Theorem 80 (Serre)**.** *Let S_0 be a noetherian ring, $S = S_0[x_0, \cdots, x_N]$ be a polynomial ring, $I \subseteq S$ be a homogeneous ideal not containing S_+, and let $X = \operatorname{Proj} S/I$. Let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there is an integer n_0 such that for all $n \geq n_0$, the sheaf $\mathcal{F}(n)$ can be generated by a finite number of global sections.*

Proof. Let $i : X \hookrightarrow \mathbb{P}^N_{S_0}$ be a closed immersion of X into a projective space such that $i^*\mathcal{O}(1) = \mathcal{O}_X(1)$. Then $i_*\mathcal{F}$ is coherent, and $i_*(\mathcal{F}(n)) = (i_*\mathcal{F})(n)$ by the projection formula. Hence, we may replace $\mathcal{F}(n)$ by $i_*\mathcal{F}(n)$. In other words, we may assume that $X = \mathbb{P}^N_{S_0}$.
Take an affine open cover of X by $D_+(x_i)$, $0 \leq i \leq N$. Since $\mathcal{F}$ is coherent, there is a finitely generated module M_i over $B_i = S_0[x_0/x_i, \cdots, x_N/x_i]$ such that $\mathcal{F}|_{D_+(x_i)} \simeq \widetilde{M_i}$. For each M_i, there are finitely many elements $s_{ij} \in M_i$ which generate M_i as a B_i-module. For a sufficiently large integer n, the section $x_i^n s_{ij}$ will extend to a global section of $\mathcal{F}(n)$ as discussed previously for the affine case. We take a single n to work for all i, j. Now $\mathcal{F}(n)$, which is still coherent, corresponds to a B_i-module M_i' on $D_+(x_i)$, and the map $\cdot x_i^n : M_i \to M_i'$ induces an isomorphism for every i. In particular, the sections $x_i^n s_{ij}$ generate M_i', and hence the set of global sections $\{t_{ij}\}$, where t_{ij} is an extension of $x_i^n s_{ij}$, generates the sheaf $\mathcal{F}(n)$ everywhere. $\square$

Corollary 81. *Same assumption as above. Then any coherent sheaf $\mathcal{F}$ can be written as a quotient of a sheaf $\mathcal{E}$, where $\mathcal{E}$ is a finite direct sum of twisted structure sheaves $\mathcal{O}(n)$.*

Proof. Let $n \gg 0$ be a sufficiently large integer so that the twist $\mathcal{F}(n)$ is generated by its global sections. Then we have a surjection $\bigoplus \mathcal{O}_X \to \mathcal{F}(n) \to 0$, where the first term is a direct sum of finite copies of $\mathcal{O}_X$. Tensoring with $\mathcal{O}_X(-n)$, we have the desired surjection. $\square$

Remark 82 (Hilbert syzygy theorem)**.** If we have a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^N$, then the above corollary gives the existence of a surjection $\phi_0 : \bigoplus \mathcal{O}(n_{0,j}) \to \mathcal{F} \to 0$. Its kernel is also coherent, we have a surjection $\phi_1 : \bigoplus \mathcal{O}(n_{1,j}) \to \ker \phi_0 \to 0$. Repeating the process, we can construct a resolution of $\mathcal{F}$ by a direct sum of finitely many line bundles

$$\cdots \bigoplus \mathcal{O}(n_{2,j}) \to \bigoplus \mathcal{O}(n_{1,j}) \to \bigoplus \mathcal{O}(n_{0,j}) \to \mathcal{F} \to 0.$$

This is the sheaf-theoretic analogue of a free resolution of a graded module over a polynomial ring. The famous Hilbert syzygy theorem states that the resolution above terminates after a finite step, of length at most $N + 1$. Applying the $\sim$ functor will give a resolution of a coherent sheaf by direct sums of twisted free modules from a usual free resolution over a polynomial ring.

In particular, we may show the finite generatedness of the global sections of a coherent sheaf on a projective variety/scheme:

Theorem 83. *Let k be a field, A be a finitely generated k-algebra, X be a projective scheme over A, and let $\mathcal{F}$ be a coherent sheaf on X. Then $\Gamma(X, \mathcal{F})$ is a finitely generated A-module. In particular, if $A = k$, then $\Gamma(X, \mathcal{F})$ is a finite dimensional k-vector space.*

We will omit its proof since we will meet a much more general statement later.

