We can immediately check the following properties, hence, we will skip the proof.

Proposition 133. *Let $\mathfrak{A}$ be an abelian category with enough injectives, and let $F : \mathfrak{A} \to \mathfrak{B}$ be a covariant left exact functor to another abelian category $\mathfrak{B}$. Then*

(1) For each $i \geq 0$, the functor R^iF is an additive functor from $\mathfrak{A}$ to $\mathfrak{B}$. It is independent of the choices of injective resolutions.

(2) There is a natural isomorphism $R^0F \simeq F$.

(3) For each short exact sequence $0 \to A' \to A \to A'' \to 0$ and for each $i \geq 0$, there is a natural morphism $\delta^i : R^iF(A'') \to R^{i+1}F(A')$, such that we have a long exact sequence

$$\cdots \to R^iF(A') \to R^iF(A) \to R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \to \cdots .$$

(4) Given a morphism of the exact sequence of (c) to another $0 \to B' \to B \to B'' \to 0$, the δ's give a commutative diagram

$$\begin{tikzcd} R^iF(A'') \arrow[r, "\delta^i"] \arrow[d] & R^{i+1}F(A') \arrow[d] \\ R^iF(B'') \arrow[r, "\delta^i"] & R^{i+1}F(B') \end{tikzcd}$$

(5) For each injective object I of $\mathfrak{A}$, we have $R^iF(I) = 0$ for every $i > 0$.

Definition 134. Let $F : \mathfrak{A} \to \mathfrak{B}$ be as above. An object $J \in \mathfrak{A}$ is *acyclic* if $R^iF(J) = 0$ for every $i > 0$.

We may use an acyclic resolution instead of an injective resolution to compute the derived functor:

Proposition 135. *Let $F : \mathfrak{A} \to \mathfrak{B}$ be as above. Let $A \in \mathfrak{A}$ be an object such that it admits a resolution*

$$0 \to A \to J^0 \to J^1 \to \cdots$$

where each J^i is acyclic for F (we say $J^\bullet$ is an F-acyclic resolution of A). Then for each $i \geq 0$, there is a natural isomorphism $R^iF(A) \simeq h^i(F(J^\bullet))$.

We will see that this definition of derived functors is very natural, by showing a universal property. We need to generalize our notion slightly.

Definition 136. Let $\mathfrak{A}, \mathfrak{B}$ be abelian categories. A covariant *δ-functor* from $\mathfrak{A}$ to $\mathfrak{B}$ is a collection of functors $T^\bullet = (T^i)_{i\geq 0}$, together with a morphism $\delta^i : T^i(A'') \to T^{i+1}(A')$ for each short exact sequence $0 \to A' \to A \to A'' \to 0$ and each $i \geq 0$ such that

(i) For each short exact sequence above, we have a long exact sequence

$$\cdots \to T^i(A') \to T^i(A) \to T^i(A'') \xrightarrow{\delta^i} T^{i+1}(A') \to \cdots .$$

(ii) For each morphism of a short exact sequence to another short exact sequence $0 \to B' \to B \to B'' \to 0$, we have a commutative diagram

$$\begin{tikzcd} T^i(A'') \arrow[r, "\delta^i"] \arrow[d] & T^{i+1}F(A') \arrow[d] \\ T^i(B'') \arrow[r, "\delta^i"] & T^{i+1}F(B') \end{tikzcd}$$

A δ-functor $T = (T^i)$ is *universal* if, given any other δ-functor T', and any given morphism of functors $f^0 : T^0 \to T'^0$, there is a unique sequence of morphisms $f^i : T^i \to T'^i$ for each $i \geq 0$, starting with the given f^0, which commute with the δ^i for each short exact sequence.

Proposition 137. *Assume that $\mathfrak{A}$ is an abelian category with enough injectives. For any left exact functor $F : \mathfrak{A} \to \mathfrak{B}$, the derived functors $(R^iF)_{i\geq 0} : \mathfrak{A} \to \mathfrak{B}$ form a universal δ-functor with $F \simeq R^0F$. Conversely, if $T = (T^i)$ is any universal δ-functor from $\mathfrak{A}$ to $\mathfrak{B}$, then T^0 is left exact, and the T^i are isomorphic to R^iT^0 for every $i \geq 0$.*

Hence, the right derived functor is a unique way to extend naturally a left exact functor. To conclude the construction, it only remains to show that the category of sheaves of abelian groups has enough injectives.

