Some of the numbers can be read off from the Hilbert function of C, in particular:

Proposition 277. *$\kappa_{1,1} = \binom{d-g-1}{2}$ and $\kappa_{r-1,2} = g$.*

Proof. Since there is no linear form in the ideal I_C, we have $\kappa_{1,1} = \dim(I_C)_2 = \dim S_2 - h^0(\mathcal{O}_C(2)) = \binom{r+2}{2} - (1-g+2d)$. Since $r = d-g$, we have the desired result.
Moreover, by Green's duality, $K_{r-1,2}(C, \mathcal{O}_C(1)) = K_{0,0}(C, K_C, \mathcal{O}_C(1)) = H^0(K_C)$ is g-dimensional as stated. □

Proposition 278. *Let C be a curve of high degree as above. If $\kappa_{i,1} = 0$, then $\kappa_{j,1} = 0$ for every $j \geq i$. If $\kappa_{i,2} = 0$, then $\kappa_{j,2} = 0$ for every $j \leq i$.*

Proof. Note that $\kappa_{i+1,1}$ counts the number of independent linear relations among minimal generators of $S(-i-1)^{\oplus \kappa_{i,1}} \subseteq F_i$ appearing in i-th term of the minimal free resolution of C. Hence, if $\kappa_{i,1} = 0$, no (linear) relations can occur, which forces $\kappa_{i+1,1} = 0$.
The second statement comes from the same argument, applied on the "dual resolution" $\mathrm{Hom}(F_\bullet, S(-r-1))$ which is a free resolution of the graded module $\bigoplus_j H^0(\omega_C(j))$. □

To sum up, the Betti table of a curve of high degree has the following shape:

	0	1	2	$\cdots$	a	$a+1$	$\cdots$	$b-1$	b	$\cdots$	$r-1$
0	1	–	–	$\cdots$	–	–	$\cdots$	–	–	$\cdots$	–
1	–	$\kappa_{1,1}$	$\kappa_{2,1}$	$\cdots$	$\kappa_{a,1}$	$\kappa_{a+1,1}$	$\cdots$	$\kappa_{b-1,1}$	–	$\cdots$	–
2	–	–	–	$\cdots$	–	$\kappa_{a+1,2}$	$\cdots$	$\kappa_{b-1,2}$	$\kappa_{b,2}$	$\cdots$	g

Question 279. It is natural to ask the following questions for a curve of high degree $C \subset \mathbb{P}^r$ of degree $d = 2g+1+p$.

(1) What is the number $a = a(C)$ so that $\kappa_{a,2} = 0$ but $\kappa_{a+1,2} \neq 0$? (such a number is called the Green-Lazarsfeld index)

(2) What is the number b so that $\kappa_{b,1} = 0$ but $\kappa_{b-1,1} \neq 0$?

As we seen above, Green's $(2g+1+p)$-theorem implies that $a(C) \geq p$. An upper bound of $a(C)$ comes from the presence of special secants:

Definition 280. A *degenerate q-secant plane* of $C \subset \mathbb{P}^r$ is a linear subspace $\Lambda \subseteq \mathbb{P}^r$ such that $length(C \cap \Lambda) \geq q$, and $\dim \Lambda \leq q-2$.

If we choose general q points $P_1, \cdots, P_q$ of C with $q < r$, then their linear span $\Lambda = \langle P_1, \cdots, P_q \rangle$ is a linear subspace of dimension $(q-1)$ which intersects C exactly at $P_1, \cdots, P_q$. Hence, a degenerate q-secant plane implies that there are q-points on C which form a special configuration in this manner. We address a known result without proofs:

Proposition 281. *Let C be as above. If C has a degenerate q-secant plane, then $a(C) \leq q-3$. Furthermore, C always has a degenerate q-secant plane for the value $q = p+3+\max\left(0, \lceil \frac{g-p-3}{2} \rceil\right)$*

For the values of $b(C)$, one has an upper bound, called $K_{p,1}$-theorem:

Theorem 282. *$b(C) \le r$, and the equality holds (*i.e., *$\kappa_{r-1,1} \neq 0$) if and only if C is a rational normal curve.*

A well-known proof of the $K_{p,1}$-theorem uses a notion of "syzygy scheme" and Castelnuovo theory, so we will skip in this lecture.
A lower bound comes from the following nonvanishing theorem by Green and Lazarsfeld.

