{1,2}(C, K_C)^\vee$ does not vanish. In particular, the ideal of the canonical curve requires a cubic equation as generators.
(\Leftrightarrow) First, consider the case \( g = 3 \) as an example. By Riemann-Roch, we have \( h^0(K_C) = 3 \) and \( h^0(nK_C) = 4n - 2 \) for \( n > 1 \). Since \( K_C \) is very ample, hence, for any point \( P \in C \) we have
\[
\begin{align*}
\begin{cases}
h^0(K_C(-P)) &= 2, \\
h^0(K_C(-2P)) &= 1.
\end{cases}
\end{align*}
\]
In particular, we are able to find a basis \( \{r, s, t\} \) of \( H^0(K_C) \) such that
\[
\begin{align*}
\begin{cases}
ord_P(r) &= 2, \\
ord_P(s) &= 1, \\
ord_P(t) &= 0.
\end{cases}
\end{align*}
\]
Since \( K_C(-P) \) is a base-point-free pencil, we have a short exact sequence
\[
0 \rightarrow -K_C(P) \rightarrow H^0(K_C(-P)) \otimes \mathcal{O}_C \rightarrow K_C(-P) \rightarrow 0.
\]
Twisting by \( K_C \) and considering the cohomology long exact sequence, one can show that the multiplication map
\[
H^0(K_C(-P)) \otimes H^0(K_C) \rightarrow H^0(2K_C(-P))
\]
is surjective. Hence, \( H^0(2K_C(-P)) \) is spanned by \( r^2, rs, rt, s^2, st \). Since \( h^0(2K_C) = 6 \), we conclude that
\[
H^0(2K_C(-P)) = \langle r^2, rs, rt, s^2, st \rangle \subset \langle r^2, rs, rt, s^2, st, t^2 \rangle = H^0(2K_C).
\]
Similarly, the multiplication map \( H^0(K_C(-P)) \otimes H^0(2K_C) \rightarrow H^0(3K_C(-P)) \) is also surjective, and we have
\[
\begin{align*}
H^0(3K_C(-P)) &= \langle r^3, r^2s, r^2t, rs^2, rst, rt^2, s^3, st^2 \rangle, \\
H^0(3K_C) &= \langle r^3, r^2s, r^2t, rs^2, rst, rt^2, s^3, st^2, t^3 \rangle.
\end{align*}
\]
We will show the statement by a similar argument. Now let \( C \) be a non-hyperelliptic curve of genus \( g \geq 3 \). Choose a general set of points \( P_1, P_2, \ldots, P_g \in C \) such that the divisor \( D = P_3 + \cdots + P_g \) satisfies
\begin{itemize}
\item \( K_C(D) \) is globally generated;
\item \( h^0(K_C(D)) = 2 \), that is, \( |K_C(D)| \) is a base-point-free \( g^1_2 \).
\end{itemize}
Since \( V_i := H^0(K_C(-P_1 - \cdots - P_i + P_j)) \subset H^0(K_C) \) is 1-dimensional for each \( 1 \leq i \leq g \), we may pick a basis \( \{\omega_1, \cdots, \omega_g\} \) of \( H^0(K_C) \) from generators of \( V_i \). We have
\[
\begin{align*}
\begin{cases}
\omega_i(P_j) &\neq 0, \\
\omega_i(P_j) &= 0 \text{ if } i \neq j,
\end{cases}
\end{align*}
\]
and \( H^0(K_C(-D)) = \langle \omega_1, \omega_2 \rangle \) (in particular, \( \omega_1 \) and \( \omega_2 \) vanish with order exactly 1 on \( P_3, \ldots, P_g \)).
We apply the base-point-free pencil trick for the following multiplicative map
\[
\mu_n : H^0(K_C(-D)) \otimes H^0((n - 1)K_C) \rightarrow H^0(nK_C(-D))
\]
for each \( n \geq 2 \), we have
• \( \omega_1^n, \ldots, \omega_g^n \in H^0(nK_C) \setminus H^0(nK_C(-D)); \)
• \( \omega_1^n, \ldots, \omega_g^n \) are linearly independent;
• \( h^0(nK_C) - h^0(nK_C(-D)) = g - 2. \)

Hence, \( H^0(nK_C) \) is spanned by forms in \( H^0(nK_C(-D)) \) and \( \omega_1^n, \ldots, \omega_g^n. \) As a consequence, for any \( n \geq 2, \) the multiplicative map
\[
H^0(K_C) \otimes H^0((n-1)K_C) \to H^0(nK_C)
\]
is surjective, which proves the "projective normality part" of M. Noether's theorem.

