The lemma below, called the projection formula, is very useful to compute a pushforward.

Lemma 51 (Projection formula)**.** *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Then there is a natural isomorphism*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \simeq f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}.$$

Proof. Since $f^*(f_*\mathcal{F} \otimes \mathcal{E}) \simeq f^*f_*\mathcal{F} \otimes f^*\mathcal{E}$, there is a natural morphism $f^*f_*\mathcal{F} \otimes f^*\mathcal{E} \to \mathcal{F} \otimes f^*\mathcal{E}$. Since f^* and f_* are adjoint to each other, there is a corresponding natural morphism $\varphi : f_*\mathcal{F} \otimes \mathcal{E} \to f_*(\mathcal{F} \otimes f^*\mathcal{E})$. Note that being φ an isomorphism is a local property, hence, we may replace Y by a sufficiently small open subset. Since f_* and the tensor product commutes with the direct sum, we may assume furthermore that $\mathcal{E} = \mathcal{O}_Y$. The conclusion follows from the fact that $f^*\mathcal{O}_Y = f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X = \mathcal{O}_X$. □

Note that the category of $\mathcal{O}_X$-modules is an abelian category: we can talk about kernel, cokernel, and so on. In particular, we are able to consider (cochain) complexes of $\mathcal{O}_X$-modules, and do homological algebra. Fiberwisely, vector bundles on $\mathcal{O}_X$ give vector spaces; and similarly, vector spaces also form an abelian category. However, when we focus on their global nature (= vector bundles themselves, or equivalently, locally free sheaves), they do not form an abelian category.

Example 52. Let $X = \mathbb{P}^1 = \operatorname{Proj} \mathbb{C}[x, y]$, and let $Z = V(x) \subset X$ be a closed subset of X. The ideal sheaf $\mathscr{I}_Z$ is indeed locally free:

> On an affine open subset $U_x \simeq \operatorname{Spec} \mathbb{C}[y/x]$, which corresponds to the set of points $[x : y] \in \mathbb{P}^1$ where $x \neq 0$, we have $\mathscr{I}_Z(U_x) = \mathbb{C}[y/x]$ since there is no condition. On the other hand, on an affine open subset $U_y \simeq \operatorname{Spec} \mathbb{C}[x/y]$, which corresponds to the set of points $[x : y] \in \mathbb{P}^1$ where $y \neq 0$, an element of $\mathscr{I}_Z(U_y)$ is a regular function on $\mathbb{P}^1 \setminus \{[1 : 0]\}$ which has zero at $Z = [0 : 1]$. In particular, $\mathscr{I}_Z(U_y) \simeq \frac{x}{y} \cdot \mathbb{C}[x/y]$. We conclude that $\mathscr{I}_Z$ is locally free, since it can be covered by open (affine) subsets such that on $\mathscr{I}_Z$ is free on each open subset.

We have a natural inclusion $\mathscr{I}_Z \to \mathcal{O}_X$, whose cokernel is isomorphic to $\mathcal{O}_Z$ which is not a locally free sheaf on $\mathcal{O}_X$.

Hence, it is natural to extend the category of locally free sheaves by adding more objects so that the cokernels of maps of locally free sheaves appear as objects in the enlarged category. In most places, we are in the (locally) Noetherian setting, so that the finite rank case will behave much better. Therefore, it is also reasonable to extend the category of locally free sheaves of finite rank in this manner. In general, there are three finiteness conditions for an A-module M:

(1) *finitely generated*, there is a surjection $A^{\oplus n} \to M \to 0$ for some integer n;

(2) *finitely presented*, so that the finite number of generators have only finitely many relations, *i.e.*, there is a right exact sequence $A^{\oplus m} \to A^{\oplus n} \to M \to 0$ for some integer m, n;

(3) *coherent*, M is finitely generated, and any homomorphism $A^{\oplus n} \to M$ (not necessarily surjective) has the finitely generated kernel.

When A is a Noetherian ring, all these conditions are equivalent. Since we will only work over Noetherian schemes (= schemes which can be covered by a finite number of open affine subschemes $\operatorname{Spec} A_i$ where A_i are Noetherian), or even better: affine/projective varieties over a field (of finite dimension), we do not need to distinguish them.

Definition 53. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is *quasi-coherent* it has a local presentation, that is, for every point $x \in X$, there is an open neighborhood $U \subseteq X$ and a right exact sequence

$$\mathcal{O}_X|_U{}^{\oplus I} \to \mathcal{O}_X|_U{}^{\oplus J} \to \mathcal{F}|_U \to 0$$

for some (possibly infinite) set I and J. In other words, a quasi-coherent sheaf is locally a cokernel of (locally) free sheaves.
An $\mathcal{O}_X$-module $\mathcal{F}$ is *coherent* if it is quasi-coherent, and locally coherent, that is, it satisfies the following properties:

(i) $\mathcal{F}$ is of finite type, *i.e.*, for every point $x \in X$, there is an open neighborhood $U \subseteq X$ and a surjection $\mathcal{O}_X|_U{}^{\oplus n} \to \mathcal{F}|_U \to 0$ for some integer n;

(ii) for any open subset $U \subseteq X$, any integer n, and any morphism $\varphi : \mathcal{O}_X|_U{}^{\oplus n} \to \mathcal{F}|_U$, the kernel $\ker \varphi$ is of finite type.

As we discussed above, the notion becomes much simpler for noetherian schemes. We start by defining the sheaf of modules $\widetilde{M}$ on $\operatorname{Spec} A$ associated to an A-module M.

