4 Sheaf of differentials and canonical line bundle

The goal of this section is to define the sheaf of (relative) differentials and the canonical line bundle. Differentials, measuring the infinitesimal nature of functions, are intuitive geometric notion which provide a lot of topological/geometric information. We will see how it is described algebraically in terms of Kähler differentials and sheaves. In the case of a nonsingular variety over $\mathbb{C}$, the sheaf of differentials is almost same as the cotangent bundle defined in complex differential geometry, which is the dual of the tangent bundle. In other words, it corresponds to the vector bundle composed of holomorphic 1-forms. In contrast to differential/complex geometry, the notion of cotangent sheaf is much more "natural" in algebraic geometry. When A is a local k-algebra, and if we have a maximal ideal $\mathfrak{m}$ of A, the *Zariski cotangent space* at $\mathfrak{m}$ is defined to be the k-vector space $\mathfrak{m}/\mathfrak{m}^2$. It looks natural than the *Zariski tangent space* $(\mathfrak{m}/\mathfrak{m}^2)^\vee$, the dual vector space of the Zariski cotangent space. Hence, this will be a dual picture to the geometric pictures what we learned from differential geometry and complex geometry. There are technical merits of the algebraic construction. First, we are able to construct the sheaf even if X is not "smooth". The cotangent sheaf may not be locally free in this case, but it will be quasi-coherent. Second, the construction naturally works in a relative case; we may define a sheaf of differentials $\Omega_f = \Omega_{X/Y}$ on X for a morphism $f : X \to Y$. Roughly speaking, this will measure the cotangent vectors of the fiber of the map so that the cotangent vectors of X are a combination of the cotangent vectors of Y and the cotangent vectors of fibers.

Example 111. Let $A = \mathbb{C}[z_1, \cdots, z_n]$ be a polynomial ring over the complex numbers. Let $\mathfrak{m} = (z_1, \cdots, z_n)$ be the maximal ideal corresponding to the origin. Let f be a regular function on $\mathbb{C}^n$, then it admits a power series representation

$$f = f(0) + \left(\sum_{i=1}^{n} a_i z_i\right) + (\text{terms of order at least } 2)\,.$$

where $a_i = \frac{\partial f}{\partial z_i}(0)$. Hence, its image in $\mathfrak{m}/\mathfrak{m}^2$ is $a_1 z_1 + \cdots + a_n z_n$; the linearization of f at the origin. If we denote $\frac{\partial}{\partial z_i}$ be the dual basis of $(\mathfrak{m}/\mathfrak{m}^2)^\vee$, then we may regard $\frac{\partial}{\partial z_i}$ as a linear functional; it reads off the i-th coefficient of the linearization by $\frac{\partial}{\partial z_i}(f) = \frac{\partial f}{\partial z_i}(0) = a_i$.

We first begin with the affine case: we will see the algebraic theory of Kähler differentials. Suppose A is a B-algebra, so we have a morphism of rings $\phi : B \to A$ and the corresponding morphism of spectra $\operatorname{Spec} A \to \operatorname{Spec} B$. For any A-module M, we may define a B-derivation as follows.

Definition 112. A *B-derivation* of A into M is a B-linear map $d : A \to M$ such that

(i) (additivity) $d(a + a') = da + da'$.

(ii) (Leibniz rule) $d(aa') = a da' + a' da$.

(iii) (triviality) $db = 0$ for every $b \in \phi(B) \subseteq A$.

The notion of B-derivations leads to the module of Kähler differentials (or, the module of relative differentials) as follows.

Definition 113. The *module of Kähler differentials* of A over B is an A-module $\Omega_{A/B}$, together with a B-derivation $d : A \to \Omega_{A/B}$ which satisfies the following universal property: for any A-module M and a B-derivation $d' : A \to M$, there is a unique A-module homomorphism $f : \Omega_{A/B} \to M$ such that $d' = f \circ d$.