We are particularly interested in quadratic forms. Indeed, the \((3g - 3)\)-dimensional vector space \( H^0(2K_C) \) is spanned by:
\[
H^0(2K_C) = \langle \omega_1^2, \omega_1\omega_2, \ldots, \omega_1\omega_g, \omega_2^2, \omega_2\omega_3, \ldots, \omega_g^2, \omega_3^2, \ldots, \omega_g^2 \rangle.
\]

Let \( i, j \in \{3, \ldots, g\} \) be distinct indices. Since \( \omega_i \) vanishes on \( P_k \neq P_i \) and \( \omega_j \) vanishes on \( P_k \neq P_j, \) their multiplication \( \omega_i \omega_j \in H^0(2K_C) \) vanishes at \( P_1, \ldots, P_g. \) Therefore, \( \omega_i^2 \text{-term cannot appear} \) (which vanishes at every \( P_1, \ldots, P_g \) but not at \( P_k \)). In other words, there exist \( \lambda_{ij}, \mu_{ij}, b_{ij} \in \mathbb{C} \) such that \( \omega_i \omega_j \) is expressed as a linear sum
\[
\omega_i \omega_j = b_{ij} \omega_1^2 + \sum_{s=3}^g (\lambda_{ij} \omega_1 + \mu_{ij} \omega_2) \omega_s.
\]

In particular, a quadratic form
\[
f_{ij} := \omega_i \cdot \omega_j - b_{ij} \omega_1 \cdot \omega_2 - \sum_{s=3}^g (\lambda_{ij} \omega_1 + \mu_{ij} \omega_2) \cdot \omega_s \in \text{Sym}^2 H^0(K_C)
\]
is in the kernel \( I \) of the natural map \( \varphi : \text{Sym} H^0(K_C) \to \bigoplus_n H^0(nK_C). \) The elements \( f_{ij} \) are linearly independent, and hence, we have \( \binom{g-2}{2} \) quadratic equations which form a basis for the ideal \( I_2 \) of \( C. \)

We are now going to construct a set of cubic relations \( G_{jk} \) such that \( f_{ij} \)'s and \( G_{jk} \)'s form a generating set for the whole \( I. \) However, the multiplication map
\[
H^0(K_C - D) \otimes H^0(2K_C - D) \to H^0(3K_C - 2D)
\]
is not surjective. Inside the \((3g - 1)\)-dimensional vector space \( H^0(3K_C - 2D), \) the image forms a \((3g - 2)\)-dimensional subspace
\[
W := \langle \omega_1^3, \omega_1^2 \omega_2, \ldots, \omega_1^2 \omega_g, \omega_1 \omega_2^2, \ldots, \omega_1 \omega_2 \omega_g, \omega_2^3, \omega_2^2 \omega_3, \ldots, \omega_2^2 \omega_g \rangle
\]
(corresponding to cubic monomials which contains \( \omega_3, \ldots, \omega_g \) at most once; we skipped a proof for their linear independence). Take \( \eta \in H^0(3K_C(-2D)) \setminus W \) so that \( H^0(3K_C(-2D)) = \langle W, \eta \rangle. \) Hence, we have a filtration
\[
W : 3g - 2 \subset H^0(3K_C - 2D) : 3g - 1 \subset H^0(3K_C - D) : 4g - 3 \subset H^0(3K_C) : 5g - 5 \]
of vector spaces. For each \( i \in \{3, \ldots, g\}, \) there is an element \( \alpha_i \in H^0(K_C(-D)) = \langle \omega_1, \omega_2 \rangle \) such that \( \alpha_i \omega_i^2 \in H^0(3K_C - 2D) \setminus W. \)
Note that both $\omega_1\omega_1^2$ and $\omega_2\omega_1^2$ has a zero of order 1 at $P_i$. Hence, by taking a suitable linear combination of them, there is a unique nonzero element $\alpha_i \in \langle \omega_1, \omega_2 \rangle$ such that $\alpha_i\omega_i^2$ has a zero of order $\geq 2$ at $P_i$. Clearly it has a zero of order 2 at the other $P_j$, $j \neq i$. In particular, $\alpha_i\omega_i^2 \in H^0(3K_C(-2D))$.

