Proposition 154. *Let X be a topological space, $\mathfrak{U}$ an open covering, and let $\mathcal{F}$ be a flasque sheaf of abelian groups on X. Then $\check{H}^p(\mathfrak{U}, \mathcal{F}) = 0$ vanishes for every $p > 0$. In particular, an injective sheaf of abelian groups on X has no nonvanishing higher Čech cohomology groups.*

Proof. Consider the Čech resolution $0 \to \mathcal{F} \to \mathscr{C}^\bullet(\mathfrak{U}, \mathcal{F})$ as above. Since $\mathcal{F}$ is flasque, and a product of flasque sheaves are flasque, all the sheaves $\mathscr{C}^p(\mathfrak{U}, \mathcal{F})$ are flasque for all p. In particular, this gives a flasque resolution of $\mathcal{F}$, hence, we may use it to compute the sheaf cohomology instead of an injective resolution of $\mathcal{F}$. But a flasque sheaf $\mathcal{F}$ has no higher cohomology: $H^p(X, \mathcal{F}) = 0$ for each $p > 0$.
On the other hand, we may recover the Čech complex $C^\bullet(\mathfrak{U}, \mathcal{F})$ by taking the global section functor $\Gamma(X, -)$. In particular, $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) = h^p(\Gamma(X, \mathscr{C}^\bullet)) = H^p(X, \mathcal{F}) = 0$. □

Lemma 155. *Let X be a topological space, and let $\mathfrak{U}$ be an open covering of X. For each $p \geq 0$, there is a natural map, functorial in $\mathcal{F}$,*

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) \to H^p(X, \mathcal{F}).$$

Proof. Let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. Being injective, we may find a morphism of complexes from $0 \to \mathcal{F} \to \mathscr{C}^\bullet(\mathfrak{U}, \mathcal{F})$ to $0 \to \mathcal{F} \to \mathcal{I}^\bullet$, which extends the identity map on $\mathcal{F}$. Applying the functors $\Gamma(X, -)$ and taking the cohomology, we have a natural induced map which is the desired map. □

Theorem 156. *Let X be a noetherian variety over a field k, $\mathfrak{U}$ be an affine open cover of X, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then the above natural map is an isomorphism for every $p \geq 0$.*

Proof. For $p = 0$, both groups coincide with the group of global sections $\Gamma(X, \mathcal{F})$. We first embed $\mathcal{F}$ into a quasi-coherent flasque sheaf $\mathcal{G}$. Let $\mathcal{R}$ be the quotient: $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{R} \to 0$.
For each $i_0 < \cdots < i_p$, the intersection of affine open subsets $U_{i_0, \cdots, i_p}$ becomes affine.

> Let U, V be two affine open subsets. Their intersection $U \cap V$ coincide with the diagonal in $U \times V$, passing by a map $U \cap V \to U \times V$ sending $x \mapsto (x, x)$. Here, $U \times V$ is an affine scheme and the diagonal is a closed subscheme, hence, $U \cap V$ is also affine. Inside a general scheme, the diagonal needs not to be closed; we need the separatedness assumption.

Since $\mathcal{F}$ is quasi-coherent, there is no H^1 on an affine open set, that is, we have a short exact sequence of abelian groups

$$0 \to \mathcal{F}(U_{i_0, \cdots, i_p}) \to \mathcal{G}(U_{i_0, \cdots, i_p}) \to \mathcal{R}(U_{i_0, \cdots, i_p}) \to 0.$$

Taking their products, we have a short exact sequence of Čech complexes

$$0 \to C^\bullet(\mathfrak{U}, \mathcal{F}) \to C^\bullet(\mathfrak{U}, \mathcal{G}) \to C^\bullet(\mathfrak{U}, \mathcal{R}) \to 0.$$

Taking the cohomology, we have a long exact sequence of Čech cohomology groups. We have $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{G}) = 0$ for every $p > 0$ since $\mathcal{G}$ is flasque. In particular, $\check{\mathrm{H}}^1(\mathfrak{U}, \mathcal{F}) \simeq H^1(X, \mathcal{F})$ since both are the cokernel of $\Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{R})$.
Note also that we have an isomorphism $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) \simeq \check{\mathrm{H}}^{p-1}(\mathfrak{U}, \mathcal{R})$ for every $p \geq 2$. Since $\mathcal{R}$ is also quasi-coherent, we may apply the induction on p. □