In particular, $\Omega_{A/B}$ is the "representing object" of the functor of B-derivations $M \mapsto \mathrm{Der}_B(A, M)$. One can construct such a module $\Omega_{A/B}$ as the free module generated by the symbols $\{da \mid a \in A\}$, and take the quotient by the submodule generated by all the expressions of the form

(i) $d(a + a') - da - da'$;

(ii) $d(aa') - ada' - a'da$;

(iii) db

so that $d : A \to \Omega_{A/B}$ which sends a to da should be a B-derivation. The pair $(\Omega_{A/B}, d)$ is unique up to a unique isomorphism.

Caution. Both A and $\Omega_{A/B}$ are A-modules, but the map $d : A \to \Omega_{A/B}$ is not A-linear.

Proposition 114. *If A is a finitely generated B-algebra, then $\Omega_{A/B}$ is a finitely generated A-module. If A is a finitely presented B-algebra, then $\Omega_{A/B}$ is a finitely presented A-module.*

Proof. Let $\{x_1, \cdots, x_n\}$ be a set of generators of A as a B-algebra, and let $\{r_j\}$ be a set of relations among x_i, that is, r_j is a polynomial such that $r_j(x_1, \cdots, x_n) = 0$. Then $\Omega_{A/B}$ is generated by $dx_1, \cdots, dx_n$, subject to the relations $dr_j = 0$. In particular, we do not need to take every single element or every single relation, but can take generators and generators of the relations. □

Example 115. Here we give a few baby examples.

(1) Let $A = B/I$. Then the module $\Omega_{A/B} = 0$.

(2) Let $A = B[x_1, \cdots, x_n]$. The module $\Omega_{A/B} = Adx_1 \oplus Adx_2 \oplus \cdots \oplus Adx_n$, a free module of rank n.

(3) Let $B = k$ is a field of characteristic $\neq 2$, and let $A = k[x, y]/(y^2 - x^3 + x)$. The module $\Omega_{A/B}$ is generated by dx and dy, subject to the relation $2ydy = (3x^2 - 1)dx$. In the locus $(y \neq 0)$, the generator dy can be expressed in terms of dx, hence, the sheaf $\widetilde{\Omega_{A/B}}$ is generated by dx, and is isomorphic to the structure sheaf. Similarly, in the locus $((3x^2 - 1) \neq 0)$, the sheaf $\widetilde{\Omega_{A/B}}$ is generated by dy, and is isomorphic to the structure sheaf. Since the curve defined by the equation $y^2 - x^3 + x = 0$ is covered by those two loci, we conclude that $\widetilde{\Omega_{A/B}}$ is an invertible sheaf.

(4) Let $A = k[x,y]/(y^2 - x^3)$. We have $\Omega_{A/B} \simeq (Adx \oplus Ady)/(2ydy - 3x^2dx)$, hence, at the origin $(x = y = 0)$, it is of rank 2. Outside of the origin, it is of rank 1. Hence, the sheaf $\widetilde{\Omega_{A/B}}$ is not a locally free sheaf.

There are two geometrically motivated exact sequences.

Proposition 116 (Relative cotangent sequence)**.** *Let $C \to B \to A$ be ring homomorphisms. There is a natural right exact sequence of A-modules*

$$A \otimes_B \Omega_{B/C} \xrightarrow{a\otimes db \mapsto adb} \Omega_{A/C} \xrightarrow{da \mapsto da} \Omega_{A/B} \longrightarrow 0.$$

Proposition 117 (Conormal sequence)**.** *Let B is a C-algebra, I an ideal of B, and $A = B/I$. Then there is a natural right exact sequence of A-modules*

$$I/I^2 = A \otimes_B I \xrightarrow{\delta = 1\otimes d} A \otimes_B \Omega_{B/C} \xrightarrow{a\otimes db \mapsto adb} \Omega_{A/C} \to 0.$$

Remark 118. The above statements generalize basic constructions in geometry. Suppose we have a nice "fibration" $\pi : X \to Y$ so that all the fibers of π are smooth. Let Z be a single point. Then we have a short exact sequence of relative tangent bundles (sheaves), which can be checked on the stalks (= tangent spaces)

$$0 \to \mathcal{T}_{X/Y} \to \mathcal{T}_{X/Z} \to \pi^*\mathcal{T}_{Y/Z} \to 0.$$

Taking the dual, we have

$$0 \to \pi^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

which has the same form of the relative cotangent sequence we stated. When we do not assume "smoothness", we may lose the left exactness in general.
Similarly, when $j : Y \subset X$ is an embedding of smooth manifolds, then the inclusion of tangent spaces induces a short exact sequence, called the normal bundle sequence,

$$0 \to \mathcal{T}_Y \to j^*\mathcal{T}_X \to \mathcal{N}_{Y/X} \to 0.$$

Its dual has the form of the conormal sequence.