If it is an element in $W$, one can express $\alpha_i\omega_i^2$ as a linear combination

$$\alpha_i\omega_i^2 = \alpha_i\omega_1\varphi_1 + \alpha_i\omega_2\varphi_2 + \omega_2^2\theta$$

for some linear forms $\varphi_1, \varphi_2, \theta \in H^0(K_C)$. Consider the (effective) divisor of zeros of $\alpha_i$. Then $(\alpha_i)_{|D} = D + P_i + D_i$; $\alpha_i$ vanishes along $P_3, \cdots, P_g$, and vanishes twice at $P_i$. Since $[K_C - D]$ is base-point-free, the divisors $D_i$ and $(\omega_2)_{|D}$ are disjoint. Hence, if $Q$ is a point such that $\alpha_i(Q) = 0$ but $\omega_2(Q) \neq 0$, then $\theta(Q) = 0$, that is, $\theta \in H^0(K_C - D_i)$. Among the elements in $H^0(K_C - D)$, the only possible choice is $\theta$ is a constant multiple of $\alpha_i$. Hence the relation reduces into

$$\alpha_i\omega_i^2 = \alpha_i(\sum \lambda_j\omega_1\omega_j + \sum \mu_j\omega_2\omega_j),$$

which gives a contradiction by observing the vanishing order at $P_i$.

We conclude that there is an element $\theta_i \in W$ such that $\alpha_i\omega_i^2 = \eta + \theta_i$ for each $i \leq 3 \leq g$.

Therefore, for any given distinct $j, k \in \{3, \cdots, g\}$, the cubic relation

$$G_{jk} := (\alpha_j \cdot \omega_j \cdot \omega_j - \theta_j) - (\alpha_k \cdot \omega_k \cdot \omega_k - \theta_k) \in \text{Sym}^3 H^0(K_C)$$

lies in the kernel $I$ of $\varphi$. In particular, $I_3$ is generated by $\omega_k \cdot f_{ij}$ and $G_{jk}$'s.

<table>
<thead>
<tr>
<th>Vector space</th>
<th>(additional) Generators</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W$</td>
<td>$\omega_1^3, \omega_1^2\omega_2, \omega_1\omega_2^2, \cdots, \omega_1^3\omega_2^3, \omega_2^6, \cdots, \omega_2^6\omega_g$</td>
</tr>
<tr>
<td>$H^0(3K_C - 2D)$</td>
<td>$\eta$</td>
</tr>
<tr>
<td>$H^0(3K_C - D)$</td>
<td>$\beta_i\omega_i^2 (3 \leq i \leq g), \beta_i \in H^0(K_C(-D)) \setminus \langle \alpha_i \rangle$</td>
</tr>
<tr>
<td>$H^0(3K_C)$</td>
<td>$\omega_i^3 (3 \leq i \leq g)$</td>
</tr>
</tbody>
</table>

When $n \geq 4$, the multiplication map

$$H^0(K_C - D) \otimes H^0((n - 1)K_C + (2 - n)D) \to H^0(nK_C + (1 - n)D)$$

becomes surjective by the bpf pencil trick, since the divisor $(n - 2)K_C + (3 - n)D$ is always nonspecial. By induction, one can compute the bases of vector spaces as in the following table:

<table>
<thead>
<tr>
<th>Vector space</th>
<th>(additional) Generators</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^0(nK_C + (1 - n)D)$</td>
<td>$\omega_1^i\omega_2^m (l + m = n), \omega_1^s\omega_2^t\omega_2, (s + t = n - 1, 3 \leq i \leq g)$, $\omega_1^h\omega_2^k\eta, (h + k = n - 3)$</td>
</tr>
<tr>
<td>$H^0(nK_C + (2 - n)D)$</td>
<td>$\beta_i^{n-2}\omega_i^2 (3 \leq i \leq g)$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$H^0(nK_C - D)$</td>
<td>$\beta_i^{n-1}\omega_i^3 (3 \leq i \leq g)$</td>
</tr>
<tr>
<td>$H^0(nK_C)$</td>
<td>$\omega_i^n (3 \leq i \leq g)$</td>
</tr>
</tbody>
</table>
This explicit computation of bases allows us to find the generators of the ideal $I$ of the canonical curve of $C$. Indeed, $I$ is generated by the $f_{ij}$ (quadratic equations) and $G_{jk}$ (cubic equations):

$$f_{ij} = \omega_i \cdot \omega_j - b_{ij} \omega_1 \omega_2 - \sum_{s=3}^{g} (\lambda_{ijs} \omega_1 + \mu_{ijs} \omega_2) \cdot \omega_s$$

$$G_{jk} = (\alpha_j \cdot \omega_j \cdot \omega_j - \theta_j) - (\alpha_k \cdot \omega_k \cdot \omega_k - \theta_k)$$

Note that $f_{ij}$ are linearly independent, but $G_{jk}$ are mostly not; it satisfies the cocycle condition $G_{jk} + G_{kl} = G_{jl}$.

First we show that they generate $I$. Consider an element $R = \sum_{ijk} \gamma_{ijk} \omega_i \cdot \omega_j \cdot \omega_k \in I_3$. Since $\omega_i^3 (3 \leq i \leq g)$ are linearly independent modulo $H^0(3K_C - D)$, we have $\gamma_{iii} = 0$ for $i = 3, \ldots, g$. Thus

$$R = \sum_i \delta_{ijk} f_{ij} \omega_k + \sum_{i=3}^{g} (\mu_i \alpha_i + \nu_i \beta_i) \omega_i^2 + w$$

where $w \in W$. Restricting to $C$ and use the relation $\alpha_i \omega_i^2 = \eta + \theta_i$, we have $\sum \mu_i = 0$ and $\nu_i = 0$, and hence we may write it as

$$R = \sum \sigma_{ijk} f_{ij} \omega_k + \sum \lambda_{jk} G_{jk} + w'$$

with some $w' \in W$. Restricting again to $C$, we see that $w' \in I_3$. However, by the construction of $W$, we have $W \cap I = \{0\}$ so $w'$ must be 0. In particular, $R$ is generated by $f_{ij}$ and $G_{jk}$’s.

Similarly, one can check that $I_n$ is generated by $f_{ij}$ and $G_{jk}$ for $n \geq 4$.

To complete the proof, we need to exhibit the syzygies among them. First assume that $g \geq 5$; a canonical curve of genus 4 is always trigonal, since it is a complete intersection of a quadric and a cubic hypersurface, so that the rulings of the quadric cut out on $C$ a $g_3^1$.

Consider the relation

$$\omega_i \omega_j = \sum_{s=3}^{g} (\alpha_{ijs}) \omega_s + b_{ij} \omega_1 \omega_2$$

determined by the quadratic equation $f_{ij}$. For any triple of distinct integers $i,j,k$, the linear form (differential) $\alpha_{ijk}$ vanishes doubly at $P_k$, so there are scalars $\rho_{ijk}$ such that $\alpha_{ijk} = \rho_{ijk} \alpha_k$. Hence, we have Petri syzygies

$$f_{ij} \omega_k - f_{ik} \omega_j = \sum_{s=3}^{g} (\alpha_{iks} f_{sj} - \alpha_{ijs} f_{sk}) + \rho_{ijk} G_{jk}$$

for any triple of distinct indices $3 \leq i,j,k \leq g$ (here, in the summation appearing in the right-hand-side, $f_{jj} = f_{kk} = 0$). One can also check that the coefficients $\rho_{ijk}$ are symmetric in $i,j,k$. 