Remark 157. On an arbitrary topological space X with an arbitrary sheaf of abelian groups $\mathcal{F}$ on X, Čech cohomology may differ from the sheaf cohomology. In general, we have a natural map

$$\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$$

for an arbitrary open cover X, which is an isomorphism when $p = 0$. If we consider finer open covers (refinements) of $\mathfrak{U}$, they form a direct system. In particular, we have a direct limit $\varinjlim \check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F})$, together with a natural map $\varinjlim \check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$ for each p. This is an isomorphism when $p = 0$ or $p = 1$.

Exercise 158. Let $(X, \mathcal{O}_X)$ be a ringed space. We denote $\operatorname{Pic} X$ by the group of isomorphism classes of invertible sheaves. Show that $\operatorname{Pic} X \simeq H^1(X, \mathcal{O}_X^\times)$, where $\mathcal{O}_X^\times$ denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U, \mathcal{O}_X)$, with the multiplication as the group operation.
[Hint : Let $\mathcal{L}$ be an invertible sheaf. Cover X by open subsets U_i on which $\mathcal{L}$ is free. Fix local isomorphisms $\varphi_i : \mathcal{O}_{U_i} \xrightarrow{\sim} \mathcal{L}|_{U_i}$. On $U_i \cap U_j$, we have an automorphism $\varphi_i^{-1} \circ \varphi_j$ of $\mathcal{O}_{U_{i,j}}$. These automorphisms give an element of $\check{\mathrm{H}}^1(\mathfrak{U}, \mathcal{O}_X^\times)$. Now use the fact that $\varinjlim \check{\mathrm{H}}^1(\mathfrak{U}, \mathcal{F}) = H^1(X, \mathcal{F})$.]

Remark 159. When X is a (compact) complex manifold, we have an exponential sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0.$$

From the cohomology long exact sequence, we have an exact sequence

$$H^1(X, \underline{\mathbb{Z}}) = H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^\times) = \operatorname{Pic} X \xrightarrow{c_1} H^2(X, \mathbb{Z}).$$

The last map c_1 is called the *first Chern class map*, and the first Chern class of a line bundle L is defined by the image $c_1(L)$ in $H^2(X, \mathbb{Z})$. In particular, the Picard group of X is composed of two parts: first, a discrete part, measured by the first Chern class; and a continuous part, coming from the complex vector space $H^1(X, \mathcal{O}_X)$ quotient by the images of $H^1(X, \mathbb{Z})$. The vector space $H^1(X, \mathcal{O}_X)$ identifies with the first order deformation space of a line bundle $\mathcal{O}_X$. When X is sufficiently good, the continuous part $\ker c_1 \subseteq \operatorname{Pic} X$ coincides with the quotient space $H^1(X, \mathcal{O}_X)$ mod out by $\mathbb{Z}$-lattices of full rank, in other words, a complex torus.

Exercise 160. Let C be a projective plane curve, defined by a single homogeneous equation $f(x, y, z) = 0$ of degree d. Assume that $(1 : 0 : 0) \notin C$, equivalently, f does not contains a x^d term.

(1) Show that C is covered by two affine open subsets $U = C \cap U_y = \{y \neq 0\}$ and $V = C \cap U_z = \{z \neq 0\}$.

(2) Compute the Čech complex explicitly.

(3) Verify that $h^0(C, \mathcal{O}_C) = 1$ and $h^1(C, \mathcal{O}_C) = \binom{d-1}{2}$.

