4 Sheaves of modules, vector bundles, and (quasi-)coherent sheaves

> *"If you believe a ring can be understood geometrically as functions its spectrum, then modules help you by providing more functions with which to measure and characterize its spectrum."* – Andrew Critch, from MathOverflow.net

So far we discussed general properties of sheaves, in particular, of rings. Similar as in the module theory in abstract algebra, the notion of sheaves of modules allows us to increase our understanding of a given ringed space (or a scheme), and to provide further techniques to play with functions, or function-like objects. There are particularly important notions, namely, quasi-coherent and coherent sheaves. They are analogous notions of the usual modules (respectively, finitely generated modules) over a given ring. They also generalize the notion of vector bundles.

Definition 38. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-modules*, or simply an $\mathcal{O}_X$-module, is a sheaf $\mathcal{F}$ on X such that

(i) the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module for each open set $U \subseteq X$;

(ii) the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structure via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

A morphism $\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is a morphism of sheaves such that the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-module homomorphism for every open $U \subseteq X$.

Example 39. Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism. Then $\ker \varphi, \operatorname{im} \varphi, \operatorname{coker} \varphi$ are again $\mathcal{O}_X$-modules. If $\mathcal{F}' \subseteq \mathcal{F}$ is an $\mathcal{O}_X$-submodule, then the quotient sheaf $\mathcal{F}/\mathcal{F}'$ is an $\mathcal{O}_X$-module. Any direct sum, direct product, direct limit, or inverse limit of $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module.

Definition 40. Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules. We denote the group of morphisms from $\mathcal{F}$ to $\mathcal{G}$ by $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ (or $\operatorname{Hom}_X(\mathcal{F}, \mathcal{G})$ or $\operatorname{Hom}(\mathcal{F}, \mathcal{G})$ if there is no confusing).
If $U \subseteq X$ is an open subset, then $\mathcal{F}|_U$ is an $\mathcal{O}_X|_U$-module. The presheaf

$$U \mapsto \operatorname{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$$

is indeed a sheaf, and we will call it the *sheaf Hom*. We denote it by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.
The sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ is called the *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. If there is no confusing, we write simply $\mathcal{F} \otimes \mathcal{G}$.

Definition 41. An $\mathcal{O}_X$-module $\mathcal{F}$ is *free* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is *locally free* if X can be covered by open subsets U for which $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module. In the case, the *rank* of $\mathcal{F}$ is the number of free copies of the structure sheaf needed. Note that the rank of a locally free sheaf is the same everywhere when X is connected. A locally free sheaf of rank 1 is called an *invertible sheaf.*
A *sheaf of ideals* on X is a sheaf of modules $\mathscr{I}$ which is a subsheaf of $\mathcal{O}_X$, that is, $\mathscr{I}(U)$ is an ideal in $\mathcal{O}_X(U)$ for each open set $U \subseteq X$.

Example 42. Let A be a ring, and $I \subset A$ be an ideal. By the definition, the set $V(I) \subseteq X = \operatorname{Spec} A$ is a closed subset. We identify $V(I)$ as an affine scheme $Y = \operatorname{Spec} A/I$, so that we have a short exact sequence of sheaves of $\mathcal{O}_X$-modules

$$0 \to \mathscr{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

which is the natural analogue of $0 \to I \to A \to A/I \to 0$.
Indeed, if there is a closed subscheme $Y \overset{f}{\hookrightarrow} X$, then the map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ induces a surjection on its global sections $A \to B$. In other words, $B \simeq A/I$ for some ideal I in A. It is straightforward that the quotient $A \to A/I$ is compatible with every localization, hence, we may cover Y by $D(f_i) \cap Y$ where $\{D(f_i)\}$ is a collection of distinguished open subsets of $X = \operatorname{Spec} A$. In particular, Y has to coincide with the affine scheme $\operatorname{Spec} A/I$. The ideal sheaf $\mathscr{I}$ is defined from local data on distinguished open sets. Namely, if $U = D(s)$ where $s \in A$, then we assign $\mathscr{I}(U) := I_s \subseteq A_s$.

Definition 43. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and let $\mathcal{G}$ be an $\mathcal{O}_Y$-module.
Note that the direct image $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module. Together with the morphism $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y, we have a natural $\mathcal{O}_Y$-module structure on $f_*\mathcal{F}$. We call it the *direct image* of $\mathcal{F}$, or the *pushforward* of $\mathcal{F}$ by f.
Also note that the inverse image sheaf $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module. Thanks to the adjoint property of f^{-1}, there is a unique morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ which corresponds $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$. To provide an $\mathcal{O}_X$-module structure, we take the tensor product

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

and call it the *inverse image* of $\mathcal{G}$, or the *pullback* of $\mathcal{G}$ by f.