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To complete the theorem, we have to ask under which condition the coefficients $\rho_{ijk}$ are zero/nonzero. Let $3 \leq j \leq g$. Denote $V_j \subseteq \mathbb{P}^{g-1}$ be the algebraic set defined by $(g-3)$ quadratic equations

$$f_{ij} = 0, \quad 3 \leq i \leq g, \quad i \neq j.$$ 

Note that $f_{ij} \in I$; hence the canonical curve of $C$ lies in $V_j$. Then one can check that $V_j$ has a unique surface component $F_j$ which contains the canonical curve $C$.

Since $V_j$ is defined by $(g-3)$ equations in $\mathbb{P}^{g-1}$, every component of $V_j$ is of dimension $\geq 2$, and there is a component which contains $C$ since $V_j \supseteq C$.

Write the equations $\{f_{ij}\}$ defining $V_j$ in the following way:

$$\sum_{s=3,s \neq j}^{g} (\delta_{is}\omega_j - \alpha_{ij}s)\omega_s = b_{ij}\omega_1\omega_2 + \alpha_{ijj}\omega_j, \quad 3 \leq i \leq g, i \neq j$$

where $\delta_{is}$ is the Kronecker delta symbol. Consider the determinant $\Delta_j$ of the $(g-3) \times (g-3)$ matrix

$$M_j = (\delta_{is}\omega_j - \alpha_{ij}s)_{3 \leq i,s \leq g,i \neq j}$$

Then $\Delta_j$ is a polynomial in $\omega_1, \omega_2, \omega_j$ such that $\Delta_j(P_j) \neq 0$. In particular, the hypersurface $(\Delta_j = 0)$ does not contain the canonical curve $C \subseteq \mathbb{P}^{g-1}$. In particular, we can solve the above system of equations in the $\omega_s$. As a result, we obtain a rational surface $F_j$ with rational parametric equations, away from the hypersurface $\Delta_j = 0$,

$$\begin{align*}
\omega_1 &= \omega_1; \\
\omega_2 &= \omega_2; \\
\omega_j &= \omega_j; \\
\omega_s &= M_j^{-1} \left( b_{ij}\omega_1\omega_2 + \alpha_{ijj}\omega_j \right)_s, \quad 3 \leq s \leq g, s \neq j.
\end{align*}$$

By construction, it is the only component of $V_j$ which is not contained in the hypersurface $V(\Delta_j)$. Since $P_j \in C \subseteq V_j$ and $P_j \notin V(\Delta_j)$, the only possibility is that $F_j$ is the only component containing $C$.

Petri’s key idea is encoded in the vanishing condition of $\rho_{ijk}$ in terms of those surfaces, namely:

Let $3 \leq j, k \leq g, j \neq k$. Two surfaces $F_j$ and $F_k$ coincide if and only if $\rho_{ijk} = 0$ for every $i \in \{3, \cdots, g\} \setminus \{j, k\}$.

We will see how the above statement concludes the proof. If $\rho_{ijk} \neq 0$, from the Petri syzygy we have

$$\rho_{ijk}G_{jk} = f_{ij}\omega_k - f_{ik}\omega_j - \sum_{s=3}^{g} (\alpha_{iks}f_{sj} - \alpha_{ijjs}f_{sk}),$$
hence $G_{jk}$ is generated by quadratic equations $\{f_{ij}\}$, which are quadratic generators of $I$. Since $\rho_{ijk}$ is symmetric up to permutations of indices, we may have the same conclusion for $G_{ij}$ and $G_{ik} = G_{ij} + G_{jk}$. Hence, it remains to see that all the coefficients $\rho_{ijk}$ cannot vanish simultaneously. Suppose not, then the surfaces $F_3 = \cdots = F_g = F$, it means that all the quadrics containing $C$ contain a surface $F$. Such an irreducible surface $F$ has the maximal number of quadric generators it can have (equals to $\binom{\text{codim}(F) + 1}{2} = \binom{g-2}{2}$).

Castelnuovo theory implies that $\deg F = g - 2$, and $F$ is either one of the following:

- a cone over a rational normal curve;
- a rational normal scroll;
- Veronese surface in $\mathbb{P}^5$.