Remark 161. The coincidence of Čech cohomology and the sheaf cohomology implies a special case of the dimensional cohomology vanishing. Let $X = \mathbb{P}^n = \operatorname{Proj} k[x_0, \cdots, x_n]$ be the projective n-space, and let $\mathcal{F}$ be any quasi-coherent sheaf on X. The sheaf cohomology groups can be computed via an affine open covering; we may take $\mathfrak{U} = \{U_i\}$, where $U_i = D_+(x_i) = \{x_i \neq 0\}$ as a collection of $(n+1)$ affine open subsets. Since the Čech complex $C^\bullet(\mathfrak{U}, \mathcal{F})$ becomes 0 after $(n+1)$-steps, all the cohomology groups $H^p(X, \mathcal{F})$ vanish when $p \geq n+1$.
More generally, let $X \subset \mathbb{P}^N$ be a projective variety of dimension n. Let $c = N - n$ be the codimension. Since X does not intersect with a general linear subspace of dimension $c-1$, hence, we may assume that X does not intersect with the linear subspace $\{[0 : \cdots : 0 : x_{n+1} : x_{n+2} : \cdots : x_{(n+1)+(c-1)=n+c=N}]\} \subseteq \mathbb{P}^N$, after a certain linear change of coordinates if necessary. In particular, X can be covered by $(n+1)$ affine open subsets $X \cap U_i = \{x_i \neq 0\}$, $0 \leq i \leq n$. The same argument shows that $H^p(X, \mathcal{F}) = 0$ for every $p \geq n+1$ and for every quasi-coherent sheaf $\mathcal{F}$ on X.

Exercise 162. Compute the cohomology groups $H^i(X, \mathcal{O}_X)$ where $X = \mathbb{A}^2_k \setminus \{(0,0)\}$ is a punctured affine plane. [Hint : compute Čech cohomology groups with respect to an affine open cover $\{U_x, U_y\}$, where $U_x \subseteq \{(x,y) \mid x \neq 0\} \subseteq \mathbb{A}^2_k$ and $U_y \subseteq \{(x,y) \mid y \neq 0\} \subseteq \mathbb{A}^2_k$.] Conclude that $\mathbb{A}^2_k \setminus \{(0,0)\}$ is not affine.

4 Cohomology on projective spaces

We will make explicit computations of the cohomology of sheaves $\mathcal{O}(n)$ on a projective space, and observe some basic properties of cohomology groups of coherent sheaves on a projective space.
Let A be a noetherian ring, $S = A[x_0, \cdots, x_r]$ $(r \geq 1)$, and let $X = \operatorname{Proj} S$ be the projective r-space $\mathbb{P}^r_A$ over A. Let $\mathcal{O}_X(1)$ be the twisting sheaf of Serre. We denote by $\Gamma_*(\mathcal{F})$ the graded S-module $\oplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F})$ for any $\mathcal{O}_X$-module $\mathcal{F}$.

Theorem 163. *Let $A, S, X, \mathcal{O}_X(1), \mathcal{F}$ be as above.*

(1) The natural map $S \to \Gamma_*(\mathcal{O}_X) = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n))$ is an isomorphism of graded S-modules.

(2) (no intermediate cohomology) $H^i(X, \mathcal{O}_X(n)) = 0$ for $0 < i < r$ and all $n \in \mathbb{Z}$.

(3) $H^r(X, \mathcal{O}_X(-r-1)) \simeq A$.

(4) The natural map $H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \to H^r(X, \mathcal{O}_X(-r-1)) \simeq A$ is a perfect pairing of finitely generated free A-modules, for each $n \in \mathbb{Z}$.

In particular, $H^0(X, \mathcal{O}_X(n))$ can be interpreted as the set of homogeneous polynomials of degree n in $x_0, \cdots, x_r$, and $H^r(X, \mathcal{O}_X(n))$ can be interpreted as the set of homogeneous Laurent polynomials of degree n in $x_0, \cdots, x_n$, where in each monomial, each x_i appears with degree at most -1.