Theorem 283 (Green-Lazarsfeld nonvanishing)**.** *Let X be a smooth projective variety, L be a very ample line bundle. Let M_1, M_2 be two line bundles such that $L \simeq M_1 \otimes M_2$ and*

$$r_i = h^0(X, M_i) - 1 \ge 1$$

for $i = 1, 2$. Then $K_{r_1+r_2-1,1}(X, L) \neq 0$.

If we are able to find certain M_1, M_2 for our curve of high degree, then $b(C) \ge r_1 + r_2$. A nonzero cohomology class in $K_{r_1+r_2-1,1}(X, L)$ provided from the above nonvanishing theorem is called a *Green-Lazarsfeld class.*

Question 284. Let C be a curve (of high degree).

(1) Do the degenerate secant planes completely determine the value $a(C)$?

(2) Do the Green-Lazarsfeld classes completely determine the value $b(C)$?

Both of the questions seem to be extremely difficult in general. There is an answer to the second question when C is a curve of sufficiently high degree. We first begin with a consequence of the Green-Lazarsfeld nonvanishing.

Corollary 285. *Let C be a k-gonal curve, and let L be a very ample line bundle on C of degree $\deg L = 2g+1+p$ for $p \ge 0$, so that $|L|$ embeds C into $\mathbb{P}^N$ where $N = g+p+1$. Then $K_{N-k,1}(C, L) \neq 0$.*

Proof. Apply the Green-Lazarsfeld nonvanishing theorem for a pair M and $L \otimes M^\vee$, where M is a line bundle which gives a $\mathfrak{g}^1_k$ on C. Since $h^0(C, L \otimes M^\vee) \ge g+p+2-k = N+1-k$, the Koszul cohomology groups $K_{i,1}(C, L)$ cannot be zero for $1 \le i \le h^0(L-M)-1$, where the range covers $i = N-k$. □

When $\deg L$ is sufficiently large, then the divisor $L - M$ becomes nonspecial, and hence the number $h^0(L-M) - 1$ coincides with $N - k$. Therefore, we may ask a natural question whether this result is sharp:

Question 286. Let L be a very ample divisor on C with $\deg L \gg 0$, so that $|L|$ embeds C into $\mathbb{P}^N$ where $N = \deg L - g$. Does $K_{p,1}(C, L) = 0$ when $p > N - k$?

The problem, once known as Green-Lazarsfeld's gonality conjecture, is turned out to be true by Ein and Lazarsfeld.

Theorem 287 (Ein-Lazarsfeld, Rathmann)**.** *Let L be any very ample divisor on C with $\deg L > 4g-4$, so that $|L|$ embeds C into $\mathbb{P}^N$ where $N = \deg L - g$. Then $K_{p,1}(C, L) = 0$ for $p > N - k$.*

In particular, we are able to read off the gonality of C from the shape of Betti table for $C \subset \mathbb{P}^N$ a curve of sufficiently high degree.

Remark 288. For a generic k-gonal curve C of genus g, Farkas and Kemeny showed that $\deg(L) \geq 2g-1+k$ is enough for the degree condition for L. Their bound is sharp; every k-gonal curve C of genus g has a line bundle of degree $2g-2+k$ which fails to verify the statement of the gonality conjecture. However, their result cannot cover every k-gonal curve of genus g; plane curves do not satisfy the statement (of course, they are NOT general in the moduli).

6 Enriques-Petri theorem and Green's conjecture

We begin with another consequence of Green-Lazarsfeld nonvanishing theorem, asking the Clifford index:

Question 289. Is it possible to read off the Clifford index of C from a certain Betti table?