Exercise 84. Let A be a ring, X be a closed subscheme of $\mathbb{P}^N_A$. We define the *homogeneous coordinate ring* $S(X)$ of X for the given embedding to be $A[x_0, \cdots, x_N]/I$, where $I = \Gamma_*(\mathscr{I}_X)$ is the (largest) ideal defining X.
A scheme X is *normal* if all the local rings $\mathcal{O}_{X,P}$ are integrally closed. A closed subscheme $X \subseteq \mathbb{P}^N_A$ is *projectively normal* (or, *arithemetically normal*) for the given embedding, if its homogeneous coordinate ring $S(X)$ is an integrally closed domain.
Now assume that k is an algebraically closed field, and that X is a connected, normal, closed subvariety of $\mathbb{P}^N_A$.

(a) Let $S(X)$ be the homogeneous coordinate ring of X, and let $R(X) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(n))$ be the section ring. Show that $S(X)$ is an integral domain, and that $R(X)$ is its integral closure.

(b) Show that $S(X)_d = R(X)_d$ for all sufficiently large d.

(c) Show that $S(X)^{(d)} := \bigoplus_n S(X)^{(d)}_n = \bigoplus_n S(X)_{nd}$ is integrally closed for sufficiently large d. This implies that the d-uple embedding of X is projectively normal when d is large enough.

(d) Show that a closed subscheme $X \subseteq \mathbb{P}^N_A$ is projectively normal if and only if it is normal, and the natural map on global sections $\Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) \to \Gamma(X, \mathcal{O}_X(n))$ is surjective for every $n \geq 0$.

(e) Let $X = \{[s^4 : s^3t : st^3 : t^4] \mid [s : t] \in \mathbb{P}^1\} \subseteq \mathbb{P}^3$ be a rational quartic curve in $\mathbb{P}^3$. Let $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^3}(1)$ be the very ample line bundle. Compute $S(X)$ and $R(X)$, and conclude that X is not linearly normal, that is, $\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \to \Gamma(X, \mathcal{O}_X(1))$ is not surjective.

2 More on: very ample and globally generated line bundles

Now we are ready to discuss a projective morphism determined by a globally generated line bundle. What we will do is to revert the following process.

Example 85. Let $S = k[x_0, \cdots, x_N]$ be the polynomial ring, and let $X = \operatorname{Proj} S$. The very ample line bundle $\mathcal{O}_X(1)$ comes from the natural identification $X \overset{\sim}{\hookrightarrow} \mathbb{P}_k^N$. The homogeneous coordinates $x_0, \cdots, x_N$ of $\mathbb{P}_k^N$ give rise to global sections $x_0, \cdots, x_N \in \Gamma(X, \mathcal{O}_X(1))$. Since their images spans the stalk $\mathcal{O}_X(1)_P$ over the local ring $\mathcal{O}_{X,P}$ for every point $P \in X$, we see that $\mathcal{O}_X(1)$ is globally generated by the sections $x_0, \cdots, x_N$. Similarly, if Y is a projective variety and $\varphi : Y \to \mathbb{P}_k^N$ is any morphism to a projective space, then the line bundle $\mathcal{L} := \varphi^* \mathcal{O}_{\mathbb{P}_k^N}(1)$ generated by global sections $s_0, \cdots, s_N$ where $s_i = \varphi^*(x_i)$ due to the same reason.

The following statement holds in a general setting, namely, a scheme over a ring, however, we only deal with the case of a variety over an algebraically closed field for the simplicity.

Theorem 86. *Let k be an algebraically closed field, and let X be a (quasi-projective) variety over k.*

(i) If $\varphi : X \to \mathbb{P}_k^N$ is a morphism to a projective space, then $\varphi^*(\mathcal{O}(1))$ is an invertible sheaf on X, which is generated by global sections $s_i = \varphi^*(x_i)$ for $i = 0, \cdots, N$.

(ii) Conversely, if $\mathcal{L}$ is a globally generated invertible sheaf, and if $s_0, \cdots, s_N \in \Gamma(X, \mathcal{L})$ are global sections which generate $\mathcal{L}$, then there is a unique morphism $\varphi : X \to \mathbb{P}_k^N$ such that $\mathcal{L} \simeq \varphi^*(\mathcal{O}(1))$ and $s_i = \varphi^*(x_i)$ for $i = 0, \cdots, N$.