Proof. Let $\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n)$ be a quasi-coherent sheaf on X. Since the cohomology functor commutes with direct sums, the cohomology of $\mathcal{F}$ will decompose into a direct sum of the cohomology of $\mathcal{O}_X(n)$'s.
For each $0 \le i \le r$, let U_i be the distinguished affine open set $D_+(x_i)$. We can compute the cohomology of $\mathcal{F}$ by using Čech cohomology with respect to the covering $\mathfrak{U} = \{U_i\}$. For any set of indices $i_0 < \cdots < i_p$, the open set $U_{i_0,\cdots,i_p} = D_+(x_{i_0} \cdots x_{i_p})$, we have $\mathcal{F}(U_{i_0,\cdots,i_p}) = S_{x_{i_0}\cdots x_{i_p}}$. Also note that the grading on $\mathcal{F}$ corresponds to the natural grading of $S_{x_{i_0}\cdots x_{i_p}}$. Thus, the Čech complex of $\mathcal{F}$ is given by

$$C^\bullet(\mathfrak{U}, \mathcal{F}) : \prod S_{x_{i_0}} \to \prod S_{x_{i_0}x_{i_1}} \to \cdots \to S_{x_0\cdots x_r}.$$

Note that $H^0(X, \mathcal{F}) = \check{\mathrm{H}}^0(\mathfrak{U}, \mathcal{F})$ is the kernel of the first map, which is just S.
Next, we consider $H^r(X, \mathcal{F})$, which is the cokernel of the last map $d^{r-1} : \prod_k S_{x_0\cdots \hat{x_k}\cdots x_r} \to S_{x_0\cdots x_r}$. We regard $S_{x_0\cdots x_r}$ as a free A-module of Laurent polynomials, with basis $x_0^{\ell_0}\cdots x_r^{\ell_r}$, $l_i \in \mathbb{Z}$. The image of d^{r-1} is the free submodule generated by those basis elements for which at least one $l_i \ge 0$ is nonnegative. Thus, $H^r(X, \mathcal{F})$ is a free A-module with basis consisting of the negative monomials

$$\{x_0^{\ell_0}\cdots x_r^{\ell_r} \mid \ell_i < 0 \text{ for every } r\},$$

equipped with the natural grading $\sum \ell_i$. There is only one such monomial of degree $-r-1$, namely $x_0^{-1}\cdots x_r^{-1}$.
To check the last statement, first note that both $H^0(X, \mathcal{O}_X(n))$ and $H^r(X, \mathcal{O}_X(-n-r-1))$ vanish when $n < 0$. For $n \ge 0$, $H^0(X, \mathcal{O}_X(n))$ identifies with the free A-module with basis

$$\{x_0^{m_0}\cdots x_r^{m_r} \mid m_i \ge 0, \sum m_i = n\}.$$

The natural pairing with $H^r(X, \mathcal{O}_X(-n-r-1))$ into $H^r(X, \mathcal{O}_X(-r-1))$ is determined by

$$(x_0^{m_0}\cdots x_r^{m_r}) \cdot (x_0^{\ell_0}\cdots x_r^{\ell_r}) = x_0^{m_0+\ell_0}\cdots x_r^{m_r+\ell_r},$$

where the object on the right becomes 0 if there is an i such that $m_i + \ell_i \ge 0$. It is clear that this gives a perfect pairing; $H^r(X, \mathcal{O}_X(-n-r-1)) \simeq H^0(X, \mathcal{O}_X(n))^\vee$ with the dual basis consisting of elements of the form

$$x_0^{-m_0-1}\cdots x_r^{-m_r-1}$$

corresponding to $x_0^{m_0}\cdots x_r^{m_r}$.
It remains to show that there is no intermediate cohomology. We will use the induction on r. When $r = 1$, there is nothing to prove, so let $r > 1$. Consider the exact sequence of graded S-modules

$$0 \to S(-1) \xrightarrow{\cdot x_r} S \to S/(x_r) \to 0.$$

This gives the exact sequence of sheaves

$$0 \to \mathcal{O}_X(-1) \overset{\cdot x_r}{\to} \mathcal{O}_X \to \mathcal{O}_H \to 0$$

where $H = (x_r = 0) = \mathbb{P}_A^{r-1}$ is the hyperplane defined by x_r. Twisting by all $n \in \mathbb{Z}$ and taking the direct sum, we have

$$0 \to \mathcal{F}(-1) \overset{\cdot x_r}{\to} \mathcal{F} \to \mathcal{F}_H \to 0$$

where $\mathcal{F}_H = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_H(n)$. Taking the cohomology long exact sequence, we have

$$\cdots \to H^i(X, \mathcal{F}(-1)) \overset{\cdot x_r}{\to} H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F}_H) = H^i(H, \mathcal{F}_H) \to H^{i+1}(X, \mathcal{F}(-1)) \to \cdots .$$