Going back to the nonvanishing theorem, we have the following consequence:

Corollary 290. *Let C be a nonhyperelliptic curve of genus $g \geq 3$ such that $\mathrm{Cliff}(C) = p$. Then the Koszul cohomology group $K_{g-2-p,1}(C, K_C) \neq 0$ does not vanish.*

Proof. Let A be an effective divisor which computes the Clifford index of C. In particular, if we denote by $d = \deg A$, $r = h^0(A) - 1 \geq 1$, then $p = d - 2r$. By Riemann-Roch, the divisor $K_C - A$ is also a special divisor such that $h^0(K_C - A) - 1 = g - d + r - 1$. Hence, the nonvanishing theorem implies that $K_{g-d+2r-2,1}(C, K_C) = K_{g-2-p,1}(C, K_C)$ does not vanish. □

We may also ask that the above nonvanishing result is sharp, which is a famous Green's conjecture:

Conjecture 291 (Green's canonical syzygy conjecture)**.** *Let C be as above. Then $K_{i,1}(C, K_C) = 0$ for $i > g - 2 - \mathrm{Cliff}(C)$.*

Passing by Green's duality theorem, we have an equivalent statement for $K_{p,2}(C, K_C)$:

Conjecture 292. *Let C be as above. Then the canonical curve satisfies the property (N_p) for $p < \mathrm{Cliff}(C)$.*

The zeroth case $p = 0$ corresponds to M. Noether's theorem. The case $p = 1$ corresponds to Enriques-Petri theorem:

Theorem 293 (Enriques-Petri)**.** *Let C be a nonhyperelliptic curve of genus $g \geq 4$. The canonical curve of C is defined only by quadric equations if and only if neither C is trigonal and nor C is isomorphic to a plane quintic.*

Proof. ($\Rightarrow$) When C is trigonal, then the divisor D associated to the $3-1$ morphism $C \to \mathbb{P}^1$ contributes to the Clifford index of C; in particular, $\mathrm{Cliff}(C) \leq \deg D - h^0(D) + 2 = 1$. Since C is not hyperelliptic, the Clifford index cannot be zero. Similarly, when C is isomorphic to a plane quintic, then the hyperplane divisor D satisfies $h^0(D) = 3$ and $\deg D = 5$, which also contributes to the Clifford index of C. In particular, $\mathrm{Cliff}(C) \leq 1$, and we conclude that $\mathrm{Cliff}(C) = 1$ by the same reason. In any cases, $\mathrm{Cliff}(C) = 1$, and hence, the canonical curve fails to satisfy the property (N_1) since the Koszul cohomology group $K_{g-3,1}(C, K_C) \simeq K_{1,2}(C, K_C)^\vee$ does not vanish. In particular, the ideal of the canonical curve requires a cubic equation as generators.

($\Leftarrow$) First, consider the case $g = 3$ as an example. By Riemann-Roch, we have $h^0(K_C) = 3$ and $h^0(nK_C) = 4n - 2$ for $n > 1$. Since K_C is very ample, hence, for any point $P \in C$ we have

$$\begin{cases} h^0(K_C(-P)) &= 2, \text{ and} \\ h^0(K_C(-2P)) &= 1. \end{cases}$$

In particular, we are able to find a basis $\{r, s, t\}$ of $H^0(K_C)$ such that

$$\begin{cases} ord_P(r) &= 2, \\ ord_P(s) &= 1, \\ ord_P(t) &= 0. \end{cases}$$

Since $K_C(-P)$ is a base-point-free pencil, we have a short exact sequence

$$0 \to -K_C(P) \to H^0(K_C(-P)) \otimes \mathcal{O}_C \to K_C(-P) \to 0.$$

Twisting by K_C and considering the cohomology long exact sequence, one can show that the multiplication map

$$H^0(K_C(-P)) \otimes H^0(K_C) \to H^0(2K_C(-P))$$

is surjective. Hence, $H^0(2K_C(-P))$ is spanned by r^2, rs, rt, s^2, st. Since $h^0(2K_C) = 6$, we conclude that

$$H^0(2K_C(-P)) = \langle r^2, rs, rt, s^2, st \rangle \subset \langle r^2, rs, rt, s^2, st, t^2 \rangle = H^0(2K_C).$$

Similarly, the multiplication map $H^0(K_C(-P)) \otimes H^0(2K_C) \to H^0(3K_C(-P))$ is also surjective, and we have

$$\begin{aligned} H^0(3K_C(-P)) &= \langle r^3, r^2s, r^2t, rs^2, rst, rt^2, s^3, s^2t, st^2 \rangle, \\ H^0(3K_C) &= \langle r^3, r^2s, r^2t, rs^2, rst, rt^2, s^3, s^2t, st^2, t^3 \rangle. \end{aligned}$$

We will show the statement by a similar argument. Now let C be a non-hyperelliptic curve of genus $g \geq 3$. Choose a general set of points $P_1, P_2, \cdots, P_g \in C$ such that the divisor $D = P_3 + \cdots + P_g$ satisfies

- $K_C(-D)$ is globally generated;
- $h^0(K_C(-D)) = 2$, that is, $|K_C(-D)|$ is a base-point-free $\mathfrak{g}^1_g$.