Proof. Only for the last statement. Let $s \in \Gamma(X, \mathcal{L})$ be a global section of $\mathcal{L}$. Since $s_P \in \mathcal{L}_P \simeq \mathcal{O}_{X,P}$, we may define $s(P)$ to be the image of s_P in $\mathcal{O}_{X,P}/m_{X,P} \simeq k$ for any closed point $P \in X$. This uniquely determines a function $X \to k$, which we will denote also by s. Define $\varphi(P) := [s_0(P) : s_1(P) : \cdots : s_N(P)] \in \mathbb{P}_k^N$. Since $s_0, \cdots, s_N$ generate $\mathcal{L}$, that is, at least one of the images $(s_0)_P, \cdots, (s_N)_P$ in $\mathcal{L}_P$, a free $\mathcal{O}_{X,P}$-module of rank 1, is not contained in the maximal ideal $m_{X,P}$. In particular, there is no $P \in X$ such that $s_i(P) = 0$ for every i simultaneously. Hence the map φ is well-defined. $\square$

Definition 87. Let $\mathcal{L}$ be an invertible sheaf on X. A point $P \in X$ is called a *base point* of $\mathcal{L}$ if $s_P = 0$ for every $s \in \Gamma(X, \mathcal{L})$. The set of base points is called the *base locus* of $\mathcal{L}$; it is a closed subset of X. We call $\mathcal{L}$ is *base point free* if it has no base points.

Remark 88. The above theorem implies that "base-point-free" is exactly same as "globally generated", and the global sections of $\mathcal{L}$ provide a morphism from X outside the base locus to a projective space. When X is projective, $\mathcal{L}$ is very ample, $s_0, \cdots, s_N$ are global sections which generate the vector space $\Gamma(X, \mathcal{L})$, then the morphism φ becomes an embedding; one can check that φ separates both the points and the tangent vectors. More precisely, for any distinct closed two points $P, Q \in X$, there is a section $s \in \Gamma(X, \mathcal{L})$ such that $s(P) = 0$ but $s(Q) \neq 0$, or vice versa; and for each closed point $P \in X$, the set $\{s \in \Gamma(X, \mathcal{L}) \mid s_P \in m_P \mathcal{L}_P\}$ spans the k-vector space $m_P \mathcal{L}_P / m_P^2 \mathcal{L}_P$.

Note that "very ample" is a relative notion; we have to pick a morphism $X \to \operatorname{Spec} A$ (or, $X \to \operatorname{Spec} k$ if X is a k-variety). Hence, it is natural to give an absolute notion which does not depend on the choice of a base morphism.

Definition 89. An invertible sheaf $\mathcal{L}$ on a noetherian scheme X is called *ample* if for every coherent sheaf $\mathcal{F}$ on X, there is an integer n_0 (depending on the choice of $\mathcal{F}$) such that for every $n \geq n_0$, the sheaf $\mathcal{F} \otimes \mathcal{L}^n$ is generated by its global sections. This is equivalent to say that: $\mathcal{L}^m$ is very ample for some $m > 0$.

Example 90. (1) Every invertible sheaf on an affine variety (or a scheme) X is ample, since every coherent sheaf on X is generated by its global sections.

(2) Let $X = \mathbb{P}^n_k$ be the projective n-space. The sheaf $\mathcal{O}_X(d)$ is ample if and only if it is very ample if and only if $d > 0$. It is globally generated if and only if $d \geq 0$.

Remark 91. Thanks to the intersection theory, being "ample" can be checked by observing the intersection numbers with subvarieties (or, curves and limits of curves; see Nakai-Moishezon criterion, or Kleiman criterion). On the other hand, being "globally generated", or "very ample" are much subtle in this viewpoint. There are several important open problems which compare those notions.

Exercise 92 (Riemann-Roch problem)**.** Let k be an algebraically closed field, X be a nonsingular projective variety over k. Let $\mathcal{L}$ be an invertible sheaf on X. We want to describe the number $\dim \Gamma(X, \mathcal{L}^n)$ as an integer-valued function of n.