Definition 54. Let A be a ring and let M be an A-module. The *sheaf associated to M*, denoted by $\widetilde{M}$, is defined by

$$D(s) \mapsto M_s$$

for each $s \in A$ and the distinguished open subset $D(s) \subseteq \operatorname{Spec} A$, together with the natural restriction maps.

In particular, the stalk at $\mathfrak{p} \in \operatorname{Spec} A$ is isomorphic to the localized module $(\widetilde{M})_{\mathfrak{p}} = M_{\mathfrak{p}}$, and $\Gamma(\operatorname{Spec} A, \widetilde{M}) = M$.
We first see that the map $M \mapsto \widetilde{M}$ is functorial:

Proposition 55. *Let A, B be rings, $\varphi : B \to A$ be a ring homomorphism, and let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be the corresponding morphism of spectra. Then,*

(1) the map $M \mapsto \widetilde{M}$ gives an exact, fully faithful functor from the category of A-modules to the category of $\mathcal{O}_{\operatorname{Spec} A}$-modules;

(2) if M and N are two A-modules, then $(M \otimes_A N)^\sim \simeq \widetilde{M} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \widetilde{N}$;

(3) if $\{M_i\}$ a family of A-modules, then $(\oplus M_i)^\sim \simeq \oplus \widetilde{M_i}$;

(4) for any A-module M, we have $f_*(\widetilde{M}) \simeq ({}_B M)^\sim$, where ${}_B M$ means M considered as B-modules;

(5) for any B-module N, we have $f^*(\widetilde{N}) \simeq (N \otimes_B A)^\sim$.

Proof. Note that the map $M \mapsto \widetilde{M}$ gives a functor since a homomorphism of A-modules $M \to N$ commutes with the localization $M_s \to N_s$ for each $s \in A$. It is exact, since it commutes with every localization at a prime ideal $\mathfrak{p} \subset A$, and the exactness can be measured at the stalks. It is also fully faithful:

- the functor $^\sim$ gives a natural map $\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec} A}}(\widetilde{M}, \widetilde{N})$;
- the global section functor $\Gamma(\operatorname{Spec} A, -)$ gives the inverse map $\operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec} A}}(\widetilde{M}, \widetilde{N}) \to \operatorname{Hom}_A(M, N)$.

In particular, we embed the category of A-modules into the category of $\mathcal{O}_{\operatorname{Spec} A}$-modules. All the other statement follow from the definitions and statements in terms of modules. □

Definition 56. Let $(X, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is *quasi-coherent* if X can be covered by affine open subsets $U_i = \operatorname{Spec} A_i$ such that $\mathcal{F}|_{U_i} \simeq \widetilde{M_i}$ for some A_i-module M_i for each i. $\mathcal{F}$ is *coherent* if furthermore each M_i can be taken to be a finitely generated A_i-module.

Example 57. (1) Let X be any scheme. The structure sheaf $\mathcal{O}_X$ is quasi-coherent and coherent.

(2) Let $X = \operatorname{Spec} A$ be an affine scheme, $Y = \operatorname{Spec} A/I$ be the closed subscheme of X defined by an ideal $I \subseteq A$, together with a closed immersion $i : Y \to X$. The sheaf $i_*\mathcal{O}_Y$ is a quasi-coherent (and coherent) $\mathcal{O}_X$-module, which is isomorphic to $\widetilde{(A/I)}$.

(3) Let Y be a closed subscheme of a scheme X with the inclusion $i : Y \hookrightarrow X$. The sheaf $\mathcal{O}_X|_Y = i^{-1}\mathcal{O}_X$ is not a quasi-coherent sheaf on Y in general. In fact, it needs not to be an $\mathcal{O}_Y$-module.

(4) Let $X = \mathbb{A}^n = \operatorname{Spec} k[x_1, \cdots, x_n]$. Let $\mathcal{K}$ be the constant sheaf associated to the group $K = k(x_1, \cdots, x_n)$ which is the function field (= the field of rational functions) of X. Then $\mathcal{K}$ is a quasi-coherent $\mathcal{O}_X$-module, but not coherent unless $n = 0$.

Maybe the following proposition seems to be too strong, and in fact, we have to check carefully that their global property comes from their localizations. The idea of proof is almost the same as using the "partition of unity" when we define the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ of a spectrum, so we will skip its proof.

Proposition 58. *Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if for every affine open subset $U = \operatorname{Spec} A$ of X, there is an A-module M such that $\mathcal{F}|_U \simeq \widetilde{M}$. If X is noetherian, then $\mathcal{F}$ is coherent if and only if furthermore we can take M as a finitely generated A-module.*

We mentioned several times that the global section functor $X \mapsto \Gamma(X, -)$ is left exact. The sheaf cohomology we will see later measures how $\Gamma(X, -)$ is different from the exact functor. One distinguished example we seen is a flasque sheaf, that is, a sheaf such that every restriction map is surjective. When $\mathcal{F}'$ is flasque, and we have a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, then we check that the induced maps on global sections again form a short exact sequence

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0.$$

Naively speaking, a flasque sheaf forces that the global section functor becomes "exact". Hence, it is the "simplest" sheaf in this viewpoint. We also give a remark that the affine schemes and quasi-coherent sheaves are "simple", following a similar manner. In other words, they will play a role of basic building blocks.

Proposition 59. *Let $X = \operatorname{Spec} A$ be an affine scheme, and let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. Assume that $\mathcal{F}'$ is quasi-coherent. Then the sequence*

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$$

is exact.