Since $(H, \mathcal{O}_H(1)) = (\mathbb{P}_A^{r-1}, \mathcal{O}_{\mathbb{P}_A^{r-1}}(1))$, we may apply the induction hypothesis so that $H^i(X, \mathcal{F}_H) = 0$ for $0 < i < r-1$.
At the beginning of the long exact sequence, we have

$$0 \to H^0(X, \mathcal{F}(-1)) \overset{\cdot x_r}{\to} H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}_H) \to 0,$$

since $H^0(X, \mathcal{F}_H) = S/(x_r)$. In particular, we have a bijection

$$0 \to H^1(X, \mathcal{F}(-1)) \overset{\cdot x_r}{\to} H^1(X, \mathcal{F}) \to 0.$$

At the end of the long exact sequence, we have

$$H^{r-1}(X, \mathcal{F}_H) \overset{\delta}{\to} H^r(X, \mathcal{F}(-1)) \overset{\cdot x_r}{\to} H^r(X, \mathcal{F}) \to 0.$$

(The surjectivity of the map $(\cdot x_r)$ is clear; we may regard both spaces as the space of negative Laurent polynomials, or we may apply the dimensional cohomology vanishing.) The kernel of the map $(\cdot x_r)$ is the free A-module generated by negative Laurent monomials $x_0^{\ell_0} \cdots x_r^{\ell_r}$ with $l_r = -1$. Since $H^{r-1}(X, \mathcal{F}_H)$ is the free A-module generated by negative Laurent monomials $x_0^{\ell_0} \cdots x_{r-1}^{\ell_{r-1}}$, the map δ identifies with the division by x_r, we observe that δ is injective. In particular, $H^{r-1}(X, \mathcal{F}(-1)) \overset{\cdot x_r}{\to} H^{r-1}(X, \mathcal{F})$ is bijective. We conclude that $(\cdot x_r) : H^i(X, \mathcal{F}(-1)) \to H^i(X, \mathcal{F})$ is bijective for each $0 < i < r$.
If we localize the Čech complex $C^\bullet(\mathfrak{U}, \mathcal{F})$ by x_r, we get the Čech complex for the sheaf $\mathcal{F}|_{U_r}$ on the affine open space U_r, with respect to the affine open covering $\{U_i \cap U_r\}_{0 \le i \le r-1}$. Since a quasi-coherent sheaf on an affine scheme does not have any nonvanishing higher cohomology group, we conclude that

$$H^i(X, \mathcal{F}|_{U_r}) = H^i(X, \mathcal{F})_{x_r} = 0$$

for each $i > 0$. In other words, every element of $H^i(X, \mathcal{F})$, for $i > 0$, is annihilated by some powers of x_r. Now the bijectivity of the map $(\cdot x_r) : H^i(X, \mathcal{F}(-1)) \to H^i(X, \mathcal{F})$ implies that $H^i(X, \mathcal{F}) = 0$ for $0 < i < r$.

□

Theorem 164 (Serre vanishing)**.** *Let X be a closed subscheme of a projective space $\mathbb{P}^r_A$ over a noetherian ring A, and let $\mathcal{O}_X(1)$ be the very ample invertible sheaf on X corresponding to the given embedding. Let $\mathcal{F}$ be a coherent sheaf on X. Then:*

(1) for each $i \geq 0$, $H^i(X, \mathcal{F})$ is a finitely generated A-module;

(2) there is an integer n_0 (depending on $\mathcal{F}$) such that for each $i > 0$ and $n \geq n_0$, $H^i(X, \mathcal{F}(n)) = 0$.