Remark 44. f_* and f^* are adjoint functors between the category of $\mathcal{O}_X$-modules and the category of $\mathcal{O}_Y$-modules.

If we restrict our focus to the case of schemes, we already have enough general notions and give the definition of quasi-coherent and coherent sheaves on schemes. For instance, on an affine scheme $\operatorname{Spec} A$, a quasi-coherent (resp., coherent) sheaf is completely determined by an A-module (resp., a finitely generated A-module) M. If there is an A-module M, then it is compatible with every localization of A. In particular, for each $s \in A$, the localized module M_s is an A_s-module and the module structure is compatible with the localization. This enables us to build a sheaf of $\mathcal{O}_{\operatorname{Spec} A}$-module. However, before we proceed forward, let us recall the notion of vector bundles and see why it is natural to consider quasi-coherent and coherent sheaves instead of vector bundles.
Let X be a real topological manifold. A vector bundle $\mathcal{E}$ of rank r on X is defined as a map $\pi : \mathcal{E} \to X$ such that for every $x \in X$, there is an open neighborhood U of x and a homeomorphism (also called a local trivialization) $\phi_U : U \times \mathbb{R}^r \to \pi^{-1}(U)$ satisfying $(\pi \circ \phi_U)(x, v) = x$ for any $x \in U$ and $v \in \mathbb{R}^r$. In particular, the fiber $\pi^{-1}(x) \simeq \mathbb{R}^r$ has the structure of a vector space. A rank 1 vector bundle is called a *line bundle*.

When we have two trivializations over U_1 and U_2, we have an isomorphism on the intersection $\phi_{U_2}^{-1} \circ \phi_{U_1} : (U_1 \cap U_2) \times \mathbb{R}^r \to (U_1 \cap U_2) \times \mathbb{R}^r$ which sends (x, v) to $(x, T_{12}(x)v)$ for some $GL(r)$-valued function $T_{12} : U_1 \cap U_2 \to GL(r) = \mathrm{Aut}(\mathbb{R}^r)$. When U_3 is another open subset of X with a local trivialization of $\mathcal{E}$, then they satisfy the cocycle condition

$$T_{23}|_{U_1 \cap U_2 \cap U_3} \circ T_{12}|_{U_1 \cap U_2 \cap U_3} = T_{13}|_{U_1 \cap U_2 \cap U_3}.$$

In fact, an open cover $\{U_i\}$ of X with local trivializations ϕ_{U_i} for each i and transition functions $\{T_{ij}\}$ recover the vector bundle $\pi : \mathcal{E} \to X$ by "gluing together" the various $U_i \times \mathbb{R}^r$ along $U_i \cap U_j$ using T_{ij}. (Compare with the gluing property of sheaves.)
Now we associate a sheaf to a vector bundle $\pi : \mathcal{E} \to X$, called the *sheaf of sections*. Note that a section of a vector bundle $\mathcal{E}$ on an open subset $U \subseteq X$ is a function $s : U \to \mathcal{E}$ such that $\pi \circ s = id_U$. Hence, if we take $\mathcal{O}_X$ as the sheaf of continuous functions on X and regard $(X, \mathcal{O}_X)$ as a ringed space, then the sheaf $\mathcal{F}$

$$U \mapsto \mathcal{F}(U) := \{ s : U \to \mathcal{E} \mid \text{ sections of } \pi \text{ over } U \ \}$$

has an $\mathcal{O}_X$-module structure. In particular, when π admits a local trivialization over U, the sections over U naturally identify with the r-tuples of functions $U \to \mathbb{R}$, and we conclude that $\mathcal{F}|_U \simeq \mathcal{O}_X|_U{}^{\oplus r}$. In other words, a vector bundle of rank r gives a locally free sheaf of rank r on X.
Let us write down a little more detail inside the above construction. Let $U \subset X$ be an open set, and let $\{U_i\}$ is an open covering of U by open subsets U_i equipped with trivializations. Let $s : U \to \mathcal{E}$ be a section over U. The restriction $s|_{U_i} : U_i \to \pi^{-1}(U_i)$ onto each subset U_i induces a vector-valued function $\vec{s_i} : U_i \to \mathbb{R}^r$, via the local trivialization $\phi_{U_i}^{-1} : \pi^{-1}(U_i) \to U \times \mathbb{R}^r$ together with the second projection. Then we have $T_{ij}\vec{s_i} = \vec{s_j}$: chase the following diagram