Proof. Only need to check that the last map is surjective. Let $s'' \in \Gamma(X, \mathcal{F}'')$. Since $\mathcal{F} \to \mathcal{F}''$ is surjective, for any point $x \in X$ there is an open neighborhood $D(f) \subseteq X$ such that $s''|_{D(f)}$ lifts to a section $s \in \mathcal{F}(D(f))$. Choose one and fix $f \in A$. We first claim that a multiple $f^N s''$ is the image of a global section $\zeta \in \Gamma(X, \mathcal{F})$ for some $N > 0$. Since $X = \operatorname{Spec} A$ is quasi-compact, we may cover X with a finitely many $D(g_i)$'s such that for each i, $s''|_{D(g_i)}$ lifts to a section $s_i \in \mathcal{F}(D(g_i))$. On the intersection $D(f) \cap D(g_i) = D(fg_i)$, both sections $s|_{D(fg_i)}$ and $s_i|_{D(fg_i)}$ map to the same $s''|_{D(fg_i)}$. In particular, $s'_i := (s|_{D(fg_i)} - s_i|_{D(fg_i)}) \in \mathcal{F}'(D(fg_i))$ if we regard $\mathcal{F}'$ as a subsheaf of $\mathcal{F}$.
Now $\mathcal{F}'$ is quasi-coherent, there is an A-module M' such that $\mathcal{F}' = \widetilde{M'}$. In particular, $\mathcal{F}'(D(fg_i)) = M'_{fg_i}$. Hence, s'_i is represented as a quotient $t_i/(fg_i)^{n_i}$ where $t_i \in M'$. Thus a section obtained by a multiplication $f^{n_i} s'_i \in M'_{fg_i}$ extends to a section $t_i/(g_i)^{n_i} \in \mathcal{F}'(D(g_i)) = M'_{g_i}$. By taking a large enough n and a suitable choice of t_i, we may assume that $f^n s'_i$ extends to a section $t_i/g_i^n \in \mathcal{F}'(D(g_i))$. Let $\xi_i := f^n s_i + t_i/g_i^n \in \mathcal{F}(D(g_i))$. Note that on $D(fg_i)$ we see that the two sections $\xi_i = f^n(s_i + t_i/(fg_i)^n) = f^n(s_i + s'_i) = f^n s$ coincide.
On every intersection $D(g_i) \cap D(g_j) = D(g_i g_j)$, note that the images of ξ_i and ξ_j in $\mathcal{F}''$ are same as $f^n s''$, since each t_i/g_i^n part will map to 0, and s_i will maps to s'' on $D(g_i)$. In other words, $\xi_i|_{D(g_i g_j)} - \xi_j|_{D(g_i g_j)} \in \mathcal{F}'(D(g_i g_j)) = M'_{g_i g_j}$. Since ξ_i and ξ_j are equal (to $f^n s|_{D(fg_i g_j)}$) on $D(fg_i g_j)$, it means that $f^{m_{ij}}(\xi_i|_{D(g_i g_j)} - \xi_j|_{D(g_i g_j)}) = 0$ for some $m_{ij} > 0$

when we regard it as an element of the module $M'_{g_ig_j}$. We also take m large enough if necessary, we may assume that the sections $f^m\xi_i$ and $f^m\xi_j$ coincide on the intersection $D(g_ig_j)$. The sections $f^m\xi_i$ will glue together, and form a global section $\zeta \in \Gamma(X, \mathcal{F})$. Its image in $\mathcal{F}''$ equals to $f^{n+m}s''$.
Now we cover X by a finitely many $f_1, \cdots, f_r$ so that $s''|_D(f_i)$ lifts to a section of $\mathcal{F}$ over $D(f_i)$ for each i. By the claim, we can find a single integer $N > 0$ and global sections $\zeta_i \in \Gamma(X, \mathcal{F})$ such that ζ_i is a lifting of $f_i^N s''$. Since the open sets $D(f_i)$ cover X, in particular, the ideal $(f_1^N, \cdots, f_r^N) = (1)$ is the unit ideal. We write $1 = \sum_{i=1}^r a_i f_i^N$ for some $a_i \in A$, and we define $\zeta = \sum_{i=1}^r a_i\zeta_i \in \Gamma(X, \mathcal{F})$. Then the image of ζ in $\mathcal{F}''$ equals to $\sum_{i=1}^r a_i f_i^N s'' = s''$, as desired. □

We check that the category of quasi-coherent (respectively, coherent) $\mathcal{O}_X$-modules is good enough to do homological algebra.

Proposition 60. *Let X be a noetherian scheme. The kernel, image, and cokernel of any morphism of quasi-coherent (resp., coherent) sheaves are quasi-coherent (resp., coherent). Any extension of quasi-coherent (resp., coherent) sheaves are quasi-coherent (resp., coherent).*

Proof. The question is local, that is, it is enough to show that the property holds for (sufficiently small) open subsets. In particular, we may assume that $X = \operatorname{Spec} A$ is affine, and A is noetherian. The kernel, image, and cokernel can be successfully translated in terms of A-modules.
Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of $\mathcal{O}_X$-modules, where $\mathcal{F}'$ and $\mathcal{F}''$ are quasi-coherent (resp., coherent). Since the induced map on global sections $0 \to M' := \Gamma(X, \mathcal{F}') \to M := \Gamma(X, \mathcal{F}) \to M'' := \Gamma(X, \mathcal{F}'') \to 0$ is exact, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{M'} & \longrightarrow & \widetilde{M} & \longrightarrow & \widetilde{M''} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0. \end{array}$$

Now the 5-lemma implies that the middle arrow is also an isomorphism. □

We also leave a statement for pushforwards and pullbacks without a proof:

Proposition 61. *Let $f : X \to Y$ be a morphism of noetherian schemes.*

(1) If $\mathcal{G}$ is a quasi-coherent (resp., coherent) $\mathcal{O}_Y$-module, then the pullback $f^*\mathcal{G}$ is a quasi-coherent (resp., coherent) $\mathcal{O}_X$-module.