Since $V_i := H^0(K_C(-P_1 - \cdots - P_g + P_i)) \subset H^0(K_C)$ is 1-dimensional for each $1 \leq i \leq g$, we may pick a basis $\{\omega_1, \cdots, \omega_g\}$ of $H^0(K_C)$ from generators of V_i. We have

$$\begin{cases} \omega_i(P_i) &\neq 0, \\ \omega_i(P_j) &= 0 \text{ if } i \neq j, \end{cases}$$

and $H^0(K_C(-D)) = \langle \omega_1, \omega_2 \rangle$ (in particular, ω_1 and ω_2 vanish with order exactly 1 on $P_3, \cdots, P_g$).

We apply the base-point-free pencil trick for the following multiplicative map

$$\mu_n : H^0(K_C(-D)) \otimes H^0((n-1)K_C) \to H^0(nK_C(-D))$$

for each $n \geq 2$, we have

- $\omega_3^n, \cdots, \omega_g^n \in H^0(nK_C) \setminus H^0(nK_C(-D))$;
- $\omega_3^n, \cdots, \omega_g^n$ are linearly independent;
- $h^0(nK_C) - h^0(nK_C(-D)) = g - 2$.

Hence, $H^0(nK_C)$ is spanned by forms in $H^0(nK_C(-D))$ and $\omega_3^n, \cdots, \omega_g^n$. As a consequence, for any $n \geq 2$, the multiplicative map

$$H^0(K_C) \otimes H^0((n-1)K_C) \to H^0(nK_C)$$

is surjective, which proves the "projective normality part" of M. Noether's theorem.
We are particularly interested in quadratic forms. Indeed, the $(3g-3)$-dimensional vector space $H^0(2K_C)$ is spanned by:

$$H^0(2K_C) = \langle \omega_1^2, \omega_1\omega_2, \cdots, \omega_1\omega_g, \omega_2^2, \omega_2\omega_3, \cdots, \omega_2\omega_g, \omega_3^2, \cdots, \omega_g^2 \rangle.$$

Let $i, j \in \{3, \cdots, g\}$ be distinct indices. Since ω_i vanishes on $P_k \neq P_i$ and ω_j vanishes on $P_k \neq P_j$, their multiplication $\omega_i\omega_j \in H^0(2K_C)$ vanishes at $P_1, \cdots, P_g$. Therefore, ω_k^2-term cannot appear (which vanishes at every $P_1, \cdots, P_g$ but not at P_k). In other words, there exist $\lambda_{ijs}, \mu_{ijs}, b_{ij} \in \mathbb{C}$ such that $\omega_i\omega_j$ is expressed as a linear sum

$$\omega_i\omega_j = b_{ij}\omega_1\omega_2 + \sum_{s=3}^{g} (\lambda_{ijs}\omega_1 + \mu_{ijs}\omega_2)\omega_s.$$

In particular, a quadratic form

$$f_{ij} := \omega_i \cdot \omega_j - b_{ij}\omega_1 \cdot \omega_2 - \sum_{s=3}^{g} (\lambda_{ijs}\omega_1 + \mu_{ijs}\omega_2) \cdot \omega_s \in \operatorname{Sym}^2 H^0(K_C)$$

is in the kernel I of the natural map $\varphi : \operatorname{Sym} H^0(K_C) \to \bigoplus_n H^0(nK_C)$. The elements f_{ij} are linearly independent, and hence, we have $\binom{g-2}{2}$ quadratic equations which form a basis for the ideal I_2 of C.
We are now going to construct a set of cubic relations G_{jk} such that f_{ij}'s and G_{jk}'s form a generating set for the whole I. However, the multiplication map

$$H^0(K_C - D) \otimes H^0(2K_C - D) \to H^0(3K_C - 2D)$$

is not surjective. Inside the $(3g-1)$-dimensional vector space $H^0(3K_C - 2D)$, the image forms a $(3g-2)$-dimensional subspace

$$W := \langle \omega_1^3, \omega_1^2\omega_2 \cdots, \omega_1^2\omega_g, \omega_1\omega_2^2, \cdots, \omega_1\omega_2\omega_g, \omega_2^3, \cdots, \omega_2^2\omega_g \rangle$$