Proposition 138. *Let X be any topological space. The category $\mathfrak{Ab}(X)$ of sheaves of abelian groups on X has enough injectives.*

Proof. We will first prove that the caegory $\mathfrak{Mod}(X)$ of $\mathcal{O}_X$-modules has enough injectives when $(X, \mathcal{O}_X)$ is a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. For each point $x \in X$, the stalk $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module. Since every module is a submodule of an injective module, there is an injection $\mathcal{F}_x \hookrightarrow I_x$ where I_x is an injective $\mathcal{O}_{X,x}$-module. Let $j_x : \{x\} \hookrightarrow X$ be the inclusion map. The sheaf $\mathcal{I} := \prod_{x\in X}(j_x)_*(I_x)$, considered I_x as a (constant) sheaf on the one point set $\{x\}$, becomes injective:

> Let $\mathcal{G}$ be any $\mathcal{O}_X$-module. We have $\mathrm{Hom}(\mathcal{G}, \mathcal{I}) = \prod_{x\in X} \mathrm{Hom}(\mathcal{G}, (j_x)_*(I_x))$ since the direct product commutes with the Hom functor. Note that each $\mathrm{Hom}(\mathcal{G}, (j_x)_*(I_x))$ is isomorphic to a module homomorphism group $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$. Since each I_x is an injective $\mathcal{O}_{X,x}$-module, the functor $\mathrm{Hom}(-, \mathcal{I})$ is a direct product of exact functors, which is also exact.

There is a natural morphism of $\mathcal{O}_X$-modules $\mathcal{F} \hookrightarrow \mathcal{I}$ obtained from the local maps $\mathcal{F}_x \hookrightarrow \mathcal{I}_x$. It is obviously injective, hence, $\mathcal{F}$ is isomorphic to a subsheaf of an injective $\mathcal{O}_X$-module $\mathscr{I}$. We conclude that the category $\mathfrak{Mod}(X)$ has enough injectives.
$\mathfrak{Ab}(X)$ also has enough injectives since $\mathfrak{Ab}(X)$ coincides with $\mathfrak{Mod}(X)$ when we assign a sheaf of rings $\mathcal{O}_X$ on a topological space X as the constant sheaf $\underline{\mathbb{Z}}$ on X. $\square$

2 Sheaf cohomology and its properties

Now we are ready to define the sheaf cohomology as derived functors. We also define the higher direct image functor, since it follows from the exactly same argument.

Definition 139. Let X be a topological space. We define *the cohomology functors* $H^i(X,-)$ to be the right derived functor of the global section functor $\Gamma(X,-)$. For any sheaf $\mathcal{F}$ of abelian groups on X, the groups $H^i(X,\mathcal{F})$ are the *cohomology groups of* $\mathcal{F}$. If $f:(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ is a morphism of ringed spaces, we define *higher direct images* $R^if_*(-)$ to be the right derived functor of the direct image functor f_*.

Remark 140. The category $\mathfrak{Mod}(X)$ of $\mathcal{O}_X$-modules does not have enough projectives in general. Let $X=\mathbb{P}^1_k$ be the projective line with the Zariski topology, and let $\mathcal{O}_X=\underline{\mathbb{Z}}$ be the constant sheaf associated to $\mathbb{Z}$. If $\mathfrak{Mod}(X)$ has enough projectives, then there would be a surjection $\varphi:\mathscr{P}\to\mathcal{O}_X$ from a projective sheaf $\mathscr{P}$. In particular, the induced map on a stalk φ_x is surjective for every $x\in X$.
Let $\mathcal{O}_U$ denote the sheaf on X obtained by restricting $\mathcal{O}_X$ onto U and then extending by zero outside of U. Let U be an open neighborhood of x, and let V be a strictly smaller open neighborhood. Consider the surjection $\mathcal{O}_{X\setminus\{x\}}\oplus\mathcal{O}_V\to\mathcal{O}_X$. Since $\mathscr{P}$ is projective, the surjection φ lifts to $\mathscr{P}\to\mathcal{O}_{X\setminus\{x\}}\oplus\mathcal{O}_V$. The map $\mathscr{P}(U)\to\mathscr{O}_X(U)$ factors through $\mathcal{O}_{X\setminus\{x\}}(U)\oplus\mathcal{O}_V(U)=0$, hence, any map $\mathscr{P}(U)\to\mathcal{O}_X(U)$ is zero for any choice of open neighborhood U of x. We conclude that $\varphi:\mathscr{P}\to\mathcal{O}_X$ is zero since it gives a zero map on each stalk.