(i) Assume that $\mathcal{L}$ is very ample, and $i : X \hookrightarrow \mathbb{P}^N_k$ is the corresponding embedding in a projective space. Show that $\dim \Gamma(X, \mathcal{L}^n) = P_X(n)$ for sufficiently large n, where P_X is the Hilbert polynomial of X.

(ii) Show that $\dim \Gamma(X, \mathcal{L}^n)$ is a polynomial function for n large enough when $\mathcal{L}$ is ample.

(iii) Show that $\dim \Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)$ is a polynomial function for n large enough when $\mathcal{L}$ is ample and $\mathcal{F}$ is coherent (Hint: use Hilbert's syzygy theorem for $i_*(\mathcal{F} \otimes \mathcal{L}^n)$).

3 Divisors

Roughly speaking, the notion of divisors is a generalization of codimension 1 subvarieties. It gives a better understanding the intrinsic geometry of a given variety itself, and classifying line bundles. In this section we will study divisors, both Weil divisors and Cartier divisors, linear equivalence, and the divisor class group. The last one is an abelian group which is an important invariant of a variety. We will also observe the connection between Weil divisors, Cartier divisors, and invertible sheaves (= line bundles).
Codimension 1 subvarieties are much easier than subvarieties of higher codimension. For instance, a codimension 1 subvariety of a projective space is defined by the vanishing locus of a homogeneous polynomial, whereas a subvariety of codimension r ($r \geq 2$) is not always defined by the vanishing locus of r homogeneous polynomials (e.g., a twisted cubic in $\mathbb{P}^3$). In general, every codimension 1 subvariety of a smooth variety is defined by a single equation, in a small open neighborhood of each point.

Example 93. Let $C \subset \mathbb{P}^2_k$ be a nonsingular projective curve of degree d over an algebraically closed field k. If we choose a line $L \subseteq \mathbb{P}^2$, then the intersection $L \cap C$ is consisted of d points, if we count the points with the multiplicity. We may write $L \cap C = D_L = \sum_{P \in C} n_P \cdot P$, where $P \in C$ is a point on a curve and n_P is the multiplicity at P. Note that this formal sum is finite; $n_P = 0$ all but finitely many P's. As L varies, we have a family of sets of d-points, parametrized by the set of all lines in $\mathbb{P}^2$. In other words, we have the following incidence set

$$\mathcal{I}_C := \{(L, D_L) \mid D_L = L \cap C\} \subseteq (\mathbb{P}^2)^* \times C^{(d)}$$

where $C^{(d)} = (C \times C \times \cdots \times C)/\mathfrak{S}_d$ is the symmetric d-th power of C. We may recover the embedding $C \subseteq \mathbb{P}^2$ as follows.

> Let $P \in \mathbb{P}^2$ be a point. Take a subset $\Xi_P := \{P\} + C^{(d-1)} \subseteq C^{(d)}$ as the locus of d-points of C which contains P as supports, that is, $n_P \geq 1$. Note that Ξ_P is nonempty only when $P \in C$. Then the set $pr_1(pr_2^{-1}\Xi_P)$ coincides with the set of all lines $L \in (\mathbb{P}^2)^*$ passing through P. This uniquely determines the point $P \in C \subseteq \mathbb{P}^2$. Varying the choice of P, we may recover $C \subseteq \mathbb{P}^2$ completely.

Let us observe a relation between two different divisors. Let L_1, L_2 be two lines, and let l_1, l_2 be the corresponding homogeneous equations, respectively. The ratio l_1/l_2 gives a rational function on $\mathbb{P}^2$, and on C when we restrict. By construction, the rational function l_1/l_2 on C has zeroes at the points of $D_1 = L_1 \cap C$, and poles at the points of $D_2 = L_2 \cap C$. We will say two divisors D_1 and D_2 are linearly equivalent if there is a rational function whose set of zeroes is D_1 and set of poles is D_2.

In this section, all the varieties X (in particular, they are irreducible) are assumed to be normal, that is, every local ring is an integrally closed domain.

