Proposition 20 (Adjoint property)**.** *Let $f : X \to Y$ be a continuous map between two topological spaces, $\mathcal{F}$ be a sheaf on X, and let $\mathcal{G}$ be a sheaf on Y. There are natural maps $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ and $\mathcal{G} \to f_*f^{-1}\mathcal{G}$. There is a natural bijection between two sets*

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

Hence, we say that f^{-1} is a left adjoint *of f_*, and f_* is a* right adjoint *of f^{-1}.*

Proof. Only a hint (excerpted from Vakil's book) for the adjoint property. Via focusing on the stalks, one can show that the both sets are bijective to the set of compatible collections $\phi = \{\phi_{U,V} \mid U \in \mathfrak{Top}(X), f(U) \subset V \in \mathfrak{Top}(Y)\}$ where

each $\phi_{U,V} : \mathcal{G}(V) \to \mathcal{F}(U)$ is a map between abelian groups satisfying

$$\begin{array}{ccc} \mathcal{G}(V) & \xrightarrow{\phi_{U,V}} & \mathcal{F}(U) \\ {\scriptstyle \rho^{\mathcal{G}}_{V,V'}}\downarrow & & \downarrow{\scriptstyle \rho^{\mathcal{F}}_{U,U'}} \\ \mathcal{G}(V') & \xrightarrow[\phi_{U',V'}]{} & \mathcal{F}(U') \end{array}$$

for any open $U' \subseteq U \subseteq X$ and $V' \subseteq V \subseteq Y$ with $f(U) \subseteq V$, $f(U') \subseteq V'$.

□

Proposition 21 (Gluing sheaves)**.** *Let X be a topological space, $\{U_i\}$ be an open cover of X. Suppose we have a sheaf $\mathcal{F}_i$ on each U_i such that there is an isomorphism $\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_j|_{U_i \cap U_j}$ for each i, j satisfying*

(0) $\varphi_{ii} = id$;

(1) (cocycle condition) $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$ for each i, j, k.

Then there is a unique sheaf $\mathcal{F}$ on X, together with isomorphisms $\psi_i : \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$ such that $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$ for each i, j.

Proof. Let $V \subseteq X$ be an open subset. Note that $\{V \cap U_i\}_{i \in I}$ is an open covering of V. We define

$$\mathcal{F}(V) := \{(s_i)_{i\in I} \mid s_i \in \mathcal{F}_i(V \cap U_i),\ \varphi_{ij}(s_i|_{V\cap U_i \cap U_j}) = s_j|_{V \cap U_i \cap U_j} \text{ for each } i, j \in I\}.$$

Note that the cocycle condition is already hidden inside: on $V \cap U_i \cap U_j \cap U_k$, a section s of $\mathcal{F}$ contains the data of sections s_i, s_j, s_k of $\mathcal{F}_i(V \cap U_i), \mathcal{F}_j(V \cap U_j), \mathcal{F}_k(V \cap U_k)$ such that $\varphi_{ij}(s_i) = s_j, \varphi_{jk}(s_j) = s_k, \varphi_{ik}(s_i) = s_k$ on the intersection of two of them. In particular, $\varphi_{ij}(s_i|_{V\cap U_i\cap U_j \cap U_k}) = s_j|_{V \cap U_i \cap U_j \cap U_k}$, and hence

$$(\varphi_{jk} \circ \varphi_{ij})(s_i|_{V\cap U_i \cap U_j \cap U_k}) = \varphi_{jk}(s_j|_{V\cap U_i\cap U_j\cap U_k}) = s_k|_{V \cap U_i \cap U_j \cap U_k}$$

for any $s \in \mathcal{F}(V)$, $i, j, k \in I$.
Let $W \subseteq V$ be a smaller open subset. We define the restriction map $\mathcal{F}(V) \to \mathcal{F}(W)$ by sending each s_i to $s_i|_{W \cap U_i}$. It is easy to check that $\mathcal{F}$ is a sheaf:

the only part which might be unfamiliar yet is to check the gluing property. Let us see just a little more detail. Let $\{V_k\}$ be another open cover of V, and let $s^k := (s_i^k)_{i\in I} \in \mathcal{F}(V_k)$ such that $s^k|_{V_k\cap V_k'} = s^{k'}|_{V_k\cap V_k'}$. For each $i \in I$ and for any k, k', we have $s_i^k|_{V_k\cap V_k'\cap U_i} = s_i^{k'}|_{V_k\cap V_k'\cap U_i} \in \mathcal{F}_i(V_k \cap V_k' \cap U_i)$. Hence, they glue together and form a single section $s_i \in \mathcal{F}_i(V \cap U_i)$. It is straightforward that $\varphi_{ij}(s_i|_{V\cap U_i\cap U_j}) = s_j|_{V\cap U_i\cap U_j}$.

To complete the proof, we need to find local isomorphisms $\psi_i : \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$. Let $V \subseteq U_i$, and let $s = (s_j)_{j\in I} \in \mathcal{F}(V)$. We define $\psi_i(s) := s_i \in \mathcal{F}_i(V)$. The definition of $\mathcal{F}(V)$ implies that $s_j = \varphi_{ij}(s_i|_{V\cap U_j})$ for every $j \in$I, hence, on any open subset $V \subseteq U_i \cap U_j$ and a section $s \in \mathcal{F}(V) = \mathcal{F}(V \cap U_i \cap U_j)$, we have

$$\varphi_{ij}\psi_i(s) = \varphi_{ij}(s_i|_{V\cap U_i\cap U_j}) = s_j|_{V\cap U_i\cap U_j} = \psi_j(s)$$

as desired. Clearly it is an isomorphism with a two-sided inverse map

$$s_i \in \mathcal{F}_i(V) \mapsto (s_j)_{j\in I} \in \mathcal{F}(V),\ s_j = \varphi_{ij}(s_i|_{V\cap U_j})$$

for any open subset $V \subseteq U_i$. □

Remark 22. Suppose that we have another $\mathcal{G}_i$ on each U_i with local isomorphism $\mathcal{G}_i|_{U_i\cap U_j} \to \mathcal{G}_j|_{U_i\cap U_j}$ for each i, j satisfying the above conditions. Suppose furthermore that we have morphisms $\phi_i : \mathcal{F}_i \to \mathcal{G}_i$ compatible with local isomorphisms. Then we are able to glue local morphisms ϕ_i and to build a global morphism $\phi : \mathcal{F} \to \mathcal{G}$ of sheaves on X.

Exercise 23 (Extension by zero)**.** Let $Z \subset X$ be a closed subset, and $U = X \setminus Z$ be the complement open set. Let $i : Z \hookrightarrow X$, $j : U \hookrightarrow X$ be the inclusions.

(1) Let $\mathcal{F}$ be a sheaf on Z. Show that the stalk of the sheaf $i_*\mathcal{F}$ on X at $P \in X$ is $\mathcal{F}_P$ if $P \in Z$, and 0 otherwise (If there is no confusing, we sometimes omit i_*.)

(2) Let $\mathcal{G}$ be a sheaf on U. Let $j_!\mathcal{G}$ be the sheaf associated to the presheaf $V \mapsto \mathcal{F}(V)$ if $V \subseteq U$, and 0 otherwise. Show that the stalk of the sheaf $j_!\mathcal{G}$ at P is $\mathcal{G}_P$ if $P \in U$, and 0 otherwise. The sheaf $j_!\mathcal{G}$ is called the sheaf obtained by *extending* $\mathcal{G}$ *by zero* outside U.

(3) Show that there is a natural bijection of sets $\mathrm{Hom}_X(j_!\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_U(\mathcal{G}, \mathcal{F}|_U)$ for any sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on U, that is, j^{-1} is a right adjoint of $j_!$ and $j_!$ is a left adjoint of j^{-1}.