The sheaf of Kähler differentials coincides with the (Zariski) cotangent space in many cases, in particular, it measures whether a given variety is nonsingular or not at a point:

Theorem 119. *Let $(B, \mathfrak{m})$ be a local ring which contains its residue field $k = B/\mathfrak{m}$. The map $\delta : \mathfrak{m}/\mathfrak{m}^2 \to k \otimes_B \Omega_{B/k}$ is an isomorphism.*
Assume furthermore that k is perfect, and B is a localization of a finitely generated k-algebra. Then $\Omega_{B/k}$ is a free B-module of rank equal to the Krull dimension $\dim B$ *if and only if B is a regular local ring.*

Exercise 120. Let $\phi : B \to A$ be a ring homomorphism, $S \subseteq A$ be a multiplicative subset of A, and let $T \subseteq B$ be a multiplicative subset of B such that $\phi(T) \subseteq S$. Along the commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & T^{-1}B \\ {\scriptstyle \phi}\downarrow & & \downarrow \\ A & \longrightarrow & S^{-1}A \end{array}$$

we have a natural base change map $S^{-1}\Omega_{A/B} \to \Omega_{S^{-1}A/T^{-1}B}$. Show that this map is an isomorphism.

Now we give a definition of the sheaf of Kähler differentials for a general setting. Let $\pi : X \to Y$ be a morphism of schemes. It can be covered by open affine subsets of the form $U = \operatorname{Spec} A$ of X and $V = \operatorname{Spec} B$ of Y such that $\pi(U) \subseteq V$. We will define $\Omega_{U/V}$ as the sheaf associated to the module $\Omega_{A/B}$. Note that the module of Kähler differentials are compatible with the localizations, and hence they glue together and form a sheaf $\Omega_{X/Y}$. Also note that the derivations $d : A \to \Omega_{A/B}$ glue together to give a map $d : \mathcal{O}_X \to \Omega_{X/Y}$ of sheaves of abelian groups on X. Also the sequences we discussed above generalize as follows.

Proposition 121.

(a) (Relative cotangent sequence) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Then there is a right exact sequence of sheaves on X

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

(b) (Conormal sequence) Let $f : X \to Y$ be a morphism, and let Z be a closed subscheme of X, with the ideal sheaf $\mathscr{I}$. Then there is a right exact sequence of sheaves on Z

$$\mathscr{I}/\mathscr{I}^2 \to \Omega_{X/Y} \otimes \mathcal{O}_Z \to \Omega_{Z/Y} \to 0.$$

We will give an example of exact sequences which will be very useful in future.

Theorem 122 (Euler sequence)**.** *Let A be a ring, $Y = \operatorname{Spec} A$, and let $X = \mathbb{P}^n_A$. There is a short exact sequence of sheaves on X*

$$0 \to \Omega_{X/Y} \to \mathcal{O}_X(-1)^{(n+1)} \to \mathcal{O}_X \to 0.$$

Proof. Let $S = A[x_0, \cdots, x_n]$ be the homogeneous coordinate ring of X. Let E be the graded S-module $S(-1)^{(n+1)}$, with basis $e_0, \cdots, e_n$ in degree 1. Sending e_i to x_i gives a degree 0 homomorphism of graded S-modules $E \to S$. We denote its kernel by M, so we have a left exact sequence