Definition 94. A *prime divisor* on X is a closed subvariety Y of codimension 1. A *Weil divisor* is an element of the free abelian group $\operatorname{Div} X$ generated by the prime divisors, that is,

$$D = \sum_Y n_Y Y$$

where Y are the prime divisors in X, and n_Y are the integers all but finitely many of them are 0. If all the $n_Y \geq 0$, we say that D is *effective* and denote by $D \geq 0$.
If Y is a prime divisor on X, then the local ring at its generic point $\mathcal{O}_{X,Y}$ (this is why we deal with schemes and their topological nature) is integrally closed domain of Krull dimension 1 by the assumption. Hence, there is a corresponding discrete valuation $v_Y : K(X) \setminus \{0\} \to \mathbb{Z}$ where $K(X)$ is the function field of X. Let $f \in K(X) \setminus \{0\}$ be any nonzero rational function on X. We define the *divisor* of f by the sum

$$(f) := \sum v_Y(f) Y$$

taken over all the prime divisors Y on X. A divisor of the form (f) for some nonzero rational function on X is called a *principal divisor*.
Since $(fg) = (f) + (g)$ for any $f, g \in K(X) \setminus \{0\}$, the image of $K(X) \setminus \{0\}$ in $\operatorname{Div} X$ forms a subgroup. Hence, it is natural to consider the equivalence classes of divisors. We say two divisors D_1 and D_2 are *linearly equivalent*, denoted by $D_1 \sim D_2$, if $D_1 - D_2$ is a principal divisor on X. The group $\operatorname{Div} X$ quotient by the subgroup of principal divisors is called the *divisor class group*, and is denoted by $\operatorname{Cl} X$.

Remark 95. This notion also makes sense in a general setting: X is a noetherian integral separated scheme which is regular in codimension 1. For instance, we may take $X = \mathbb{Z}$. In this case, a prime divisor is a nonzero prime ideal in $\mathbb{Z}$. Hence, any divisor is a formal sum of finitely many primes with the integer coefficient, namely, $D = \sum n_i \cdot p_i$. Hence, D can be identified with a rational number $q_D := \prod p_i^{n_i} \in \mathbb{Q}$. When we apply this process to a Dedekind domain, we will have a group of fractional ideals of the ring of integers, and will lead to the ideal class group by taking the quotient by the subgroup of principal ideals.

The divisor class group measures the failure of unique factorizations as in the ideal class group for Dedekind domains:

Proposition 96. *Let A be a noetherian domain. Then A is a unique factorization domain if and only if $X = \operatorname{Spec} A$ is normal and $\operatorname{Cl} X = 0$.*

Proof. Note that A is a UFD if and only if every height 1 prime ideal is principal. We leave the details in Hartshorne's book II.6.2 and references therein. ☐

Proposition 97 (Algebraic Hartogs's lemma)**.** *Let A be an integrally closed noetherian domain. Then*

$$A = \bigcap_{\operatorname{ht} \mathfrak{p} = 1} A_{\mathfrak{p}}$$

where the intersection is taken over all prime ideals of height 1.

It is worthwhile to consider a parallel statement in complex geometry. Roughly speaking, a rational function $f \in K(A)$ does not lie in $A_\mathfrak{p}$ means that f has a pole at $[\mathfrak{p}]$. Hence, A, the set of regular functions, coincides with the intersection of $A_\mathfrak{p}$ implies that a rational function which does not have a pole at any height 1 primes (= codimension 1 subvarieties) is indeed regular. In particular, if we have a rational function, has pole possibly on a closed (algebraic) subset of codimension at least 2, then we may extend it as a regular function.

Example 98. Let $X = \operatorname{Spec} k[x_1, \cdots, x_n]$ be an affine n-space over a field k. Then $\operatorname{Cl} X = 0$ since the polynomial ring is a UFD.

Proposition 99. *Let $X = \mathbb{P}^n_k$ be a projective n-space over a field k. Let $D = \sum n_i Y_i$ be a divisor. We define the degree of D by $\deg D = \sum n_i(\deg Y_i)$, where $\deg Y_i$ is the degree of the hypersurface Y_i. Let H be the hyperplane $(x_0 = 0)$. Then:*

(i) if D is a divisor of degree d, then $D \sim dH$;

(ii) for any $f \in K(X) \setminus \{0\}$, $\deg(f) = 0$;

(iii) the degree map $\deg : \operatorname{Cl} X \to \mathbb{Z}$ is an isomorphism.