Proof. Let $i : X \hookrightarrow \mathbb{P}^r_A$ be the given embedding, so that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. Since $i_*\mathcal{F}$ is coherent and the cohomology is the same, thus, we may reduce to the case $X = \mathbb{P}^r_A$. Note that the statements we want to show are obvious when $\mathcal{F} = \mathcal{O}_X(q)$ for some $q \in \mathbb{Z}$. For $i > r$, we have $H^i(X, \mathcal{F}) = 0$ and hence there is nothing to prove. Given a coherent sheaf $\mathcal{F}$ on X, there is a surjection $\mathcal{E} = \bigoplus \mathcal{O}_X(q_i) \to \mathcal{F} \to 0$ where $\mathcal{E}$ is a finite direct sum of line bundles. Let $\mathcal{R}$ be the kernel, which is also coherent. Using the descending induction hypothesis on i, we have the finite generatedness of $H^i(X, \mathcal{F})$ since there is an exact sequence of A-modules

$$\cdots \to H^i(X, \mathcal{E}) \to H^i(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{R}) \to \cdots$$

with both terms on the left and on the right are finitely generated.
To show the second statement, we just twist the above sequence by a sufficiently large number $n >> 0$; the module on the left vanishes unless $i = 0$ because $\mathcal{E}$ is a direct sum of $\mathcal{O}_X(q_i)$, and the module on the right vanishes thanks to the induction hypothesis. Hence $H^i(X, \mathcal{F}(n)) = 0$ for $n \gg 0$. Since there are only finitely many i's involved in the sequence, namely $0 < i \leq r$, we may take n_0 as the maximum of integers what we obtained separately for each i. □

As an application, we give a cohomological criterion of the ampleness of an invertible sheaf:

Proposition 165. *Let X be a closed subscheme of a projective space $\mathbb{P}^r_A$ over a noetherian ring A. Let $\mathcal{L}$ be an invertible sheaf on X. Then the following are equivalent:*

(i) $\mathcal{L}$ is ample;

(ii) For each coherent sheaf $\mathcal{F}$ on X, there is an integer n_0 (depending on $\mathcal{F}$) such that for each $i > 0$ and $n \geq n_0$, we have the vanishing cohomology group $H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$.

Proof. ($\Rightarrow$) Note that $\mathcal{L}^m$ is very ample for some $m > 0$. We take the corresponding projective morphism, so that we have another embedding $i : X \hookrightarrow \mathbb{P}^N_A$ into a projective space such that $i^*\mathcal{O}_{\mathbb{P}^N_A}(1) = \mathcal{L}^m$. Now apply the Serre vanishing for $\mathcal{F}, \mathcal{F} \otimes \mathcal{L}, \cdots, \mathcal{F} \otimes \mathcal{L}^{m-1}$.
($\Leftarrow$) We need to show that for any coherent sheaf $\mathcal{F}$ on X, there is an integer m_0 such that $\mathcal{F} \otimes \mathcal{L}^m$ is globally generated for each $m \geq m_0$.

Let P be a closed point of X, and let $\mathscr{I}_P$ be the ideal sheaf of the closed subset $\{P\}$. We have a short exact sequence

$$0 \to \mathscr{I}_P\mathcal{F} \to \mathcal{F} \to \mathcal{F} \otimes k(P) \to 0$$

where $k(P)$ is the skyscraper sheaf $\mathcal{O}_X/\mathscr{I}_P$. Tensoring with $\mathcal{L}^m$, we have

$$0 \to \mathscr{I}_P\mathcal{F} \otimes \mathcal{L}^m \to \mathcal{F} \otimes \mathcal{L}^m \to \mathcal{F} \otimes \mathcal{L}^m \otimes k(P) \to 0.$$

Since $\mathscr{I}_P\mathcal{F}\otimes\mathcal{L}^m$ is also coherent, there is an integer m_0 such that $H^1(X, \mathscr{I}_P\mathcal{F}\otimes\mathcal{L}^m) = 0$ for each $m \geq m_0$. In particular,

$$\Gamma(X, \mathcal{F} \otimes \mathcal{L}^m) \to \Gamma(X, \mathcal{F} \otimes \mathcal{L}^m \otimes k(P))$$

is surjective for each $m \geq m_0$. Thanks to Nakayama's lemma, this implies that the stalk of $\mathcal{F} \otimes \mathcal{L}^m$ at P is generated by global sections. Since it is coherent, there is an open neighborhood U of P (it may depend on m) such that the global sections of $\mathcal{F} \otimes \mathcal{L}^m$ generate the sheaf at every point in U.