(corresponding to cubic monomials which contains $\omega_3, \cdots, \omega_g$ at most once; we skipped a proof for their linear independence). Take $\eta \in H^0(3K_C(-2D)) \backslash W$ so that $H^0(3K_C(-2D)) = \langle W, \eta \rangle$. Hence, we have a filtration

$$W^{3g-2} \subset H^0(3K_C - 2D)^{3g-1} \subset H^0(3K_C - D)^{4g-3} \subset H^0(3K_C)^{5g-5}$$

of vector spaces. For each $i \in \{3, \cdots, g\}$, there is an element $\alpha_i \in H^0(K_C(-D)) = \langle \omega_1, \omega_2 \rangle$ such that $\alpha_i\omega_i^2 \in H^0(3K_C - 2D) \setminus W$.

Note that both $\omega_1\omega_i^2$ and $\omega_2\omega_i^2$ has a zero of order 1 at P_i. Hence, by taking a suitable linear combination of them, there is a unique nonzero element $\alpha_i \in \langle\omega_1, \omega_2\rangle$ such that $\alpha_i\omega_i^2$ has a zero of order ≥ 2 at P_i. Clearly it has a zero of order 2 at the other P_j, $j \neq i$. In particular, $\alpha_i\omega_i^2 \in H^0(3K_C(-2D))$. If it is an element in W, one can express $\alpha_i\omega_i^2$ as a linear combination

$$\alpha_i\omega_i^2 = \alpha_i\omega_1\varphi_1 + \alpha_i\omega_2\varphi_2 + \omega_2^2\theta$$

for some linear forms $\varphi_1, \varphi_2, \theta \in H^0(K_C)$. Consider the (effective) divisor of zeros of α_i. Then $(\alpha_i)_0 = D + P_i + D_i$; α_i vanishes along $P_3, \cdots, P_g$, and vanishes twice at P_i. Since $|K_C - D|$ is base-point-free, the divisors D_i and $(\omega_2)_0$ are disjoint. Hence, if Q is a point such that $\alpha_i(Q) = 0$ but $\omega_2(Q) \neq 0$, then $\theta(Q) = 0$, that is, $\theta \in H^0(K_C - D_i)$. Among the elements in $H^0(K_C - D)$, the only possible choice is: θ is a constant multiple of α_i. Hence the relation reduces into

$$\alpha_i\omega_i^2 = \alpha_i(\sum \lambda_j\omega_1\omega_j + \sum \mu_j\omega_2\omega_j),$$

which gives a contradiction by observing the vanishing order at P_i.

We conclude that there is an element $\theta_i \in W$ such that $\alpha_i\omega_i^2 = \eta + \theta_i$ for each $i \leq 3 \leq g$. Therefore, for any given distinct $j, k \in \{3, \cdots, g\}$, the cubic relation

$$G_{jk} := (\alpha_j \cdot \omega_j \cdot \omega_j - \theta_j) - (\alpha_k \cdot \omega_k \cdot \omega_k - \theta_k) \in \operatorname{Sym}^3 H^0(K_C)$$

lies in the kernel I of φ. In particular, I_3 is generated by $\omega_k \cdot f_{ij}$ and G_{jk}'s.