Next, we see that any injective $\mathcal{O}_X$-module on a ringed space is acyclic. In particular, an injective resolution is an example of an acyclic resolution for the global section functor. We will have a small detour; passing through the notion of flasque sheaves.

Lemma 141. *Let $(X,\mathcal{O}_X)$ be any ringed space. Any injective $\mathcal{O}_X$-module $\mathcal{I}$ is flasque,* i.e.*, every restriction map $\mathcal{I}(U)\to\mathcal{I}(V)$ is surjective.*

Proof. Let $U\subseteq X$ be an open subset. Let $\mathcal{O}_U$ denote the sheaf $j_!\mathcal{O}_X|_U$, which is the restriction of $\mathcal{O}_X$ to U, extended by zero outside of U. If we have any inclusion of open sets $V\subseteq U$, we have an inclusion $0\to\mathcal{O}_V\to\mathcal{O}_U$. Hence, for any injective $\mathcal{O}_X$-module $\mathcal{I}$, we have a surjection

$$\mathrm{Hom}(\mathcal{O}_U,\mathcal{I})\to\mathrm{Hom}(\mathcal{O}_V,\mathcal{I})\to 0.$$

On the other hand, the functor $\mathrm{Hom}(\mathcal{O}_U,-)$ is same as the global section functor $\Gamma(U,-)$ taking on U, in particular, the above surjection identifies with the restriction map $\mathcal{I}(U)\to\mathcal{I}(V)$. □

Then it is easy to see that any flasque sheaf is acyclic:

Proposition 142. *Let $\mathcal{F}$ be a flasque sheaf on a topological space X. Then $H^i(X,\mathcal{F})=0$ for all $i>0$.*

Proof. Let $\mathcal{I}$ be an injective sheaf in $\mathfrak{Ab}(X)$ containing $\mathcal{F}$, and let $\mathcal{G}$ be its quotient. We have a short exact sequence
$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0.$$
Since $\mathcal{F}$ is flasque, we have a short exact sequence on sections
$$0 \to \mathcal{F}(U) \to \mathcal{I}(U) \to \mathcal{G}(U) \to 0$$
for every open set $U \subseteq X$. Now $\mathcal{I}$ is also flasque, one can immediately check that $\mathcal{G}$ is also flasque. Since $\mathcal{I}$ is injective, we have $H^i(X, \mathcal{I}) = 0$ for all $i > 0$.
Therefore, from the long exact sequence of cohomology, we have $H^1(X, \mathcal{F}) = 0$ and $H^{i+1}(X, \mathcal{F}) \simeq H^i(X, \mathcal{G})$ for every $i \geq 1$. But $\mathcal{G}$ is also flasque, hence, we get the desired result by the induction on i. □

In particular, we can calculate the cohomology using a flasque resolution, which does not depend on the choice of the sheaf of rings $\mathcal{O}_X$. It means, there is no difference between the right derived functor of $\Gamma(X, -) : \mathfrak{Ab}(X) \to \mathfrak{Ab}$ (= the sheaf cohomology functor $H^i(X, -)$) and the right derived functor of $\Gamma(X, -) : \mathfrak{Mod}(X) \to \mathfrak{Ab}$. It does not depend on the choice of the ring structure on a topological space X.
On the other hand, if we have a ringed space $(X, \mathcal{O}_X)$ with a ring $A = \Gamma(X, \mathcal{O}_X)$, then any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ admits an A-module structure on $\Gamma(X, \mathcal{F})$. Thus, any cohomology group $H^i(X, \mathcal{F})$ also has a natural A-module structure.

The cohomology we defined has nice properties we may expect. We will introduce some of them, without their proofs if too complicated. The following dimensional cohomology vanishing implies that the cohomology groups are constrained by the dimension of the underlying space X.