(2) If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then the pushforward $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module.

(3) When f is projective(e.g., both X and Y are projective varieties), a pushforward $f_*\mathcal{F}$ of a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module.

Example 62. When $X = \operatorname{Spec} A$ is an affine scheme, we discussed that the affine schemes of the form $Y = \operatorname{Spec} A/I$ for some ideal $I \subset A$ are the closed subschemes of X. In particular, the morphism $i^{\#} : \mathcal{O}_X \to i_*\mathcal{O}_Y$ associated to the inclusion $i : Y \hookrightarrow X$ is equivalent to the data consisted of the ring quotient map $A \to A/I$ and its localizations. In general, by patching local affine open pieces, a closed subscheme Y of a scheme X is defined as a morphism $i : Y \hookrightarrow X$ of locally ringed spaces such that the morphism of sheaves $i^{\#} : \mathcal{O}_X \to i_*\mathcal{O}_Y$ is surjective. The kernel of $i^{\#}$, denoted by $\mathscr{I}_Y$, is called the *ideal sheaf* of Y (Check: compare with the definition of an ideal sheaf in previous lectures). When X is a noetherian scheme, any ideal sheaf is coherent. Conversely, any (quasi-)coherent sheaf of ideals on X is the ideal sheaf of a uniquely determined closed subscheme of X.

Example 63. Let X be a scheme, $\mathcal{E}$ be a coherent sheaf on X. As we seen above, $\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is called the *dual* of $\mathcal{E}$. It is easy to check that $\mathcal{E}^\vee$ is also a coherent sheaf on X.
One can check that there is a natural morphism to its double dual $\mathcal{E} \to (\mathcal{E}^\vee)^\vee$. If $\mathcal{E}$ is a vector bundle of finite rank, it is an isomorphism. In general, it needs not to be an isomorphism. For instance, let $X = \operatorname{Spec} k[x]$, $\mathcal{E}$ is a sheaf associated to the $k[x]$-module $k \simeq k[x]/(x)$. Since there is no nontrivial $k[x]$-module homomorphism from $k[x]/(x) \to k[x]$, the dual $\mathcal{E}^\vee$ is zero. A coherent sheaf $\mathcal{E}$ for which the double dual morphism is an isomorphism is called a *reflexive sheaf.* A locally free sheaf of finite rank is immediately reflexive, however, the converse is not true in general. The canonical map $\mathcal{E} \otimes \mathcal{E}^\vee \to \mathcal{O}_X$ is called the *trace map.*

Exercise 64 (Geometric Nakayama's lemma)**.** Let X be a scheme, $\mathcal{F}$ be a coherent sheaf. Let $U \subset X$ be an open neighborhood of a point $P \in X$, and let $s_1, \cdots, s_n \in \mathcal{F}(U)$ be sections such that their images $s_1|_P, \cdots, s_n|_P \in \mathcal{F}|_P := \mathcal{F}_P \otimes (\mathcal{O}_{X,P}/\mathfrak{m}_{X,P})$ generate the geometric fiber $\mathcal{F}|_P$. Show that there is an affine open neighborhood $P \in \operatorname{Spec} A \subseteq U$ such that $s_1|_{\operatorname{Spec} A}, \cdots, s_n|_{\operatorname{Spec} A}$ generate $\mathcal{F}|_{\operatorname{Spec} A}$ in the following sense:

(i) $s_1|_{\operatorname{Spec} A}, \cdots, s_n|_{\operatorname{Spec} A}$ generate $\mathcal{F}|_{\operatorname{Spec} A}$ as an A-module;

(ii) for any $Q \in \operatorname{Spec} A$, $s_1, \cdots, s_n$ generate the stalk $\mathcal{F}_Q$ as an $\mathcal{O}_{X,Q}$-module.

As a special case, we want to associate a (quasi-)coherent sheaf on $\operatorname{Proj} S$ from a graded S-module over a graded ring S. As in the case of Spec, it comes from a module structure over a ring, however, we should choose only degree 0 elements as similar as we construct $\operatorname{Proj} S$.

Definition 65. Let S be a graded ring and let M be a graded S-module. We define the sheaf associated to M on $\operatorname{Proj} S$, denoted by $\widetilde{M}$, by assigning $\widetilde{M}(D_+(f)) = M_{(f)}$, where $M_{(f)} \subseteq M_f$ denotes the group of degree 0 elements in the localized module M_f. One can check that $\widetilde{M}$ is a quasi-coherent $\mathcal{O}_{\operatorname{Proj} S}$-module. If S is noetherian and M is finitely generated, then $\widetilde{M}$ is coherent.

On $\operatorname{Proj} S$, we have the notion of "twisting", which does not change the ring S itself but translate only the grading structure on S.

Definition 66. Let $S = \oplus S_d$ be a graded ring, and let M be any graded S-module. For any $n \in \mathbb{Z}$, we define its n-twist $M(n)$ by

$$M(n)_d := M_{d+n}.$$

It is clear that $M(n)$ is a graded S-module.