$$0 \to M \to E \to S$$

of graded S-modules. Taking the $\sim$ functor, we have a short exact sequence

$$0 \to \widetilde{M} \to \mathcal{O}_X(-1)^{(n+1)} \to \mathcal{O}_X \to 0,$$

where the last map becomes surjective since the corresponding map $E \to S$ is surjective for every positive degree.
It remains to show that $\widetilde{M} \simeq \Omega_{X/Y}$. On the distinguished open subset $U_i = D_+(x_i)$, $\widetilde{M}$ comes from a free module M_{x_i} which is generated by $\{e_j/x_i - (x_j/x_i^2)e_i \mid j \neq i\}$.
Since $U_i \simeq \operatorname{Spec} A[x_0/x_i, \cdots, x_n/x_i]$, the sheaf $\Omega_{X/Y}|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module generated by $d(x_0/x_i), \cdots, d(x_n/x_i)$. Via the identification φ_i defined by

$$d(x_j/x_i) \mapsto (1/x_i^2)(x_i e_j - x_j e_i),$$

we see that $\Omega_{X/Y}|_{U_i} \simeq \widetilde{M}|_{U_i}$. On the intersection $U_i \cap U_j$, we have $(x_k/x_i) = (x_k/x_j) \cdot (x_j/x_i)$ for any k. Hence, in $\Omega_{X/Y}|_{U_i \cap U_j}$, we have

$$d(x_k/x_i) - x_k/x_j d(x_j/x_i) = x_j/x_i d(x_k/x_j).$$

Applying φ_i on the left hand side, and φ_j on the right hand side, we immediately check that both are identified with a single element $(1/x_i x_j)(x_j e_k - x_k e_j)$. Hence φ_i's glue together and give an isomorphism $\Omega_{X/Y} \simeq \widetilde{M}$. □

The conormal sequence measures the "smoothness" of a subvariety in some sense:

Theorem 123. *Let X be a nonsingular variety over an algebraically closed field k. Let $Y \subseteq X$ be a closed subvariety defined by a sheaf of ideals $\mathscr{I}$. Then Y is nonsingular if and only if*

(1) $\Omega_{Y/k}$ is locally free, and

(2) the conormal sequence $0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_{X/k} \otimes \mathcal{O}_Y \to \Omega_{Y/k} \to 0$ is exact on the left.

Furthermore, in this case, $\mathscr{I}$ is locally generated by $r = \operatorname{codim}(Y, X)$ elements, and the sheaf $\mathscr{I}/\mathscr{I}^2$ is a locally free sheaf of rank r on Y; called the conormal bundle of Y in X.

Proof. See Hartshorne's book (II.8.17). □

Definition 124. Let X be a nonsingular variety over an algebraically closed field k of dimension n, and let Y be a subvariety of X defined by a sheaf of ideals $\mathscr{I}$. The *tangent sheaf* of X is defined to be $\mathcal{T}_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$. The *canonical sheaf* of X is defined to be the invertible sheaf $\omega_X := \det(\Omega_{X/k}) = \wedge^n \Omega_{X/k}$. When X is projective, the nonnegative integer $p_g := \Gamma(X, \omega_X)$ is called the *geometric genus* of X, which is a very important birational invariant of X.
The sheaf $\mathscr{I}/\mathscr{I}^2$ is called the *conormal sheaf* of Y in X. Its dual $\mathscr{N}_{Y/X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathscr{I}/\mathscr{I}^2, \mathcal{O}_Y)$ is called the *normal sheaf* of Y in X. When Y is nonsingular, then they are locally free of rank $r = \operatorname{codim}(Y, X)$.

Note that the dual of conormal sequence $0 \to \mathcal{T}_Y \to \mathcal{T}_X|_Y \to \mathscr{N}_{Y/X} \to 0$ generalizes the comparison of tangent spaces of Y inside of the one of X.

Proposition 125 (Adjunction formula). *Let Y be a nonsingular subvariety of codimension r in a nonsingular variety X over k. Then $\omega_Y \simeq \omega_X \otimes \wedge^r \mathcal{N}_{Y/X}$.*

Proof. Take the highest exterior powers in the exact sequence of locally free sheaves

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \to 0$$

where $\mathscr{I}$ is the ideal sheaf of Y in X. We find that $\omega_X \otimes \mathcal{O}_Y \simeq \omega_Y \otimes \wedge^r(\mathscr{I}/\mathscr{I}^2)$. □