Theorem 143 (Grothendieck vanshing / dimensional cohomology vanishing)**.** *Let X be a noetherian topological space of dimension n. Then for any $i > n$ and for any sheaf of abelian groups $\mathcal{F}$ on X, the cohomology group $H^i(X, \mathcal{F}) = 0$ vanishes.*

Proposition 144. *When X is a paracompact Hausdorff space which is locally contractible (e.g. X is a manifold or a CW-complex), then the singular cohomology group $H^i_{sing}(X, A)$ of X with coefficients in the abelian group A coincides with the sheaf cohomology group $H^i(X, \underline{A})$, where $\underline{A}$ is the constant sheaf on X associated to A.*

Caution. Let X be an irreducible complex projective manifold, or a variety over $\mathbb{C}$, together with the constant sheaf $\underline{\mathbb{C}}$. If we regard X as a topological space with the Zariski topology, then it has too few number of open subsets, so that the constant sheaf $\underline{\mathbb{C}}$ becomes flasque. In particular, the sheaf cohomology groups of X are
$$H^i(X, \underline{\mathbb{C}}) = \begin{cases} \mathbb{C} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$
Of course, it is different from the singular cohomology with coefficients in $\mathbb{C}$ in general. To correct the difference, we need to add "more open subsets" in the topology of X, which leads to a motivation of *étale cohomology.*

Remark 145. There is a notion of "cohomology with supports" for a given closed subset Y of X. The functor $\Gamma_Y(X, -)$ which collects the sections with support in Y is a left exact functor, hence, taking its right derived functor provides a cohomology functor H^i_Y. Using this, one can generalize the excision and the Mayer-Vietoris sequence in singular (co)homology; see Exercise III-2.3 and III-2.4 of Hartshorne's book.

Affine schemes and quasi-coherent sheaves on them play as building blocks; in particular, they are analogous to the contractible space in cohomological point of view. This "affine cohomology vanishing" can also be used to determine whether a given underlying space is affine or not.

Theorem 146 (Serre)**.** *For a noetherian scheme X, the following are equivalent:*

(i) X is affine;

(ii) $H^i(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ and all $i > 0$;

(iii) $H^1(X, \mathscr{I}) = 0$ for all coherent sheaves of ideals $\mathscr{I}$.

Proof. When $X = \operatorname{Spec} A$ is affine and $\mathcal{F}$ is quasi-coherent, then we have an A-module $M = \Gamma(X, \mathcal{F})$ such that $\widetilde{M} \simeq \mathcal{F}$. We take an injective A-resolution of M as $0 \to M \to I^\bullet$ in the category of A-modules. One can show that each sheaf $\widetilde{I^i}$ is flasque (Hartshorne's book, (III.3.4)), and hence we can compute the cohomology groups of $\mathcal{F}$ by a flasque resolution $\widetilde{I^\bullet}$ of $\mathcal{F}$. Applying the functor Γ, we recover the original injective A-resolution $0 \to M \to I^\bullet$, in particular, $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.
$(iii) \Rightarrow (i)$ is a little bit tricky; we refer Hartshorne's book (III.3.7) for the details. We leave only a brief idea. Let $A = \Gamma(X, \mathcal{O}_X)$. Since H^1 vanishes for every ideal sheaf, for each point $P \in X$ one can choose an element $f_P \in A$ such that $\operatorname{Spec} A_f$ becomes an affine open neighborhood of P. Thanks to the quasi-compactness, we can choose a finite affine open covering $\{\operatorname{Spec} A_{f_1}, \cdots, \operatorname{Spec} A_{f_r}\}$ of X. Now the affineness of X is equivalent to the fact that the elements $f_1, \cdots, f_r$ generate the unit ideal in A. Let $\alpha : \mathcal{O}_X^{\oplus r} \to \mathcal{O}_X$ be the map corresponding to the matrix $(f_1 \cdots f_r)$. Its kernel $(\ker \alpha)$ admits a filtration where each quotient has a form of a sheaf of ideals in $\mathcal{O}_X$, one can check that $H^1(X, \ker \alpha) = 0$. In other words, $H^0(X, \mathcal{O}_X^{\oplus r}) \to H^0(X, \mathcal{O}_X) = A$ is surjective, and hence $f_1, \cdots, f_r$ generate the unit ideal in A. □

Remark 147. This is an algebraic version of Cartan's theorem B in complex geometry. Cartan's theorem A and B states that if $\mathcal{F}$ is an analytic coherent sheaf on a Stein manifold X, then $\mathcal{F}$ is generated by its global sections and $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. By the GAGA principle, this is equivalent to say that the higher cohomology groups vanishes for affine varieties over the complex numbers.