$$\begin{array}{ccccccc}
U_i \cap U_j & \xrightarrow{s|_{U_i \cap U_j}} & \pi^{-1}(U_i \cap U_j) & \xrightarrow{\phi_{U_i}^{-1}} & (U_i \cap U_j) \times \mathbb{R}^r & \xrightarrow{pr_2} & \mathbb{R}^r \\
\| & & \| & & \downarrow{\scriptstyle id_{U_i \cap U_j} \times T_{ij}} & & \downarrow{\scriptstyle T_{ij}} \\
U_i \cap U_j & \xrightarrow{s|_{U_i \cap U_j}} & \pi^{-1}(U_i \cap U_j) & \xrightarrow{\phi_{U_j}^{-1}} & (U_i \cap U_j) \times \mathbb{R}^r & \xrightarrow{pr_2} & \mathbb{R}^r
\end{array}$$

where the composition of the horizontal maps in the top row describes $\vec{s_i}$, and the one for the bottom row describes $\vec{s_j}$. Hence, the locally free sheaf $\mathcal{F}$ of sections of $\pi : \mathcal{E} \to X$ provides the same transition functions $T_{ij} \in GL(r, \mathcal{O}(U_i \cap U_j))$, and of course, they satisfy the cocycle condition. Thus, we conclude that the data of a locally free sheaf $\mathcal{F}$ of rank n equal to the data of a vector bundle $\pi : \mathcal{E} \to X$.

Exercise 45. Let $\pi : \mathcal{E} \to X$ be a vector bundle of rank r on a manifold X, and let $\mathcal{F}$ be the sheaf of sections of $\mathcal{E}$. What is the espace étalé $\mathrm{Spé}(\mathcal{F}) \to X$?

Exercise 46. Let X be a smooth manifold, and let $\mathcal{O}_X$ be the sheaf of differentiable functions on X. Describe a vector bundle on X corresponding to $\mathcal{O}_X$.
Let $\mathcal{T}_X$ be a tangent bundle on X. Describe the locally free sheaf corresponding to $\mathcal{T}_X$. What is the stalk $\mathcal{T}_{X,x}$ at a point $x \in X$?

Definition 47. Let $\mathcal{E}$ be a locally free sheaf on a ringed space $(X, \mathcal{O}_X)$. The sheaf $\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is called the *dual* of $\mathcal{E}$.

Exercise 48. Let $\mathcal{E}, \mathcal{F}$ be locally free sheaves on a ringed space $(X, \mathcal{O}_X)$. Show that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ and $\mathcal{E} \otimes \mathcal{F}$ are locally free sheaves. What are their ranks?

Exercise 49. Let $\mathcal{F}$ be an invertible sheaf on $(X, \mathcal{O}_X)$. Show that $\mathcal{F} \otimes \mathcal{F}^\vee \simeq \mathcal{O}_X$.

Proposition 50. *Let $\mathcal{E}$ be a locally free sheaf of finite rank, and let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. Then $\mathcal{H}om(\mathcal{F}, \mathcal{E} \otimes \mathcal{G}) \simeq \mathcal{H}om(\mathcal{E}^\vee \otimes \mathcal{F}, \mathcal{G})$.*

Proof. We just leave a sketch of proof. Let $U \subseteq X$ be an open subset which is small enough, in particular,

- $\mathcal{E}(U)$ and $\mathcal{E}^\vee(U)$ are free $\mathcal{O}_X(U)$-modules of rank r;
- $(\mathcal{E} \otimes \mathcal{G})(U) = \mathcal{E}(U) \otimes \mathcal{G}(U)$;
- $(\mathcal{E}^\vee \otimes \mathcal{F})(U) = \mathcal{E}(U) \otimes \mathcal{F}(U)$.

Let $\{e_1, \cdots, e_r\}$ be a free basis of $\mathcal{E}(U)$, and let $\{f_1, \cdots, f_r\}$ be the dual basis of $\mathcal{E}^\vee(U)$. Let $\varphi \in \mathcal{H}om(\mathcal{F}, \mathcal{E} \otimes \mathcal{G})(U)$. It sends a section $s \in \mathcal{F}(U)$ to $\varphi(s) = \sum_{i=1}^r e_i \otimes g_i \in \mathcal{E}(U) \otimes \mathcal{G}(U)$. We assign $\psi \in \mathcal{H}om(\mathcal{E}^\vee \otimes \mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}(\mathcal{E}^\vee(U) \otimes \mathcal{F}(U), \mathcal{G}(U))$ as follows:

> We first assign $\psi(f_i \otimes s) := g_i$ for $i = 1, \cdots, r$, where $s \in \mathcal{F}(U)$ and $\varphi(s) = \sum_{i=1}^r e_i \otimes g_i$. Since $\{f_1, \cdots, f_r\}$ generates a free module $\mathcal{E}^\vee(U)$, we can naturally extend ψ to a homomorphism of $\mathcal{O}_X(U)$-modules.

One can show that the mapping $\varphi \mapsto \psi$ induces an isomorphism of $\mathcal{O}_X$-modules. □