Proof. Let $S = k[x_0, \cdots, x_n]$ be the homogeneous coordinate ring of X. If $g \in S$ is a homogeneous polynomial of degree d, we can factor it into a product of polynomials $g = g_1^{n_1} \cdots g_r^{n_r}$. Note that g_i defines a hypersurface Y_i of degree $d_i = \deg g_i$. The divisor of g is $(g) = \sum n_i Y_i$, which is of degree $d = \sum n_i d_i$. Since a rational function f is a quotient g/h of homogeneous polynomials of the same degree, and $(f) = (g) - (h)$, we have $\deg(f) = \deg(g) - \deg(h) = 0$.
If D is a divisor of degree d, then we may write it as a difference $D_1 - D_2$ of effective divisors of degree d_1 and d_2 with $d_1 - d_2 = d$ (collect terms with positive coefficients and call it D_1, and collect terms with negative coefficients and call it D_2, for instance). Since any effective divisor $\sum n_i Y_i$ is a divisor of a polynomial $\prod g_i^{n_i}$ where Y_i is defined by g_i, we may write $D_1 = (g_1)$ and $D_2 = (g_2)$. Now $D - dH = (f)$ where $f = g_1/x_0^d g_2$ is a rational function on X, that is, $D \sim dH$. The last statement follows from the above and the fact $\deg H = 1$. □

Definition 100. Let D be a Weil divisor on X. We define the sheaf $\mathcal{O}_X(D)$ by

$$\Gamma(U, \mathcal{O}_X(D)) := \{t \in K(X) \setminus \{0\} \mid div|_U t + D|_U \geq 0\} \cup \{0\}.$$

In other words, $\mathcal{O}_X(D)$ is the sheaf of rational functions which can have poles at most "D". A positive coefficient of D allows a pole of that order, and a negative coefficient of D forces a zero of that order. Away from the support of D, it is isomorphic to the structure sheaf. This is a quasi-coherent sheaf on X.

The sheaf $\mathcal{O}_X(D)$ becomes an invertible sheaf in many examples, however, it is not always an invertible sheaf since the irreducible subvarieties of codimension 1 does not work very well. However, any invertible sheaf can be understood as a divisor in the following way:

Lemma 101. *Let $\mathcal{L}$ be an invertible sheaf, s be a rational section of $\mathcal{L}$, that is, a section over a dense open subset of X (note that two rational sections are equivalent when they coincide on a further dense open subset), which do not vanish everywhere on any irreducible component of X. Then s determines a Weil divisor*

$$div(s) := \sum_Y v_Y(s)[Y].$$

as usual. Then $\mathcal{O}(div(s)) \simeq \mathcal{L}$.

Proof. Only for a rough sketch. For a small enough open subset $U \subseteq X$, we may define $\phi_U : \mathcal{O}(div(s))(U) \to \mathcal{L}(U)$ by sending a rational function t (with zeroes and poles are constrained by $div(s)$) to $s \cdot t$. Since $div(s) + div(t)$ have no pole in U, we see that the rational section $s \cdot t$ is well-defined on every point of U, that is, a section of $\mathcal{L}$ on U. Now check that this map induces an isomorphism of sheaves $\mathcal{O}(div(s)) \simeq \mathcal{L}$. □

This leads to the notion of Cartier divisors.

Definition 102. Let X be a scheme. For each open subset U, let $S(U)$ denote the set of elements in $\Gamma(U, \mathcal{O}_X)$ which are not zero divisors in each local ring $\mathcal{O}_{X,x}$ for $x \in U$. The rings $S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ form a presheaf, and we call its sheafification $\mathscr{K}$ the *sheaf of total quotient rings* of $\mathcal{O}$. We denote by $\mathscr{K}^\times$ the sheaf (of multiplicative groups) of invertible elements in $\mathscr{K}$. We denote $\mathcal{O}^\times$ the sheaf of invertible elements in $\mathcal{O}_X$. Since all the regular functions are rational, we have the following short exact sequence

$$0 \to \mathcal{O}^\times \to \mathscr{K}^\times \to \mathscr{K}^\times/\mathcal{O}^\times \to 0.$$

A global section of $\mathscr{K}^\times/\mathcal{O}^\times$ is called a *Cartier divisor* on X. Hence, a Cartier divisor is represented by a collection of pairs (U_i, f_i), where $\{U_i\}$ is an open covering of X, and $f_i \in \Gamma(U_i, \mathscr{K}^\times)$ is a rational element such that $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^\times)$. In other words, it is "locally" defined by a single rational function, and the ratio of such rational functions on the intersection is regular.
A Cartier divisor is *principal* if it is in the image of $\Gamma(X, \mathscr{K}^\times)$, in other words, it is globally defined by a single rational function. Two Cartier divisors are *linearly equivalent* if their ratio is principal. A Cartier divisor is *effective* if it can be represented by $\{(U_i, f_i)\}$ where all the $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$.