3 Sheaves of rings, ringed spaces, and schemes

Even though we are not going to study the scheme theory, it is important to mention that how it plays a significant role in the modern algebraic geometry. A philosophy what we learned is that an affine (resp. projective) variety is determined by its affine(resp. homogeneous) coordinate ring. Suppose that we have an affine variety $X \subset \mathbb{A}^N$. Nullstellensatz implies that we have a correspondence between X and a prime ideal $I(X) = \{f \in k[x_1, \cdots, x_N] \mid f(P) = 0 \text{ for every } P \in X\}$, or equivalently, with an affine coordinate ring $A(X) = k[x_1, \cdots, x_N]/I(X)$. Moreover, a morphism between two affine varieties $\varphi : X \to Y$ corresponds to a ring homomorphism $A(\varphi) : A(Y) \to A(X)$. In particular, the functor $A : X \mapsto A(X)$ induces an equivalence of categories between the category of affine varieties and the opposite category of finitely generated integral domains. Hence, a natural generalization of an affine variety should be a geometric object corresponds to a ring (or a k-algebra) with the functorial property.

Question 24. Let A, B be commutative rings (with unity), $\psi : B \to A$ be a ring homomorphism. Let $I \subset A$ be an ideal.

(1) Is $\psi^{-1}(I)$ an ideal of B?

(2) Suppose that I is prime. Is $\psi^{-1}(I)$ also prime?

(3) Suppose that I is maximal. Is $\psi^{-1}(I)$ also maximal?

(Hint : Consider the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ for instance.)

From the strong Nullstellensatz, the only maximal ideal of an affine variety $X \subset \mathbb{A}^N$ has the form $(x_1 - k_1, \cdots, x_N - k_N)$ for some $k_1, \cdots, k_N \in k$. In other words, X is same as the set of maximal ideals in $k[x_1, \cdots, x_N]$ containing $I(X)$, or equivalently, the set of maximal ideals in $A(X)$. Unfortunately, the set of maximal ideals (called maximal spectrum) does not always have the functorial property. Instead of considering only the maximal ideals, we should also collect the prime ideals in $A(X)$.

Definition 25. As a set, $\operatorname{Spec} A$ is the set of all prime ideals of A. If $I \subsetneq A$ is an ideal of A, we define the subset $V(I) \subseteq \operatorname{Spec} A$ be the set of all prime ideals containing I. For an element $s \in A$, we denote by $D(s)$ the complement of $V((s))$ in $\operatorname{Spec} A$.

The following lemma is elementary but useful.

Lemma 26. *Let A be a ring (= commutative ring with unity).*

(i) If I and J are two ideals of A, then $V(IJ) = V(I) \cup V(J)$.

(ii) If $\{I_i\}$ is any set of ideals of A, then $V(\sum I_i) = \cap V(I_i)$.

(iii) Let I and J be ideals of A. $V(I) \subseteq V(J)$ if and only if $\sqrt{I} \supseteq \sqrt{J}$.

An open subset of $\operatorname{Spec} A$ has the form $\operatorname{Spec}(A) \setminus V(I)$ for some ideal $I \subseteq A$. This makes $\operatorname{Spec} A$ as a topological space.

Lemma 27. *The collection* $\{D(s) \mid s \in A\}$ *forms a basis of* $\operatorname{Spec} A$.

Proof. Let $U = D(I)$ be an open subset of $\operatorname{Spec} A$ for some ideal $I \subseteq A$. Let $\{f_i\}$ be a set of generators of I. For each $f_i \in I$, we have $V((f_i)) \supseteq V(I)$ since $(f_i) \subseteq I$. In particular, $D(f_i) \subseteq D(I)$. Note also that the union $\bigcup_{i \in I} D(f_i) = D(I)$ since $V(I) = \bigcap_{i \in I} V((f_i))$. □

Remark 28. While on proving the lemma, we see that $\bigcup_{i \in I} D(f_i) = D(1) = \operatorname{Spec}(A)$ if and only if $\{f_i\}$ generates the unit ideal $(1) = A$.

We define the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ as follows. Assign $\mathcal{O}_{\operatorname{Spec} A}(D(s)) = A_s$, where A_s is the localization of A by s. To conclude that $\mathcal{O}_{\operatorname{Spec} A}$ is a sheaf, we need to verify that it satisfies a gluing property. Let $t \in A$ be another element. On the intersection $D(s) \cap D(t) = D(st)$, we have an isomorphism on the intersection, followed by a further localization. Since $\operatorname{Spec} A$ is covered by distinguished open subsets $\{D(s) \mid s \in A\}$, they glue together and form a sheaf $\mathcal{O}_{\operatorname{Spec} A}$ on $\operatorname{Spec} A$. In particular, a stalk $\xi_{\mathfrak{p}} \in \mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}}$ is represented by a pair $(D(s), a/s^m)$ for some $a \in A, s \in A \setminus \mathfrak{p}$ and for some integer m. (Do you agree that this looks similar as the definition of regular functions for (affine) varieties?)