Remark 148. It is a special case of the following statement:

> If X is a variety which can be covered by n affine open subsets, then $H^i(X, \mathcal{F}) = 0$ for every quasi-coherent sheaf $\mathcal{F}$ and $i \geq n$.

For instance, when $X = \mathbb{P}^n_k$, it is covered by $(n+1)$ distinguished open subsets determined by the coordinates. Hence, the statement gives a special case of the dimension cohomology vanishing. In this point of view, the cohomology also encodes the information that how far the underlying space X is from being affine.

3 Čech cohomology

Čech cohomology is a cohomology theory for a sheaf of abelian groups on a topological space X, with respect to a given open covering of X. When X is a noetherian separated scheme, and an open covering are sufficiently nice, then the Čech cohomology groups for a quasi-coherent sheaf coincide with the sheaf cohomology groups. In particular, we need not to compute the cohomology via an injective resolution, which is almost impossible to reach in practice.

Let X be a topological space, and let $\mathfrak{U} = (U_i)_{i\in I}$ be an open covering of X. Choose and fix a well-ordering of the index set I. For any finite set of indices $i_0, \cdots, i_p \in I$, we denote the intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ by $U_{i_0,\cdots,i_p}$.

Let $\mathcal{F}$ be a sheaf of abelian groups on X. We define a complex $C^\bullet(\mathfrak{U}, \mathcal{F})$ of abelian groups as follows. For each $p \geq 0$, let

$$C^p(\mathfrak{U}, \mathcal{F}) := \prod_{i_0<\cdots<i_p} \mathcal{F}(U_{i_0,\cdots,i_p}).$$

An element $\alpha \in C^p(\mathfrak{U}, \mathcal{F})$ is determined by giving an element $\alpha_{i_0,\cdots,i_p} \in \mathcal{F}(U_{i_0,\cdots,i_p})$ for each $(p+1)$-tuple $i_0 < \cdots < i_p$ of elements in I. We define the coboundary map $d : C^p \to C^{p+1}$ by

$$(d\alpha)_{i_0,\cdots,i_p,i_{p+1}} := \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\cdots,\hat{i_k},\cdots,i_{p+1}}|_{U_{i_0,\cdots,i_{p+1}}}$$

where $\hat{i_k}$ means we omit the index i_k. Since $\alpha_{i_0,\cdots,\hat{i_k},\cdots,i_{p+1}}$ is an element in $\mathcal{F}(U_{i_0,\cdots,\hat{i_k},\cdots,i_{p+1}})$, we may take its restriction onto $U_{i_0,\cdots,i_{p+1}}$. One can easily check that $d^2 = 0$, that is, we have a complex of abelian groups.

Definition 149. Let X be a topological space and let $\mathfrak{U}$ be an open covering of X. For any sheaf of abelian groups $\mathcal{F}$ on X, we define the *p-th Čech cohomology group of $\mathcal{F}$ with respect to the covering* $\mathfrak{U}$, to be $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) := h^p(C^\bullet(\mathfrak{U}, \mathcal{F}))$.

Caution. In general, the Čech cohomology functor $\check{\mathrm{H}}^p(\mathfrak{U}, -)$ is not a δ-functor; we do not get a long exact sequence of Čech cohomology groups from a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$. For example, if $\mathfrak{U}$ is consisted of the single open set X, we do not have higher Čech cohomology groups, whereas $\check{\mathrm{H}}^0(\mathfrak{U}, -)$ coincides with the global section functor $\Gamma(X, -)$ which is not exact in general.