Definition 67. Let S be a graded ring, $X = \operatorname{Proj} S$, and let $n \in \mathbb{Z}$ be an integer. We define the sheaf $\mathcal{O}_X(n) := \widetilde{S(n)}$. The sheaf $\mathcal{O}_X(1)$ is called the *twisting sheaf of Serre*. For any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-module, we denote by $\mathcal{F}(n)$ the twisted sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Proposition 68. *Let S be a graded ring and let $X = \operatorname{Proj} S$. Assume that S is generated by S_1 as an S_0-algebra.*

(1) The sheaf $\mathcal{O}_X(n)$ is an invertible sheaf on X.

(2) For any graded S-module M, we have $\widetilde{M}(n) \simeq \widetilde{M(n)}$. In particular, $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \simeq \mathcal{O}_X(m+n)$.

Proof. Since S is generated by S_1 as an S_0-algebra, X is covered by the distinguished open sets $D_+(f)$ where $f \in S_1$. Consider the restriction $\mathcal{O}_X(n)|_{D_+(f)}$. Since $D_+(f) \simeq \operatorname{Spec} S_{(f)}$, this is isomorphic to $\widetilde{S(n)_{(f)}}$. It is a free $S_{(f)}$-module of rank 1 since there is an isomorphism sending a degree 0 element $s \in S_{(f)}$ to a degree n element $f^n s \in S_f$, where the later one is a degree 0 element in $S(n)_{(f)}$. Hence, $\mathcal{O}_X(n)$ is locally free of rank 1.
The second statement follows from the fact that $(M \otimes_S N)_{(f)} = M_{(f)} \otimes_{S_{(f)}} N_{(f)}$. Since $D_+(f), f \in S_1$ covers X, we are done. □

The twisting operation allows us to recover a graded S-module from a sheaf of $\mathcal{O}_{\operatorname{Proj} S}$-modules. Note that we made it just by taking the global section of a sheaf of modules on an affine scheme. On the other hand, on Proj, taking the global section only recovers the degree 0 elements, not for the whole graded module. Hence, we have to adjust its grading and collect all the elements of various degrees.

Definition 69. Let S be a graded ring, $X = \operatorname{Proj} S$, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define the *graded S-module associated to $\mathcal{F}$* to be $\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$. If $s \in S_d$ is a degree d element, it gives a global section $s \in \Gamma(X, \mathcal{O}_X(d))$ in a natural way. Hence, for any $t \in \Gamma(X, \mathcal{F}(n))$, we define the product $s \cdot t \in \Gamma(X, \mathcal{F}(n+d))$ by taking the image of $s \otimes t$ under the natural isomorphism $\mathcal{F}(n) \otimes \mathcal{O}_X(d) \simeq \mathcal{F}(n+d)$.

Proposition 70. *Let A be a ring, $S = A[x_0, \cdots, x_r], r \geq 1$ be a polynomial ring over A, and let $X = \operatorname{Proj} S$ be a projective r-space over A. Then $\Gamma_*(\mathcal{O}_X) \simeq S$.*

Proof. Note that a global section $t \in \Gamma(X, \mathcal{O}_X(n))$ is equivalent to a finite collection of sections $t_i \in \mathcal{O}_X(n)(D_+(x_i)) = S(n)_{(x_i)}$ which agree on the intersection $D_+(x_i x_j)$. Since the localization maps $S \to S_{x_i}$, $S_{x_i} \to S_{x_i x_j}$ are injective, all these rings are subrings of $S' = S_{x_0 \cdots x_r}$. In particular, an element of $\Gamma_*(\mathcal{O}_X)$ is a finite collection of sections $t_i \in S_{x_i}$ whose images in S' are same. Since any homogeneous element in S' can be uniquely expressed by a product $x_0^{m_0} \cdots x_r^{m_r} f(x_0, \cdots, x_r)$, where $m_i \in \mathbb{Z}$ and f is a homogeneous polynomial not divisible by any of x_i. It is an element of S_{x_i} if and only if the exponent m_j is nonnegative for $j \neq i$. Hence, $\Gamma_*(\mathcal{O}_X) = \bigcap_{i=0}^{r} S_{x_i} = S$. □

Caution. If S is a graded ring which is not a polynomial ring, then $\Gamma_*(\mathcal{O}_X)$ does not coincide with S in general.

Let $\mathcal{F} = \widetilde{M}$ for some graded S-module M. In general, it is not true that the module of twisted sections $\Gamma_*(\mathcal{F})$ and the original S-module M coincide. However, in most cases, they define the same sheaf of $\mathcal{O}_{\operatorname{Proj} S}$-modules.

Proposition 71. *Let S be a graded ring, which is finitely generated by S_1 as an S_0-algebra. Let $X = \operatorname{Proj} S$, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then there is a natural isomorphism $\beta : \Gamma_*(\mathcal{F})^\sim \to \mathcal{F}$.*