Remark 29. Here is a little more detail on the gluing property what we skipped in the above paragraph, sometimes called a "partition of unity". In classical topology, or in differential geometry, we divide the unity function as a sum of functions which are supported locally. For instance, we define the integration over a manifold which naturally extends the integration over Euclidean spaces, or to show the existence of a Riemannian metric. In particular, we are able to extend the notions/properties, which make sense locally, to global notions/properties.
Suppose that there is an open covering $\bigcup_{i \in I} D(f_i)$ of $\operatorname{Spec} A$. Then there is a finite subset, namely, $D(f_1), \cdots, D(f_r)$ such that $\operatorname{Spec} A = \bigcup_{i=1}^r D(f_i)$, that is, $A = (1) = (f_1, \cdots, f_r)$ (quasi-compactness of $\operatorname{Spec} A$). Suppose we have a section $s \in A$ such that $s|_{D(f_i)} = 0 \in A_{f_i}$ for every $1 \leq i \leq r$. In other words, there is an integer m_i such that $f_i^{m_i} s = 0$. Let m be the maximum of m_i. Since $(f_1^m, \cdots, f_r^m) = A$ (Hint: write $1 = \sum_{i=1}^r a_i f_i$ and consider an expression for 1^M with $M \gg 0$), there are $g_i \in A$ such that $\sum_{i=1}^r g_i f_i^m = 1$. Hence,

$$s = \left(\sum_{i=1}^r g_i f_i^m \right) s = \sum_{i=1}^r g_i (f_i^m s) = 0$$

as desired. We may check this property for any distinguished open subset $D(s)$ and open covering by distinguished open subsets of $D(s)$, by replacing A by A_s.
We next check the gluing property. Assume that we have a finite covering by distinguished open subsets $D(f_1), \cdots, D(f_r)$ of $\operatorname{Spec} A$. Suppose that we have elements $s_i = a_i / f_i^{m_i} \in A_{f_i}$ which agree on the intersection $A_{f_i f_j}$. The assumption s_i and s_j agree on the intersection $D(f_i) \cap D(f_j) = D(f_i f_j)$ means that there is an integer m_{ij} such that

$$(f_i f_j)^{m_{ij}} (f_j^{m_j} a_i - f_i^{m_i} a_j) = 0$$

in A. By taking enough large m, we may write it as:

$$(f_i f_j)^m (f_j^{m_j} a_i - f_i^{m_i} a_j) = f_j^{m+m_j}(a_i f_i^m) - f_i^{m+m_i}(a_j f_j^m) = 0.$$

Since $A = (f_1^{m+m_1}, \cdots, f_r^{m+m_r})$, we may represent $1 = \sum_{i=1}^r g_i f_i^{m+m_i}$ for some $g_i \in A$. We define an element $s = \sum_{i=1}^r g_i(a_i f_i^m)$. Note that

$$s f_j^{m+m_j} = \sum_{i=1}^r g_i(a_i f_i^m) f_j^{m+m_j} = \sum_{i=1}^r g_i(a_j f_j^m) f_i^{m+m_i} = a_j f_j^m,$$

in other words, $s = (a_j f_j^m)/(f_j^{m+m_j}) = a_j / f_j^{m_j}$ on $D(f_j)$. The gluing property also holds for an arbitrary infinite covering $\bigcup_{i \in I} D(f_i)$ of $\operatorname{Spec} A$ via an easy observation. First we take a finite subcover $D(f_1), \cdots, D(f_r)$, and an open subset $D(f_c)$ not contained in the subcover we chose. We have two sections $a, a' \in A$, corresponding to finite covers $D(f_1), \cdots, D(f_r)$ and $D(f_1), \cdots, D(f_r), D(f_c)$, respectively. Since the restrictions of a and a' coincide on $D(f_1), \cdots, D(f_r)$, we conclude that $a = a' \in A$ and hence $a|_{D(f_c)} = a'|_{D(f_c)}$, for arbitrary choice of $D(f_c)$. In other words, the construction of the "glued section" a does not depend on the choice of finite subcovers.