Note that a Cartier divisor is Weil; a rational function f_i can have poles or zeroes only along the codimension 1 subsets, hence, the divisor $div(f_i)$ on U_i is well-defined and gives a Weil divisor on U_i. On the intersection $U_i \cap U_j$, we see that the both the ratios f_i/f_j and f_j/f_i are regular. In particular, a regular function f_i/f_j does not have any zeroes or poles, hence, defines a zero Weil divisor. This implies that we can glue the Weil divisors $div(f_i)$ on U_i's, and we obtain the Weil divisor on X as a result. Following this way, an effective Cartier divisor gives an effective Weil divisor, which also coincides with the closed subscheme defined by the sheaf of ideals $\mathscr{I}$ which is locally generated by f_i.

A Weil divisor D is Cartier if and only if the sheaf $\mathcal{O}_X(D)$ is invertible. If we have a Cartier divisor $D = \{(U_i, f_i)\}$, the map $\mathcal{O}_{U_i} \to \mathcal{O}_X(D)|_{U_i}$ defined by $1 \mapsto f_i^{-1}$ becomes an isomorphism. However, the three notions: Weil divisors, Cartier divisors, and invertible sheaves coincide in many cases.

Theorem 103. *Let X be a normal variety whose local rings are unique factorization domains (X locally factorial). There are isomorphisms between three groups:*

(i) $\operatorname{Cl} X$*, the group of Weil divisors on X modulo linear equivalences;*

(ii) $\operatorname{CaCl} X$*, the group of Cartier divisors on X modulo linear equivalences;*

(iii) $\operatorname{Pic}(X)$*, the group of invertible sheaves under $\otimes$ modulo isomorphisms.*

Proof. See Hartshorne's book (II.6.11) and (II.6.14). □

Corollary 104. *If $X = \mathbb{P}^n_k$ for some field k, then every invertible sheaf on X is isomorphic to $\mathcal{O}(d)$ for some $d \in \mathbb{Z}$.*

Proposition 105. *Let D be an effective Cartier divisor on X, and let Y be the associated locally principal subscheme. Then $\mathscr{I}_Y = \mathcal{O}(-Y)$.*

Proof. $\mathcal{O}(-Y)$ is the subsheaf of $\mathscr{K}$ generated locally by f_i. Since D is effective, this is actually a subsheaf of $\mathcal{O}_X$. □

The following proposition is quite useful to compute the divisor class group.

Proposition 106. *Let Z be a proper closed subset of X, and let $U = X \setminus Z$. Then there is a surjective homomorphism $\operatorname{Cl} X \to \operatorname{Cl} U$ sending $D = \sum n_i Y_i$ to $\sum n_i (Y_i \cap U)$, where we ignore $Y_i \cap U$ if it is empty.*
In particular, when Z is of codimension at least 2, the above homomorphism is an isomorphism. When Z is irreducible of codimension 1, then there is an exact sequence

$$\mathbb{Z} \to \operatorname{Cl} X \to \operatorname{Cl} U \to 0$$

where the first map is defined by $1 \mapsto 1 \cdot Z$ (sometimes called the excision exact sequence for class groups).

Proof. The first statement is clear since every prime divisor of U is the restriction of its closure in X. Removing a closed subset of codimension at least 2 does not change the class group; it does not change Weil divisors and principal divisors. When Z is an irreducible closed subset of codimension 1, then we have the following exact sequence

$$0 \to \mathbb{Z} \to \operatorname{Div} X \to \operatorname{Div} U \to 0.$$

By taking the quotient by principal divisors, we may lose the exactness on the left. □