Proof. Let $f \in S_1$. A section of $\Gamma_*(\mathcal{F})^\sim$ on $D_+(f)$ is represented by a fraction m/f^d, where $m \in \Gamma(X, \mathcal{F}(d))$ for some $d \geq 0$. We may think $1/f^d$ as a section of $\mathcal{O}_X(-d)$, defined on $D_+(f)$, hence, we obtain a section $m \otimes f^{-d} \in \Gamma(D_+(f), \mathcal{F})$. This defines β.
Now let $\mathcal{F}$ be quasi-coherent. To show that β is an isomorphism, it is sufficient to identify the module $\Gamma_*(\mathcal{F})_{(f)}$ with the sections $\mathcal{F}(D_+(f))$.
Note that f is a global section of an invertible sheaf $\mathcal{O}_X(1)$. Since S is finitely generated by elements in S_1, we may choose finitely many elements $f_0, \cdots, f_r \in S_1$ such that X is covered by open affine subsets $D_+(f_0), \cdots, D_+(f_r)$. Similar as in the affine case, any section $t \in \Gamma(D_+(f), \mathcal{F})$ admits an extension from $f^n t \in \Gamma(D_+(f), \mathcal{F}(n))$ to a global section s of $\mathcal{F} \otimes \mathcal{O}_X(n)$ for a sufficiently large n (using the partition of unity given by powers of f_i). In other words, a section $t \in \Gamma(D_+(f), \mathcal{F})$ can be expressed as s/f^n, a section of $\Gamma_*(\mathcal{F})^\sim$ on $D_+(f)$. We conclude that $\mathcal{F}(D_+(f)) \simeq \Gamma_*(\mathcal{F})_{(f)}$ as desired. □

Corollary 72. *Let A be a ring, $S = A[x_0, \cdots, x_r]$, and let $X = \operatorname{Proj} S$. If Y is a closed subscheme of $X = \mathbb{P}_A^r$, then there is a homogeneous ideal $I \subseteq S$ such that Y is a closed subscheme determined by I.*

Proof. Let $\mathscr{I}_Y$ be the ideal sheaf of Y on X. Since the global section functor is left exact, and the twisting functor is exact, the graded module $\Gamma_*(\mathscr{I}_Y)$ is a submodule of $\Gamma_*(\mathcal{O}_X) = S$. In particular, $I := \Gamma_*(\mathscr{I}_Y)$ is a homogeneous ideal of S. Now I induces a closed subscheme $V(I) \simeq \operatorname{Proj} S/I$, whose sheaf of ideals will be $\widetilde{I}$. Since $\mathscr{I}_Y$ is quasi-coherent, we have $\widetilde{I} \simeq \mathscr{I}_Y$. In fact, $I = \Gamma_*(\mathscr{I}_Y)$ is the largest ideal in S defining Y. □

Exercise 73. Let X be a scheme, and $\mathscr{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras, that is, a sheaf of rings which is a quasi-coherent $\mathcal{O}_X$-module simultaneously.

(1) Show that there is a unique scheme Y and a morphism $f : Y \to X$ such that for every affine open $U \subseteq X$, $f^{-1}(U) \simeq \operatorname{Spec} \mathscr{A}(U)$, and for every inclusion $V \hookrightarrow U$ of open affines of X, the morphism $f^{-1}(V) \hookrightarrow f^{-1}(U)$ corresponds to the restriction homomorphism $\mathscr{A}(U) \to \mathscr{A}(V)$. The scheme Y is called the global Spec, or the relative Spec, and denoted by $\mathbf{Spec}\ \mathscr{A}$.

(2) Let $\mathcal{E}$ be a locally free sheaf of rank r on a scheme X. Take $\mathscr{A} = \operatorname{Sym} \mathcal{E}^\vee$ be the sheaf associated to the presheaf $U \mapsto \operatorname{Sym}(\mathcal{E}^\vee(U))$. Show that this is a sheaf of $\mathcal{O}_X$-algebra.

(3) Check that the morphism $\mathbf{Spec}\ \mathscr{A} \to X$ gives a vector bundle of rank r. This is called the *total space of a locally free sheaf* $\mathcal{E}$ of a finite rank, or the *vector bundle associated to a locally free sheaf* $\mathcal{E}$.

Exercise 74. Let X be a noetherian scheme, and $\mathscr{S}$ be a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras, that is, $\mathscr{S} \simeq \bigoplus_{d \geq 0} \mathscr{S}_d$ where $\mathscr{S}_d$ is the homogeneous part of degree d. Assume that $\mathscr{S}_0 = \mathcal{O}_X$, and $\mathscr{S}_1$ is a coherent $\mathcal{O}_X$-module, and that $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as an $\mathcal{O}_X$-algebra. Complete the details of the following construction.

(1) For each open affine open subset $U = \operatorname{Spec} A \subseteq X$, let $\mathscr{S}(U)$ be the graded A-algebra $\Gamma(U, \mathscr{S}|_U)$. We have a natural morphism $\pi_U : \operatorname{Proj} \mathscr{S}(U) \to U$. Check that this is compatible with a further localization, that is, $\operatorname{Proj} \mathscr{S}(U_f) \simeq \pi_U^{-1}(U_f)$ where $U_f = \operatorname{Spec} A_f \subseteq U$ for some element $f \in A$.

(2) Let U, V be two affine open subsets of X. Check that $\pi_U^{-1}(U \cap V) \simeq \pi_V^{-1}(U \cap V)$. In particular, the schemes $\operatorname{Proj} \mathscr{S}(U)$ glue together and form a scheme $\mathbf{Proj}\ \mathscr{S}$, together with a morphism $\pi : \mathbf{Proj}\ \mathscr{S} \to X$.

(3) Check that the invertible sheaves $\mathcal{O}(1)$ on each $\operatorname{Proj} \mathscr{S}(U)$ are also compatible under the above construction. They give rise to an invertible sheaf $\mathcal{O}(1)$ on $\mathbf{Proj}\ \mathscr{S}$.