Together with the underlying space, we call $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ the *spectrum* of A.

Exercise 30. Let A be a ring, and $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ be its spectrum. Check that

(1) For any $\mathfrak{p} \in \operatorname{Spec} A$, the stalk $\mathcal{O}_\mathfrak{p}$ of $\mathcal{O}_{\operatorname{Spec} A}$ is isomorphic to the local ring $A_\mathfrak{p}$.

(2) The global section $\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is isomorphic to A.

Since the structure sheaf is constructed from the localizations, a morphism between two spectrums $\varphi : (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \to (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$ should contain both data:

(i) set-theoretically, $\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$ is a continuous map between two topological spaces;

(ii) a ring homomorphism $B \to A$ compatible with the localizations.

Definition 31. A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A *morphism of ringed spaces* is a pair $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ where $f : X \to Y$ is a continuous map and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves of rings on Y. A ringed space $(X, \mathcal{O}_X)$ is called a *locally ringed space* if the stalk $\mathcal{O}_{X,P}$ is a local ring for every $P \in X$. A morphism of locally ringed spaces is a morphism of ringed spaces whose induced map $f_P^\# : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a local homomorphism of local rings for every $P \in X$, that is, the preimage of the maximal ideal $\mathfrak{m}_{X,P}$ is the maximal ideal $\mathfrak{m}_{Y,f(P)}$. An *affine scheme* is a locally ringed space which is isomorphic to the spectrum of a ring. A *scheme* is a locally ringed space which is locally isomorphic to an affine scheme.

Proposition 32. *If $\varphi : B \to A$ be a homomorphism of rings, then φ naturally induces a morphism of locally ringed spaces*

$$(f, f^{\#}) : (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \to (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}).$$

Moreover, any morphism of locally ringed spaces from $\operatorname{Spec} A$ *to* $\operatorname{Spec} B$ *is induced by a homomorphism of rings $B \to A$.*

Proof. We define a map $f : \operatorname{Spec} A \to \operatorname{Spec} B$ by $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. If $\mathfrak{b} \subset B$ is an ideal, then the preimage $f^{-1}(V(\mathfrak{b}))$ of a closed subset coincides with a closed subset $V(\varphi(\mathfrak{b})) \subseteq \operatorname{Spec} A$, hence f is continuous. For each open subset $D(s) \subseteq \operatorname{Spec} B$, we have a homomorphism of rings

$$f^{\#} : \mathcal{O}_{\operatorname{Spec} B}(D(s)) = B_s \to \mathcal{O}_{\operatorname{Spec} A}(f^{-1}(D(s))) = \mathcal{O}_{\operatorname{Spec} A}(D(\varphi(s))) = A_{\varphi(s)}$$

by the definition. This is compatible with a further localization, and in particular, with a local homomorphism

$$\varphi_{\mathfrak{p}} : B_{\varphi^{-1}(\mathfrak{p})} \to A_{\mathfrak{p}}$$

obtained by localizing φ at a point $\mathfrak{p} \in \operatorname{Spec} A$. This gives a morphism of sheaves $f^{\#} : \mathcal{O}_{\operatorname{Spec} B} \to f_* O_{\operatorname{Spec} A}$, with the induced map on the stalk at $\mathfrak{p} \in \operatorname{Spec} A$ is just $\varphi_{\mathfrak{p}}$. Conversely, suppose that we have a morphism of locally ringed space $(f, f^{\#})$ from $\operatorname{Spec} B$ to $\operatorname{Spec} A$. Taking the global sections, $f^{\#}$ gives a homomorphism of rings

$$\varphi = \Gamma(\operatorname{Spec} B, f^{\#}) : \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) = B \to \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A.$$