Example 107. Let $X = \operatorname{Spec} k[x,y,z]/(xy - z^2)$ be a quadric cone. For the simplicity, assume that the characteristic of the field k is not equal to 2. Let D be the line $(x = z = 0)$. Note that D is defined by x set-theoretically, however, the divisor of x is $2D$ since $x = 0$ implies $z^2 = 0$, and z (not z^2) generates the maximal ideal of the local ring at the generic point of D. The natural map $\operatorname{Cl} X \to \operatorname{Cl}(X \setminus D)$ is surjective, whose kernel is consisted of Weil divisors whose support is contained in D. Hence, we have the following right exact sequence

$$\mathbb{Z} \to \operatorname{Cl} X \to \operatorname{Cl}(X \setminus D) \to 0,$$

where the first map sends $1 \mapsto 1 \cdot D$. Note also that $X \setminus D \simeq \operatorname{Spec} k[x, x^{-1}, y, z]/(xy - z^2) \simeq k[x, x^{-1}, z]$, which is a spectrum of a UFD, hence $\operatorname{Cl}(X \setminus D) = 0$. Therefore, $\operatorname{Cl} X$ is generated by D and $2D = 0$, that is, either $\operatorname{Cl} X = 0$ or $\operatorname{Cl} X = \mathbb{Z}/2\mathbb{Z}$.
We show that D is not a principal divisor. Let $\mathfrak{m} = (x, y, z)$ be the maximal ideal of $k[x,y,z]/(xy - z^2)$ corresponding to the origin. Note that $\mathfrak{m}/\mathfrak{m}^2$ is a 3-dimensional vector space generated by $\overline{x}, \overline{y}, \overline{z}$. Now the image of $I_D = (x, z)$ in $\mathfrak{m}/\mathfrak{m}^2$ contains $\overline{x}, \overline{z}$; hence it forms at least 2-dimensional subspace. In particular, the ideal I_D cannot be principal; if it was, then the image in $\mathfrak{m}/\mathfrak{m}^2$ is defined by the image of a generator (an application of Krull's Hauptidealsatz). We conclude that $\operatorname{Cl} X = \mathbb{Z}/2\mathbb{Z}$.
The above argument also proves that D is not a Cartier divisor. If it was, then it must coincide with a principal divisor at a neighborhood of the origin. However, there is no principal ideal whose image in $\mathfrak{m}/\mathfrak{m}^2$ coincides with the image of $I_D = (x, z)$ in $\mathfrak{m}/\mathfrak{m}^2$. In particular, $\operatorname{CaCl} X \simeq \operatorname{Pic}(X) = 0$.

Exercise 108. Let $X = \mathbb{P}^1_k \times \mathbb{P}^1_k = \operatorname{Proj} k[x,y,z,w]/(xy - zw)$ be a smooth quadric surface in $\mathbb{P}^3_k$. Let $D_1 = \{pt\} \times \mathbb{P}^1_k$, and let $D_2 = \mathbb{P}^1_k \times \{pt\}$.

(i) Show that there is a surjection $\mathbb{Z} \oplus \mathbb{Z} \to \operatorname{Cl} X$ (Hint : remove D_1 and D_2 from X. The complement is $X \setminus (D_1 \cup D_2) \simeq \mathbb{A}^2$, a spectrum of a UFD, hence has the trivial divisor class group.)

(ii) Show that $\mathcal{O}(D_1)$ restricts to $\mathcal{O}$ on $D_1 \simeq \mathbb{P}^1_k$, and to $\mathcal{O}(1)$ on $D_2 \simeq \mathbb{P}^1_k$. Similarly, show that $\mathcal{O}(D_2)$ restricts to $\mathcal{O}(1)$ on D_1, and to $\mathcal{O}$ on D_2.

(iii) Conclude that the homomorphism $\mathbb{Z} \oplus \mathbb{Z} \to \operatorname{Cl} X$ defined by $(a, b) \mapsto aD_1 + bD_2$ is an isomorphism.

Exercise 109. Show that $\mathbb{P}^2_k$ and $\mathbb{P}^1_k \times \mathbb{P}^1_k$ are birational but not isomorphic (Hint : what are the divisor class group of them?). Can you find two varieties, which are birational but not isomorphic, having the same divisor class group?

Exercise 110 (Torsion Picard group)**.** Let $Y \subseteq \mathbb{P}^n_k$ be an irreducible hypersurface of degree d. Show that $\operatorname{Pic}(\mathbb{P}^n_k \setminus Y) \simeq \mathbb{Z}/d\mathbb{Z}$. It is related to the fact that the fundamental group $\pi_1(\mathbb{P}^n_k \setminus Y) \simeq \mathbb{Z}/d\mathbb{Z}$.