Example 150. Let $X = \mathbb{P}^1_k = \operatorname{Proj} k[X, Y]$ be a projective line, and let $\mathcal{F} = \omega$ be the sheaf of differentials. Let $\mathfrak{U}$ be the open covering by the two affine open sets $U = \mathbb{A}^1$ with affine coordinate $x = X/Y$ and $V = \mathbb{A}^1$ with affine coordinate $y = Y/X = 1/x$. The Čech complex has only two terms:

$$\begin{aligned} C^0 &= \Gamma(U, \omega) \times \Gamma(V, \omega) \\ C^1 &= \Gamma(U \cap V, \omega). \end{aligned}$$

Note that $\Gamma(U,\omega) = k[x]dx$, $\Gamma(V,\omega) = k[y]dy$, and $\Gamma(U\cap V,\omega) = k\left[x,x^{-1}\right]dx$. Also note that the map $d : C^0 \to C^1$ is given by

$$\begin{array}{rcl} x & \mapsto & x \\ y & \mapsto & 1/x \\ dy & \mapsto & d\left(\frac{1}{x}\right) = -\frac{1}{x^2}dx. \end{array}$$

So $\check{\mathrm{H}}^0(\mathfrak{U},\omega) = \ker d$ is the set of pairs $(f(x)dx, g(y)dy)$ such that $f(x) = -1/x^2 g(1/x)$. This can happen only if $f = g = 0$ since the left-hand-side is a polynomial in x, and the right-hand-side is a nonconstant polynomial in x^{-1} unless $g = 0$.
$\check{\mathrm{H}}^1(\mathfrak{U},\omega) = \operatorname{coker} d$ is obtained by mod out all the expressions

$$\left(f(x) + \frac{1}{x^2}g\left(\frac{1}{x}\right)\right)dx,$$

where f, g are polynomials. In particular, the image of d is generated by $x^n dx$, where $n \neq 1$. Therefore $\check{\mathrm{H}}^1(\mathfrak{U},\omega) \simeq k$, generated by the image of $x^{-1}dx$.

Example 151. Let S^1 be the unit circle in the Euclidean plane, and let $\underline{\mathbb{Z}}$ be the constant sheaf. Let $\mathfrak{U}$ be the open covering by two connected open sets $U = S^1 \setminus \{(0,1)\}$ and $V = S^1 \setminus \{(0,-1)\}$. In particular, $U \cap V$ can be identified with two open intervals. We have

$$\begin{array}{rl} C^0 = & \Gamma(U,\underline{\mathbb{Z}}) \times \Gamma(V,\underline{\mathbb{Z}}) = \mathbb{Z}\times\mathbb{Z} \\ C^1 = & \Gamma(U\cap V,\underline{\mathbb{Z}}) = \mathbb{Z}\times\mathbb{Z}, \end{array}$$

together with the map $d : C^0 \to C^1$ which sends (a,b) to $(b-a, b-a)$. In particular, $\check{\mathrm{H}}^0(\mathfrak{U},\underline{\mathbb{Z}}) \simeq \mathbb{Z}$ and $\check{\mathrm{H}}^1(\mathfrak{U},\underline{\mathbb{Z}}) \simeq \mathbb{Z}$. This coincides with the singular cohomology groups of X with the integer coefficients.

We will see some properties of the Čech cohomology groups.

Lemma 152. *For any $X, \mathfrak{U}, \mathcal{F}$, we have $\check{H}^0(\mathfrak{U},\mathcal{F}) \simeq \Gamma(X,\mathcal{F})$.*

Proof. Note that $\check{\mathrm{H}}^0(\mathfrak{U},\mathcal{F}) = \ker\left[d : C^0(\mathfrak{U},\mathcal{F}) \to C^1(\mathfrak{U},\mathcal{F})\right]$. Let $\alpha \in C^0$ be an element $\{\alpha_i \in \mathcal{F}(U_i)\}$. For each $i < j$, we have $(d\alpha)_{i,j} = \alpha_j - \alpha_i \in \mathcal{F}(U_{i,j})$. In other words, $d\alpha = 0$ means two sections α_i and α_j coincide on the intersection $U_i \cap U_j$. Thanks to sheaf axioms, such a collection of sections will glue together and give a unique element in $\Gamma(X,\mathcal{F})$. $\square$

When X is a (quasi-projective) variety, it is quite natural to take an open cover $\mathfrak{U}$ as an affine open cover. We will see that the Čech cohomology groups and the sheaf cohomology groups coincide for any quasi-coherent sheaf $\mathcal{F}$ in this case. A standard idea uses a spectral sequence from a double complex – which is similar as the “Hodge-de Rham” theorem. The key facts what we need are:

(i) If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module on a ringed space $(X,\mathcal{O}_X)$, together with a finite open covering $\mathfrak{U} = \{U_i\}$ of X, then the Čech cohomology groups $\check{\mathrm{H}}^p(\mathfrak{U}, I)$ vanish for all $p > 0$;

(ii) If X is an affine scheme, and $\mathcal{F}$ is any quasi-coherent sheaf on X, then $H^p(X, \mathcal{F}) = R^i\Gamma(X, \mathcal{F}) = 0$ for all $p > 0$.