(4) Let $\mathcal{E}$ be a locally free sheaf of rank $(r+1)$ on X. Take $\mathscr{S} = \bigoplus_{d \geq 0} \operatorname{Sym}^d \mathcal{E}^\vee$ be the symmetric algebra. Check that this is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. We denote by $\mathbb{P}(\mathcal{E}) := \mathbf{Proj}\ \mathscr{S}$ be the *projective space bundle*, together with a natural projection $\pi : \mathbb{P}(\mathcal{E}) \to X$. If $\mathcal{E}$ is free over a sufficiently small open affine subset $U \simeq \operatorname{Spec} A$, then $\pi^{-1}(U) \simeq \mathbb{P}^r_A = \operatorname{Proj} A[x_0, \cdots, x_{r+1}]$.

(5) Let Y be a closed subscheme of X, and let $\mathscr{I}$ be the ideal sheaf. Take $\mathscr{S} = \bigoplus_{d \geq 0} \mathscr{I}^d$, where $\mathscr{I}^0 = \mathcal{O}_X$. Check that $\mathscr{S}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The scheme $\widetilde{X} := \mathbf{Proj}\ \mathscr{S}$, together with a natural morphism $\pi_{\mathscr{I}} : \widetilde{X} \to X$, is called the *blowing-up of X with respect to $\mathscr{I}$*. Compute it when X is an affine n-space, and Y is the origin of X.

Remark 75. Using the above constructions, one may define "affineness" and "projectiveness" of a morphism of schemes. A morphism $\pi : X \to Y$ is *affine* if there is an isomorphism

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & \mathbf{Spec}\ \mathscr{A} \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & Y & \end{array}$$

for some quasi-coherent sheaf $\mathscr{A}$ of $\mathcal{O}_X$-algebras. Similarly, A morphism $\pi : X \to Y$ is *projecive* if there is an isomorphism

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & \mathbf{Proj}\ \mathscr{S} \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & Y & \end{array}$$

for some quasi-coherent sheaf $\mathscr{S}$ of graded $\mathcal{O}_X$-algebras, finitely generated by the degree 1 part.

Example 76. Let $V = \mathbb{C}^{\oplus(N+1)}$ be a vector space of dimension $N+1$ with basis $e_0, \cdots, e_N$. Its projectivization $\mathbb{P}V := \text{Proj Sym}(V^\vee)$ becomes a projective N-space over $\mathbb{C}$. If we denote $x_0, \cdots, x_N$ be the dual basis, then $\mathbb{P}V = \text{Proj}\, S$, where $S = \mathbb{C}[x_0, \cdots, x_N]$ be the polynomial ring with variables $x_0, \cdots, x_N$. This space is a scheme-theoretical analogue of the quotient space $V \setminus \{0\}/\mathbb{C}^\times$, which parametrizes the set of lines passing through the origin $0 \in V$. Hence, a point $P = [a_0 : \cdots : a_N] \in \mathbb{P}V$ corresponds to a line passing through the origin $0 \in V$ and the point $(a_0, \cdots, a_N) \in V$. One may consider the incidence correspondence

$$\mathcal{I} := \{([a_0 : \cdots : a_N], v) \in \mathbb{P}V \times V \mid v = c(a_0, \cdots, a_N) \text{ for some } c \in \mathbb{C}\}.$$

One can check that this is a line bundle, together with the first projection $pr_1 : \mathcal{I} \to \mathbb{P}V$, and is called the *tautological line bundle* $\mathcal{L}$ on the projective space. $\mathcal{L}$ is not the twisting sheaf $\mathcal{O}(1)$ of Serre, since it does not have any nonzero global section:

> If then, a section $s : \mathbb{P}V \to \mathcal{I} \subset \mathbb{P}V \times V$ is a map which sends a point $x \in \mathbb{P}V$ to a pair $(x, \sigma_s(x))$ where $\sigma_s : \mathbb{P}V \to V$ is a regular map. Since a projective space $\mathbb{P}V$ is complete, and thanks to the open mapping theorem, σ_s must be a constant map. In particular, $v := \sigma_s(x)$ must be a point which lies on every line determined by a choice of a point $x \in \mathbb{P}V$, hence, v must be 0.

On the other hand, $\mathcal{L} \simeq \mathcal{O}(-1) = (\mathcal{O}(1))^\vee$ as in the following way:

> Let $U_i = D_+(x_i)$. $\mathcal{I}$ trivializes over U_i by a local trivialization $\phi_{U_i} : U_i \times \mathbb{C} \to pr_1^{-1}(U_i)$ by sending
>
> $$([x_0 : \cdots : x_N], c) = \left([x_0 : \cdots : x_N], \left(c\frac{x_0}{x_i}, \cdots, c = c\frac{x_i}{x_i}, \cdots, c\frac{x_N}{x_i}\right)\right).$$
>
> Hence, the transition function on $D_+(x_i) \cap D_+(x_j)$ is : $T_{ij} = x_i/x_j$. One can immediately see that this is the multiplicative inverse of the transition function for $\mathcal{O}(1)$, in particular, $\mathcal{L} \simeq O(1)^\vee = O(-1)$.

One can check that if we do not take the symmetric algebra over a dual invertible sheaf in the construction of global Spec, namely, take a global Spec construction applied into $\operatorname{Sym}\mathcal{O}(1)$, then the sheaf of sections we will obtain is not $\mathcal{O}(1)$, but $\mathcal{O}(-1)$.