Vector space	(additional) Generators
W	$\omega_1^3, \omega_1^2\omega_2 \cdots, \omega_1^2\omega_g, \omega_1\omega_2^2, \cdots, \omega_1\omega_2\omega_g, \omega_2^3, \cdots, \omega_2^2\omega_g$
$H^0(3K_C - 2D)$	η
$H^0(3K_C - D)$	$\beta_i\omega_i^2 (3 \leq i \leq g), \beta_i \in H^0(K_C(-D)) \setminus \langle\alpha_i\rangle$
$H^0(3K_C)$	$\omega_i^3 (3 \leq i \leq g)$

When $n \geq 4$, the multiplication map

$$H^0(K_C - D) \otimes H^0((n-1)K_C + (2-n)D) \to H^0(nK_C + (1-n)D)$$

becomes surjective by the bpf pencil trick, since the divisor $(n-2)K_C + (3-n)D$ is always nonspecial. By induction, one can compute the bases of vector spaces as in the following table:

Vector space	(additional) Generators
$H^0(nK_C + (1-n)D)$	$\omega_1^l\omega_2^m \quad (l + m = n)$, $\omega_1^s\omega_2^t\omega_i \quad (s + t = n - 1, 3 \leq i \leq g)$, $\omega_1^h\omega_2^k\eta \quad (h + k = n - 3)$
$H^0(nK_C + (2-n)D)$	$\beta_i^{n-2}\omega_i^2 \quad (3 \leq i \leq g)$
$\vdots$	$\vdots$
$H^0(nK_C - D)$	$\beta_i\omega_i^{n-1} \quad (3 \leq i \leq g)$
$H^0(nK_C)$	$\omega_i^n \quad (3 \leq i \leq g)$

This explicit computation of bases allows us to find the generators of the ideal I of the canonical curve of C. Indeed, I is generated by the f_{ij} (quadratic equations) and G_{jk} (cubic equations):

$$\begin{aligned} f_{ij} &= \omega_i \cdot \omega_j - b_{ij}\omega_1\omega_2 - \sum_{s=3}^{g}(\lambda_{ijs}\omega_1 + \mu_{ijs}\omega_2) \cdot \omega_s \\ G_{jk} &= (\alpha_j \cdot \omega_j \cdot \omega_j - \theta_j) - (\alpha_k \cdot \omega_k \cdot \omega_k - \theta_k) \end{aligned}$$

Note that f_{ij} are linearly independent, but G_{jk} are mostly not; it satisfies the cocycle condition $G_{jk} + G_{kl} = G_{jl}$.

First we show that they generate I. Consider an element $R = \sum \gamma_{ijk}\omega_i \cdot \omega_j \cdot \omega_k \in I_3$. Since $\omega_i^3 (3 \leq i \leq g)$ are linearly independent modulo $H^0(3K_C - D)$, we have $\gamma_{iii} = 0$ for $i = 3, \cdots, g$. Thus

$$R = \sum \delta_{ijk} f_{ij}\omega_k + \sum_{i=3}^{g}(\mu_i\alpha_i + \nu_i\beta_i)\omega_i^2 + w$$

where $w \in W$. Restricting to C and use the relation $\alpha_i\omega_i^2 = \eta + \theta_i$, we have $\sum \mu_i = 0$ and $\nu_i = 0$, and hence we may write it as

$$R = \sum \sigma_{ijk} f_{ij}\omega_k + \sum \lambda_{jk}G_{jk} + w'$$

with some $w' \in W$. Restricting again to C, we see that $w' \in I_3$. However, by the construction of W, we have $W \cap I = \{0\}$ so w' must be 0. In particular, R is generated by f_{ij} and G_{jk}'s.

Similarly, one can check that I_n is generated by f_{ij} and G_{jk} for $n \geq 4$.

To complete the proof, we need to exhibit the syzygies among them. First assume that $g \geq 5$; a canonical curve of genus 4 is always trigonal, since it is a complete intersection of a quadric and a cubic hypersurface, so that the rulings of the quadric cut out on C a $\mathfrak{g}_3^1$.

Consider the relation

$$\omega_i\omega_j = \sum_{s=3}^{g}(\alpha_{ijs})\omega_s + b_{ij}\omega_1\omega_2$$

determined by the quadratic equation f_{ij}. For any triple of distinct integers i, j, k, the linear form (differential) α_{ijk} vanishes doubly at P_k, so there are scalars ρ_{ijk} such that $\alpha_{ijk} = \rho_{ijk}\alpha_k$. Hence, we have Petri syzygies

$$f_{ij}\omega_k - f_{ik}\omega_j = \sum_{s=3}^{g}(\alpha_{iks}f_{sj} - \alpha_{ijs}f_{sk}) + \rho_{ijk}G_{jk}$$

for any triple of distinct indices $3 \leq i, j, k \leq g$ (here, in the summation appearing in the right-hand-side, $f_{jj} = f_{kk} = 0$). One can also check that the coefficients ρ_{ijk} are symmetric in i, j, k.