Example 126. From the Euler sequence $0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{(n+1)} \to \mathcal{O}_{\mathbb{P}^n} \to 0$, one immediately checks that $\omega_{\mathbb{P}^n} = \mathcal{O}(-n-1)$. If we have a smooth hypersurface Y of degree d, then the ideal sheaf $\mathscr{I} = \mathcal{O}(-Y) = \mathcal{O}(-d)$ becomes locally free. In particular, $\mathscr{I}/\mathscr{I}^2 = \mathscr{I} \otimes \mathcal{O}_{\mathbb{P}^n}/\mathscr{I} = \mathcal{O}_Y(-d)$, and hence $\omega_Y = \omega_X \otimes (\mathscr{I}/\mathscr{I}^2)^\vee = \mathcal{O}_Y(d-n-1)$.

Remark 127. Let Y be a closed subscheme of a nonsingular variety X over k. Y is called a *local complete intersection* if the ideal sheaf $\mathscr{I}$ of Y can be locally generated by $r = \operatorname{codim}(Y, X)$ elements at every point. When Y itself is nonsingular, then it must be a local complete intersection. Being a local complete intersection is an intrinsic property; it does not depend on the choice of a nonsingular variety X containing Y.

Exercise 128. Let $Y \subseteq \mathbb{P}^n_k$ be a complete intersection, that is, the homogeneous ideal I of Y in $S = k[x_0, \cdots, x_n]$ can be generated by $r = \operatorname{codim}(Y, \mathbb{P}^n_k)$ elements.

(a) Let Y be a closed subscheme of codimension r in $\mathbb{P}^n$. Show that Y is a complete intersection if and only if there are hypersurfaces $H_1, \cdots, H_r$ such that $Y = H_1 \cap H_2 \cap \cdots \cap H_r$ as schemes, that is, $\mathscr{I}_Y = \mathscr{I}_{H_1} + \cdots + \mathscr{I}_{H_r}$.

(b) If Y is a complete intersection of dimension ≥ 1, and if Y is normal, then Y is projectively normal.

(c) If Y is a nonsingular complete intersection $Y = H_1 \cap \cdots \cap H_r$, and $d_i = \deg H_i$, then $\omega_Y \simeq \mathcal{O}_Y(\sum d_i - n - 1)$.

(d) If Y is a nonsingular plane curve of degree d, then $p_g(Y) = (d-1)(d-2)/2$.

Exercise 129. Let X be a nonsingular projective variety over k. For any $n > 0$, we define the n-th plurigenus of X to be $P_n := \dim_k \Gamma(X, \omega_X^{\otimes n})$. For any $0 \leq q \leq \dim X$, we define the Hodge number at $(q, 0)$ to be the integer $h^{q,0} := \dim_k \Gamma(X, \wedge^q \Omega_{X/k})$ where $\wedge^q \Omega_{X/k}$ is a sheaf of regular q-forms on X. They are generalizations of the geometric genus since $P_1 = h^{\dim X, 0} = p_g$. Show that they are birational invariants of X, that is, if X' is birational to X, then $P_n(X') = P_n(X)$ and $h^{q,0}(X') = h^{q,0}(X)$.

Topic 3 – Derived functors and cohomology

In this section we define the cohomology of a sheaf of abelian groups on a topological space. Of course, the most important objects are the cohomology groups of quasi-coherent and coherent sheaves on a variety. It is well known that J. Leray had invented the notion of sheaves, sheaf cohomology, and spectral sequences at the prisoner camp. His definitions were simplified and clarified in 1950s. The first person who brought this notion to algebraic geometry is J.-P. Serre. In his celebrating paper "Faisceaux Algebriques Coherents", one could easily find the notion of coherent sheaves on algebraic varieties which is almost same as the one in present. In this paper, he used Čech cohomology instead of the sheaf cohomology we use in present. The central figure of the study follows a general idea of A. Grothendieck, in particular, is well-described in his 1957 Tohoku paper "Sur quelques points d'algèbre homologique". Naively speaking, sheaf cohomology describes an obstruction to solving a global geometric problem which can be solved locally. Unfortunately, computing the sheaf cohomology as the derived functor is almost impossible in practice, we should go back to the Čech cohomology. Fortunately, two cohomologies agree in many cases, for instance, for quasi-coherent sheaves on affine/projective varieties.