Since $(f, f^{\#})$ is a morphism of locally ringed space, we have a local homomorphism on the stalks $f^{\#}_{\mathfrak{p}} : B_{f(\mathfrak{p})} \to A_{\mathfrak{p}}$ for each $\mathfrak{p} \in \operatorname{Spec} A$ which must be compatible with φ and the localization. We have the following commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & A \\ \downarrow & & \downarrow \\ B_{f(\mathfrak{p})} & \xrightarrow{f^{\#}_{\mathfrak{p}}} & A_{\mathfrak{p}} \end{array}$$

Since $f^{\#}$ is a local homomorphism, $f(\mathfrak{p})$ should coincide with $\varphi^{-1}(\mathfrak{p})$. Hence, the above diagram exactly coincides with the localization of $\varphi : B \to A$ at $\mathfrak{p}$, for any choice of $\mathfrak{p}$. We conclude that the morphism $(f, f^{\#})$ indeed comes from the ring homomorphism $\varphi : B \to A$. □

Now we will define an important class of schemes, analogous to projective varieties. Let $S = \bigoplus S_d$ be the graded ring, $S_+ = \bigoplus_{d>0} S_d$ be the distinguished ideal.

Definition 33. We define $\operatorname{Proj} S$ the set of all homogeneous prime ideals $\mathfrak{p}$ which do not contain S_+. For a homogeneous element $f \in S_+$, we take $D_+(f) = \{\mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p}\}$. The topology of $\operatorname{Proj} S$ is induced by the basis $\{D_+(f)\}_{f \in S_+}$. For each $D_+(f)$, we assign a subring $S_{(f)} \subset S_f$ consisted of degree 0 elements in the localized ring. This induces a sheaf of rings $\mathcal{O}_{\operatorname{Proj} S}$.

Exercise 34. $(D_+(f), \mathcal{O}_{\operatorname{Proj} S}|_{D_+(f)}) \simeq \operatorname{Spec} S_{(f)}$. In particular, $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ is a scheme.

Example 35. (a) An affine scheme $\operatorname{Spec} A[x_1, \cdots, x_n]$ is called an affine n-space over a ring A. When A is an algebraically closed field, then the set of closed points in $\operatorname{Spec} A[x_1, \cdots, x_n]$ identifies naturally to the affine space $\mathbb{A}^n_A$.

(b) A scheme $\operatorname{Proj} A[x_0, \cdots, x_n]$ is called a projective n-space over a ring A. When A is an algebraically closed field, then the set of closed points in $\operatorname{Proj} A[x_0, \cdots, x_n]$ identifies naturally to the projective space $\mathbb{P}^n_A$.

(c) (Coordinate-free description) Let V be an $(n+1)$-dimensional vector space over a field k. We define the *projectivization* of V, denoted by $\mathbb{P}V$, as $\operatorname{Proj}(\operatorname{Sym}^\bullet V^\vee)$ where

$$\operatorname{Sym}^\bullet V^\vee = k \oplus V^\vee \oplus \operatorname{Sym}^2 V^\vee \oplus \cdots$$

is the symmetric algebra of the dual of V. In particular, if we take $\{x_0, \cdots, x_n\}$ as a basis for $V^\vee$, then $\operatorname{Sym}^\bullet V^\vee = k[x_0, \cdots, x_n]$ coincides with the usual polynomial ring.

(d) Let A be a ring, $I \subset A$ be an ideal. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} A/I$. The natural ring homomorphism $A \to A/I$ induces a morphism of schemes $i : Y \hookrightarrow X$ which is a *closed immersion* (that is, it gives a homeomorphism onto a closed subset of the topological space X, and the induced map $i^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective). The map i is a homeomorphism of Y onto the closed subset $V(I) \subseteq X$ of X, and the induced map $i^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective since it is surjective on stalks, which are localizations of A and A/I.

Exercise 36. Describe affine schemes $\operatorname{Spec} \mathbb{Z}, \operatorname{Spec} \mathbb{R}[x], \operatorname{Spec} \mathbb{C}[x, y]$ as topological spaces.

Exercise 37. Show that the projective space $\mathbb{P}^n_k$ $(n > 0)$ over a field k is not affine. (Hint : what is the global section of the structure sheaf $\mathcal{O}_{\mathbb{P}^n_k}$?)