To complete the theorem, we have to ask under which condition the coefficients ρ_{ijk} are zero/nonzero. Let $3 \le j \le g$. Denote $V_j \subseteq \mathbb{P}^{g-1}$ be the algebraic set defined by $(g-3)$ quadratic equations

$$f_{ij} = 0, \quad 3 \le i \le g, \quad i \ne j.$$

Note that $f_{ij} \in I$; hence the canonical curve of C lies in V_j. Then one can check that V_j has a unique surface component F_j which contains the canonical curve C;

> Since V_j is defined by $(g-3)$ equations in $\mathbb{P}^{g-1}$, every component of V_j is of dimension ≥ 2, and there is a component which contains C since $V_j \supset C$. Write the equations $\{f_{ij}\}$ defining V_j in the following way:
>
> $$\sum_{s=3, s\ne j}^{g} (\delta_{is}\omega_j - \alpha_{ijs})\omega_s = b_{ij}\omega_1\omega_2 + \alpha_{ijj}\omega_j, \quad 3 \le i \le g, i \ne j$$
>
> where δ_{is} is the Kronecker delta symbol. Consider the determinant Δ_j of the $(g-3) \times (g-3)$ matrix
>
> $$M_j = (\delta_{is}\omega_j - \alpha_{ijs})_{3 \le i,s \le g, i \ne j, s \ne j}$$
>
> Then Δ_j is a polynomial in $\omega_1, \omega_2, \omega_j$ such that $\Delta_j(P_j) \ne 0$. In particular, the hypersurface $(\Delta_j = 0)$ does not contain the canonical curve $C \subseteq \mathbb{P}^{g-1}$. In particular, we can solve the above system of equations in the ω_s. As a result, we obtain a rational surface F_j with rational parametric equations, away from the hypersurface $\Delta_j = 0$,
>
> $$\begin{cases} \omega_1 &= \omega_1; \\ \omega_2 &= \omega_2; \\ \omega_j &= \omega_j; \\ \omega_s &= \left[M_j^{-1} \left(\ b_{ij}\omega_1\omega_2 + \alpha_{ijj}\omega_j \ \right) \right]_s, \quad 3 \le s \le g, s \ne j. \end{cases}$$
>
> By construction, it is the only component of V_j which is not contained in the hypersurface $V(\Delta_j)$. Since $P_j \in C \subseteq V_j$ and $P_j \notin V(\Delta_j)$, the only possibility is that F_j is the only component containing C.

Petri's key idea is encoded in the vanishing condition of ρ_{ijk} in terms of those surfaces, namely:

> Let $3 \le j, k \le g, j \ne k$. Two surfaces F_j and F_k coincide if and only if $\rho_{ijk} = 0$ for every $i \in \{3, \cdots, g\} \setminus \{j, k\}$.

We will see how the above statement concludes the proof. If $\rho_{ijk} \ne 0$, from the Petri syzygy we have

$$\rho_{ijk} G_{jk} = f_{ij}\omega_k - f_{ik}\omega_j - \sum_{s=3}^{g} (\alpha_{iks} f_{sj} - \alpha_{ijs} f_{sk}),$$

hence G_{jk} is generated by quadratic equations $\{f_{ij}\}$, which are quadratic generators of I. Since ρ_{ijk} is symmetric up to permutations of indices, we may have the same conclusion for G_{ij} and $G_{ik} = G_{ij} + G_{jk}$. Hence, it remains to see that all the coefficients ρ_{ijk} cannot vanish simultaneously. Suppose not, then the surfaces $F_3 = \cdots = F_g = F$, it means that all the quadrics containing C contain a surface F. Such an irreducible surface F has the maximal number of quadric generators it can have (equals to $\binom{\mathrm{codim}(F)+1}{2} = \binom{g-2}{2}$). Castelnuovo theory implies that $\deg F = g - 2$, and F is either one of the following:

- a cone over a rational normal curve;
- a rational normal scroll;
- Veronese surface in $\mathbb{P}^5$.