1 Derived Functors

> *Ça me semble extrêmement plaisant de ficher comme ça beaucoup de choses, pas drôles quand on les prend séparément, sous le grand chapeau des foncteurs dérivés.*
>
> *I find it very agreeable to stick all sorts of things, which are not much fun when taken individually, together under the heading of derived functors.*
>
> – A. Grothendieck, a letter to J.-P. Serre, Feb. 18th, 1955. Excerpted from: Grothendieck-Serre correspondence

In modern algebraic geometry, several cohomology theories are defined as the derived functors of some functors which appears naturally. A good motivation is that a short exact sequence often gives rise to a long exact sequence. For instance, if we have a short exact sequence of abelian groups (= $\mathbb{Z}$-modules)

$$0 \to A \to B \to C \to 0,$$

then taking a tensor product with another $\mathbb{Z}$-module M gives us a right exact sequence

$$A \otimes M \to B \otimes M \to C \otimes M \to 0.$$

Taking projective resolutions of A, B, C, then tensoring by M also affects on the projective resolution, and the "changes" are measured by Tor groups. In particular, we have a long exact sequence

$$\cdots \to \mathrm{Tor}^2(C, M) \to \mathrm{Tor}^1(A, M) \to \mathrm{Tor}^1(B, M) \to \mathrm{Tor}^1(C, M) \to A\otimes M \to B\otimes M \to C\otimes M \to 0.$$

Here, one also should notice that if the functor $(-) \otimes M$ is good enough (for instance, the case when M is flat), then the functor becomes exact; hence the Tor groups will vanish since they measure how the functor $(-) \otimes M$ is far from being "exact".
The sheaf cohomology is quite similar in this manner; if we have a short exact sequence of sheaves of abelian groups $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ on a topological space X, then the global section functor gives a left exact sequence

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'').$$

We may ask how to continue this sequence to the right. We first meet two differences between the Tor functor: the global section functor $\Gamma(X, -)$ is not right exact, but left exact; and a sheaf of abelian groups on X may not have a projective resolution.
Almost every category we discussed carefully is indeed *abelian*; so that the objects and morphisms can be added and "subtracted", that is, kernels and cokernels always exist. For instance, $\mathfrak{Ab}(X)$ the category of sheaves of abelian groups on a topological space X, $\mathfrak{Mod}(X)$ the category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$, $\mathfrak{Qco}$ the category of quasi-coherent $\mathcal{O}_X$-modules on a scheme $(X, \mathcal{O}_X)$, or $\mathfrak{Coh}(X)$ the category of coherent $\mathcal{O}_X$-modules on a noetherian scheme $(X, \mathcal{O}_X)$ are important examples of abelian categories.
Let us recall some basic notions in homological algebra very briefly.

Definition 130. Let $\mathfrak{A}$ is an abelian category. A *complex* $A^\bullet$ in $\mathfrak{A}$ is a collection of objects A^i, $i \in \mathbb{Z}$ and morphisms $d^i : A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for every i. If the objects A^i are specified only in a certain range, then we regard $A^i = 0$ for all other i's. A *morphism of complexes* $f : A^\bullet \to B^\bullet$ is a set of morphisms $f^i : A^i \to B^i$ which makes the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \xrightarrow{d_A^i} & A^{i+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\
\cdots & \longrightarrow & B^{i-1} & \xrightarrow[d_B^{i-1}]{} & B^i & \xrightarrow[d_B^i]{} & B^{i+1} & \longrightarrow & \cdots
\end{array}$$

The *i-th cohomology* $h^i(A^\bullet)$ of the complex $A^\bullet$ is defined to be $\ker d^i / \operatorname{im} d^{i-1}$. If $f : A^\bullet \to B^\bullet$ is a morphism of complexes, then f induces a natural map $h^i(f) : h^i(A^\bullet) \to h^i(B^\bullet)$ for each i.
If $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is a short exact sequence of complexes (note that the category of complexes in $\mathfrak{A}$ is again abelian, so we are able to say the exactness), then

there are natural maps $\delta^i : h^i(C^\bullet) \to h^{i+1}(A^\bullet)$, called *connecting morphisms*, which give rise to a cohomology long exact sequence

$$\cdots \to h^i(A^\bullet) \to h^i(B^\bullet) \to h^i(C^\bullet) \xrightarrow{\delta^i} h^{i+1}(A^\bullet) \to \cdots .$$