However, the first case does not appear (by a projection argument). If $F = v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ is the Veronese surface, then $C \subset F \subset \mathbb{P}^5$ implies that the genus $g(C) = 6$, and $C$ is isomorphic to a plane curve since $F \approx \mathbb{P}^2$. Therefore, $C$ must be isomorphic to a plane quintic curve.

Finally, suppose that $F$ is a rational normal scroll $F \simeq \mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ ($n \geq 0$), embedded in $\mathbb{P}^g$ by a complete linear series $|H| = |\sigma + (n + 1 + k)f|$ for some $k \geq 0$, where $f$ is the fiber of the ruling $F_n \rightarrow \mathbb{P}^1$ and $\sigma$ is the unique irreducible section so that $\sigma^2 = -n$. Note that the canonical divisor is given by $K_{\mathbb{F}_n} = -2\sigma - (n + 2)f$. Since $\deg F = H^2 = g - 2$, we have

$$g - 2 = \sigma^2 + 2(n + 1 + k)\sigma.f + (n + 1 + k)^2f^2 = n + 2k + 2.$$ 

Since $C \subset F$ is a divisor, $C \in |r\sigma + sf|$ for integers $r, s$, given by the intersection number $r = C.f$ and $s = C.\sigma + rn$. By the adjunction formula, we have

$$\deg K_C = 2g - 2 = -r(r - 2)n + (r - 2)s + r(s - n - 2).$$

On the other hand, $C$ is a canonical curve, hence its degree $2g - 2$;

$$\deg C = C.H = 2g - 2 = -rn + s + r(n + 1 + k).$$

Therefore, we have

$$k = \frac{g - n - 4}{2},$$

$$s = \frac{(2g - 2) - r}{2}(g - n - 2),$$

$$0 = (g - 2)r^2 + (8 - 5g)r + (6g - 6) = (r - 3)[(g - 2)r - 2(g - 1)].$$

Since $g \geq 4$, the only integral solution is $r = 3$. Thus the fiber of the ruling $\pi : F \rightarrow \mathbb{P}^1$ intersects 3 times with $C$, induces a triple cover $\pi_C : C \rightarrow \mathbb{P}^1$. Therefore $C$ is trigonal.
Remark 294. One can imitate the above arguments for a curve which is not trigonal nor isomorphic to a plane quintic. Then, each $G_{jk}$ is generated by quadratic equations $\{f_{ij}\}$ as in the proof of the Enriques-Petri theorem, and hence give a linear syzygy among $\{f_{ij}\}$ from the cocycle condition on $\{G_{jk}\}$. To classify potential counterexamples, there are more subcases; for instance, $C$ is contained in a threefold of minimal degree. All the possible cases of curves $C$ are classified by Ehbauer.

Remark 295. For general and special curves, there are several attempts on Green’s conjecture. The next case to the Enriques-Petri theorem, describing the property $(N_2)$ with the exceptional cases of tetragonal curves/plane sextics, is known to be true by Voisin and Schreyer independently. During 90s, several people containing Bayer and Eisenbud studied a degeneration of curves and observed the behavior of syzygies. They tried to solve Green’s conjecture for general curves, by considering a family of curves whose limit is fairly easy to compute; for instance, tends to be a hyperelliptic curve, or a degenerate hyperelliptic curve (ribbon). Unfortunately, it was not very much successful at the time.

Voisin showed that Green’s conjecture holds for a general curve of genus $g$ (as a general element of the moduli space $\mathcal{M}_g$). Using Lefschetz theorem on Koszul cohomology groups, the syzygies $K_{p,q}(C, K_C)$ can be computed from the syzygies of K3 surfaces. Together with computational techniques using the Hilbert scheme of points on K3 surfaces, she found a K3 surface whose general hyperplane section is a canonical curve of genus $g$ for each $g$. Aprodu showed that the conjecture holds for a general $k$-gonal curve of genus $g$ (as an element of the gonality strata $\mathcal{M}_{g,k} \subset \mathcal{M}_g$). Aprodu and Farkas showed that the conjecture holds for arbitrary smooth curves on K3 surfaces. However, when $g > 11$, a general curve of genus $g$ is not embedded in any K3 surface.