Let X be a (noetherian separated) scheme, $\mathcal{F}$ be a quasi-coherent sheaf on X, and let $\mathfrak{U} = \{U_i\}$ be a finite affine open cover of X. We may associate a double complex for a given injective resolution $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \cdots$ as

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
0 \arrow[r] \arrow[u] & \prod_i \mathcal{I}_1(U_i) \arrow[r] \arrow[u] & \prod_{i<j} \mathcal{I}_1(U_{i,j}) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] \arrow[u] & \prod_i \mathcal{I}_0(U_i) \arrow[r] \arrow[u] & \prod_{i<j} \mathcal{I}_0(U_{i,j}) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

If we take the rightward filtration, we obtain a "vertical strip" on the 0-th column which converges to the sheaf cohomology. On the other hand, if we take the upward filtration, we obtain a "horizontal strip" on the 0-th row which converges to the Čech cohomology. However, we are not much familiar with spectral sequences, so we will follow a description in Hartshorne's book. To do this, we need a "sheafified" version of the Čech complex. For any open set $V \subseteq X$, let $f : V \hookrightarrow X$ denote the inclusion map. We define the complex $\mathscr{C}^\bullet(\mathfrak{U}, \mathcal{F})$ as

$$\mathscr{C}^p(\mathfrak{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_p} f_*(\mathcal{F}|_{U_{i_0, \cdots, i_p}})$$

and define $d : \mathscr{C}^p(\mathfrak{U}, \mathcal{F}) \to \mathscr{C}^{p+1}(\mathfrak{U}, \mathcal{F})$ by the same formula as the usual Čech complex. Note that $C^p(\mathfrak{U}, \mathcal{F}) = \Gamma(X, \mathscr{C}^p(\mathfrak{U}, \mathcal{F}))$ for each p.

Lemma 153. *For any sheaf of abelian groups $\mathcal{F}$ on X, the complex $\mathscr{C}^p(\mathfrak{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$, that is, there is a natural map $\epsilon : \mathcal{F} \to \mathscr{C}^0$ such that the sequence*

$$0 \to \mathcal{F} \xrightarrow{\epsilon} \mathscr{C}^0 \to \mathscr{C}^1 \to \cdots$$

is exact.

Proof. We define ϵ by taking the product of the natural maps $\mathcal{F} \to f_*(\mathcal{F}|_{U_i})$ for $i \in I$. The exactness at $\mathcal{F}$ and $\mathscr{C}^0$ follows from the sheaf axioms for $\mathcal{F}$. To check the exactness of the complex on the remaining parts, we will see that the induced sequence on stalks is exact. Let $x \in X$ be a point. Suppose that $x \in U_j$. For each $p \geq 1$, we define a map $k : \mathscr{C}^p(\mathfrak{U}, \mathcal{F})_x \to \mathscr{C}^{p-1}(\mathfrak{U}, \mathcal{F})_x$ as follows:

> Given $\alpha_x \in \mathscr{C}^p(\mathfrak{U}, \mathcal{F})_x$, which is represented by a section $\alpha \in \Gamma(V, \mathscr{C}^p(\mathfrak{U}, \mathcal{F}))$ over a neighborhood V of x, which we may choose small enough so that $V \subseteq U_j$. Now for any p-tuple $i_0 < \cdots < i_{p-1}$, we define $(k\alpha)_{i_0, \cdots, i_{p-1}} := \alpha_{j, i_0, \cdots, i_{p-1}}$. Then take the stalk of $k\alpha$ at x to get the desired map k.

Now for any $p \geq 1, \alpha \in \mathscr{C}_x^p$, we have

$$(dk + kd)(\alpha) = \alpha.$$

In other words, the identity map is homotopic to the zero map. In particular, all the cohomology groups $h^p(\mathscr{C}_x^\bullet)$ vanish for $p \geq 1$. $\square$