Two morphisms $f, g : A^\bullet \to B^\bullet$ are *homotopic* if there is a collection of morphisms $k^i : A^i \to B^{i-1}$ such that $f - g = d_B k + k d_A$. If f and g are homotopic, then they will induce the same morphism $h^i(A^\bullet) \to h^i(B^\bullet)$ on the cohomology. A *quasi-isomorphism* is a morphism $f : A^\bullet \to B^\bullet$ of complexes whose induced maps $h^i(f) : h^i(A^\bullet) \to h^i(B^\bullet)$ are isomorphisms for all i.
A covariant(resp., contravariant) functor $F : \mathfrak{A} \to \mathfrak{B}$ is *additive* if for any two objects $A, A' \in \mathfrak{A}$, the induced map $\mathrm{Hom}(A, A') \to \mathrm{Hom}(FA, FA')$ (resp., $\mathrm{Hom}(A, A') \to \mathrm{Hom}(FA', FA)$) is a homomorphism of abelian groups. F is *left exact* if it is additive and for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathfrak{A}$, the sequence $0 \to FA' \to FA \to FA''$ (resp., $0 \to FA'' \to FA \to FA'$) is exact in $\mathfrak{B}$. If we can write 0 on the right instead of the left, we say F is *right exact.* If F is both left and right exact, then we say it is *exact.* If only the middle part is exact, we say F is *exact in the middle.*

Example 131. (1) The global section functor $\Gamma(X, -) : \mathfrak{Ab}(X) \to \mathfrak{Ab}$ is a covariant left exact functor.

(2) Let $f : X \to Y$ be a continuous map between two topological spaces. The direct image functor $f_* : \mathfrak{Ab}(X) \to \mathfrak{Ab}(Y)$ is a covariant left exact functor.

(3) Let $\mathfrak{A}$ be an abelian category, and let A be an object. The functor $\mathrm{Hom}(A, -) : \mathfrak{A} \to \mathfrak{Ab}$ which is defined by $B \mapsto \mathrm{Hom}(A, B)$ is a covariant left exact functor. The functor $\mathrm{Hom}(-, A)$ is a contravariant left exact functor.

Definition 132. Let $\mathfrak{A}$ be an abelian category. An object $I \in \mathfrak{A}$ is *injective* if the functor $\mathrm{Hom}(-, I)$ is exact. An object $P \in \mathfrak{A}$ is *projective* if the functor $\mathrm{Hom}(P, -)$ is exact.
An *injective resolution* of an object $A \in \mathfrak{A}$ is a complex $I^\bullet$, defined in degrees $i \geq 0$, together with a morphism $A \to I^0$ such that each I^i is injective and the sequence

$$0 \to A \to I^0 \to I^1 \to \cdots$$

is exact. A *projective resolution* of an object $A \in \mathfrak{A}$ is a complex $P^\bullet$, defined in degrees $i \leq 0$, together with a morphism $P^0 \to A$ such that each P^i is projective and the sequence

$$\cdots \to P^{-1} \to P^0 \to A \to 0$$

is exact.
If every object $A \in \mathfrak{A}$ is isomorphic to a subobject of an injective object, then we say $\mathfrak{A}$ *has enough injectives.* If $\mathfrak{A}$ has enough injectives, then every object has an injective resolution. When we have a morphism $f : A \to B$, $I_A^\bullet$ an injective resolution of A, and

$I_B^\bullet$ an injective resolution of B, then f can be "extended" to a map $f : I_A^\bullet \to I_B^\bullet$ of complexes. In particular, any two injective resolutions are homotopy equivalent.
Let $\mathfrak{A}$ be an abelian category with enough injectives, and let $F : \mathfrak{A} \to \mathfrak{B}$ be a covariant left exact functor. We define the *right derived functors* R^iF, $i \geq 0$ of F by $R^iF(A) := h^i(F(I^\bullet))$ where $I^\bullet$ is an injective resolution of A.