Topic 2 – Line bundles, sections, and divisors

Line bundles, or equivalently, invertible sheaves are one of the most important objects in algebraic geometry. A ringed space $(X, \mathcal{O}_X)$ can be regarded as a triple $(X, \mathcal{O}_X, \mathcal{O}_X)$, together with a choice of a line bundle. In the last section, we learned one of the most important line bundles in projective geometry, namely, the twisting sheaf of Serre $\mathcal{O}_X(1)$ on a projective variety (scheme) X. We want to look X as a triple $(X, \mathcal{O}_X, \mathcal{O}_X(1))$. This idea leads us to projective geometry inside a projective space, in other words, X as a projective variety embedded in a certain projective space. We will see a systematic way to study line bundles by observing its global sections. We will also study divisors, in the both sense of Weil and Cartier, and how they are related to line bundles.

1 Global sections of coherent sheaves

We will call a scheme X is *projective* over $\operatorname{Spec} A$ if it is isomorphic to $\operatorname{Proj} S$ for some graded ring S, where $S_0 = A$ and S is finitely generated by S_1 as an A-algebra. In particular, S is a quotient of a polynomial ring over A by a homogeneous ideal in S.

Definition 77. Let X is a scheme over a ring A, that means, there is a morphism $X \to \operatorname{Spec} A$. An invertible sheaf $\mathcal{L}$ is *very ample* if there is an immersion $i : X \to \mathbb{P}^r_A$ for some r, such that $i^*\mathcal{O}_{\mathbb{P}^r_A}(1) \simeq \mathcal{L}$. Note that a morphism $i : X \to Z$ is an *immersion* if it gives an isomorphism of X with an open subscheme of a closed subscheme of Z.
We say $\mathcal{F}$ is *generated by global sections*, or *globally generated* if there is a family of sections $s_i \in \Gamma(X, \mathcal{F})$ such that the images of s_i generate the stalk $\mathcal{F}_P$ as an $\mathcal{O}_{X,P}$-module for every point $P \in X$. Note that $\mathcal{F}$ is globally generated if and only if $\mathcal{F}$ can be written as a quotient of a free sheaf, that is, there is a surjection

$$\bigoplus \mathcal{O}_X \to \mathcal{F} \to 0.$$

Hence, an A-scheme X is projective if and only if it is "proper" (universally closed), and there is a very ample invertible sheaf $\mathcal{L}$ on X.

Example 78. (1) Let A be a ring, $S = A[x_0, x_1, \cdots, x_n]$, and let $X = \operatorname{Proj} S$ be the projective n-space over A. The twisting sheaf of Serre $\mathcal{O}_X(1)$ is very ample since the identity map $id : X \to \mathbb{P}^n_A$ is an immersion such that $id^*\mathcal{O}_{\mathbb{P}^n_A}(1) = \mathcal{O}_{\mathbb{P}^n_A}(1) = \mathcal{O}_X(1)$.

(2) Let X be as above. The invertible sheaf $\mathcal{O}_X(d), d > 0$ is very ample. Here we give a naive explanation. We first take the d-uple embedding $v_d : X \hookrightarrow \mathbb{P}^r_A$, where

$r = \binom{n+d}{d} - 1$. Note that X is isomorphic to a closed subset of $\mathbb{P}^r_A$, it only suffices to show that $v_d^* \mathcal{O}_{\mathbb{P}^r_A}(1) \simeq \mathcal{O}_X(d)$. Since the sections of $\mathcal{O}_{\mathbb{P}^r_A}(1)$ are spanned by coordinate functions of the projective r-space, whose pullbacks on X are monomials of degree d in $x_0, x_1, \cdots, x_n$. In particular, they generate $\Gamma(X, \mathcal{O}_X(d))$.

Precisely, one can show that $X \simeq \operatorname{Proj} S^{(d)}$ where $S^{(d)}_n := S_{nd}$. Clearly, $S^{(d)}$ is isomorphic to a quotient of a polynomial ring $A[Y_{i_0, i_1, \cdots, i_n}]$, $(i_0 + i_1 + \cdots + i_n) = d$ by an ideal I generated by the elements of the form

$$Y_{i_0, i_1, \cdots, i_n} Y_{j_0, j_1, \cdots, j_n} - Y_{k_0, k_1, \cdots, k_n} Y_{l_0, l_1, \cdots, l_n},$$

where $(i_0, \cdots, i_n) + (j_0, \cdots, j_n) = (k_0, \cdots, k_n) + (l_0, \cdots, l_n)$. In particular, the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj} S^{(d)}$ equals to $\mathcal{O}_X(d)$.

(3) A very ample invertible sheaf is globally generated. Indeed, by the definition, we may regard X as an open subscheme of a closed subscheme Z of $\mathbb{P}^r_A$ via an immersion i. Let $S = A[x_0, \cdots, x_r]$, and let $Z = \operatorname{Proj} S/I$ for some homogeneous ideal $I \subseteq S$. Since $\mathcal{O}_Z(1)$ is generated by the images of $x_0, \cdots, x_r \in S_1$ in S_1/I_1, their restrictions on X also generate the sheaf $i^* \mathcal{O}_{\operatorname{Proj} S}$. Hence, by definition, the restrictions of images of the coordinate functions will generate a very ample invertible sheaf $\mathcal{L}$.

(4) Any quasi-coherent sheaf on an affine scheme is globally generated. If $\mathcal{F} = \widetilde{M}$ for an A-module M, then any set of generators of M as an A-module will generate $\mathcal{F}$.

Exercise 79. Let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on X. Suppose that $\mathcal{F}$ is globally generated, and there is a surjection $\mathcal{F} \to \mathcal{G} \to 0$. Show that $\mathcal{G}$ is also globally generated.