However, the first case does not appear (by a projection argument). If $F = v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ is the Veronese surface, then $C \subset F \subset \mathbb{P}^5$ implies that the genus $g(C) = 6$, and C is isomorphic to a plane curve since $F \simeq \mathbb{P}^2$. Therefore, C must be isomorphic to a plane quintic curve.
Finally, suppose that F is a rational normal scroll $F \simeq \mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ $(n \geq 0)$, embedded in $\mathbb{P}^{g-1}$ by a complete linear series $|H| = |\sigma + (n+1+k)f|$ for some $k \geq 0$, where f is the fiber of the ruling $\mathbb{F}_n \to \mathbb{P}^1$ and σ is the unique irreducible section so that $\sigma^2 = -n$. Note that the canonical divisor is given by $K_{\mathbb{F}_n} = -2\sigma - (n+2)f$. Since $\deg F = H^2 = g - 2$, we have

$$g - 2 = \sigma^2 + 2(n+1+k)\sigma.f + (n+1+k)^2 f^2 = n + 2k + 2.$$

Since $C \subset F$ is a divisor, $C \in |r\sigma + sf|$ for integers r, s, given by the intersection number $r = C.f$ and $s = C.\sigma + rn$. By the adjunction formula, we have

$$\deg K_C = 2g - 2 = -r(r-2)n + (r-2)s + r(s - n - 2).$$

On the other hand, C is a canonical curve, hence its degree $2g - 2$;

$$\deg C = C.H = 2g - 2 = -rn + s + r(n+1+k).$$

Therefore, we have

$$\begin{aligned} k &= \frac{g-n-4}{2}, \\ s &= (2g-2) - \frac{r}{2}(g-n-2), \\ 0 &= (g-2)r^2 + (8-5g)r + (6g-6) = (r-3)[(g-2)r - 2(g-1)]. \end{aligned}$$

Since $g \geq 4$, the only integral solution is $r = 3$. Thus the fiber of the ruling $\pi : F \to \mathbb{P}^1$ intersects 3 times with C, induces a triple cover $\pi_C : C \to \mathbb{P}^1$. Therefore C is trigonal.

□

Remark 294. One can imitate the above arguments for a curve which is not trigonal nor isomorphic to a plane quintic. Then, each G_{jk} is generated by quadratic equations $\{f_{ij}\}$ as in the proof of the Enriques-Petri theorem, and hence give a linear syzygy among $\{f_{ij}\}$ from the cocycle condition on $\{G_{jk}\}$. To classify potential counterexamples, there are more subcases; for instance, C is contained in a threefold of minimal degree. All the possible cases of curves C are classified by Ehbauer.

Remark 295. For general and special curves, there are several attempts on Green's conjecture. The next case to the Enriques-Petri theorem, describing the property (N_2) with the exceptional cases of tetragonal curves/plane sextics, is known to be true by Voisin and Schreyer independently. During 90s, several people containing Bayer and Eisenbud studied a degeneration of curves and observed the behavior of syzygies. They tried to solve Green's conjecture for general curves, by considering a family of curves whose limit is fairly easy to compute; for instance, tends to be a hyperelliptic curve, or a degenerate hyperelliptic curve (ribbon). Unfortunately, it was not very much successful at the time.
Voisin showed that Green's conjecture holds for a general curve of genus g (as a general element of the moduli space $\mathcal{M}_g$). Using Lefschetz theorem on Koszul cohomology groups, the syzygies $K_{p,q}(C, K_C)$ can be computed from the syzygies of K3 surfaces. Together with computational techniques using the Hilbert scheme of points on K3 surfaces, she found a K3 surface whose general hyperplane section is a canonical curve of genus g for each g. Aprodu showed that the conjecture holds for a general k-gonal curve of genus g (as an element of the gonality strata $\mathcal{M}_{g,k} \subset \mathcal{M}_g$). Aprodu and Farkas showed that the conjecture holds for arbitrary smooth curves on K3 surfaces. However, when $g > 11$, a general curve of genus g is not embedded in any K3 surface.