In particular, taking $\mathcal{F} = \mathcal{O}_X$, we find that there is an integer $m_1 > 0$ and an open neighborhood V of P such that $\mathcal{L}^{m_1}$ is globally generated by global sections over V. On the other hand, the above argument gives a neighborhood U_j of P such that $\mathcal{F} \otimes \mathcal{L}^{m_0+j}$ is generated by global sections over U_j for each $0 \leq j \leq m_1 - 1$. Now let $U_P := V \cap U_0 \cap \cdots \cap U_{m_1-1}$. Then over U_P, all of the sheaves $\mathcal{F} \otimes \mathcal{L}^m$, $m \geq m_0$ are generated by global sections since it can be expressed as a product of globally generated invertible sheaves

$$(\mathcal{F} \otimes \mathcal{L}^{m_0+j}) \otimes (\mathcal{L}^{m_1})^k$$

for some $0 \leq j \leq m_1 - 1$ and $k \geq 0$.

Now cover X by a finitely number of the open sets U_P, for various closed points P (since X is quasi-compact). Let n_0 be the maximum of the m_0's corresponding to those points P. Then $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections over all of X, for all $n \geq n_0$. □

Exercise 166. Let X be a projective scheme over a field k, $\mathcal{O}_X(1)$ a very ample invertible sheaf, and let $\mathcal{F}$ be a coherent sheaf on X. We define the *Euler characteristic* of $\mathcal{F}$ by

$$\chi(\mathcal{F}) := \sum (-1)^i \dim_k H^i(X, \mathcal{F}).$$

(1) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of coherent sheaves on X, show that $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$.

(2) Show that there is a polynomial $P(z) \in \mathbb{Q}[z]$ such that $\chi(\mathcal{F}(n)) = P(n)$ for all $n \in \mathbb{Z}$. This is called the *Hilbert polynomial* of $\mathcal{F}$ with respect to $\mathcal{O}_X(1)$.

(3) Show that there is an integer n_0 such that $h^0(X, \mathcal{F}(n)) = P(n)$ for all $n \geq n_0$.

Exercise 167. Let X be a projective scheme over a noetherian ring A, and let $\mathcal{F}^1 \to \mathcal{F}^2 \to \cdots \mathcal{F}^r$ be an exact sequence of coherent sheaves on X. Show that there is an integer n_0 such that for all $n \geq n_0$, the sequence of global sections

$$\Gamma(X, \mathcal{F}^1) \to \Gamma(X, \mathcal{F}^2) \to \cdots \to \Gamma(X, \mathcal{F}^r)$$

is exact.

5 Ext groups, sheaves, and higher direct images

Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, there are two objects: $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$ the group of $\mathcal{O}_X$-module homomorphisms, and $\mathcal{H}om_X(\mathcal{F}, \mathcal{G})$ the sheaf Hom. For a fixed $\mathcal{F}$, they give left exact covariant functors: $\mathrm{Hom}_X(\mathcal{F}, -) : \mathfrak{Mod}(X) \to \mathfrak{Ab}$, $\mathcal{H}om_X(\mathcal{F}, -) : \mathfrak{Mod}(X) \to \mathfrak{Mod}(X)$. Since $\mathfrak{Mod}(X)$ has enough injectives, we may consider their right derived functors as follows.

Definition 168. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define the functors $\mathrm{Ext}^i_X(\mathcal{F}, -)$ as the right derived functors of $\mathrm{Hom}_X(\mathcal{F}, -)$, and $\mathcal{E}xt^i_X(\mathcal{F}, -)$ as the right derived functors of $\mathcal{H}om_X(\mathcal{F}, -)$.

An additive functor $F : \mathfrak{A} \to \mathfrak{B}$ is called *effaceable* if for each object A in $\mathfrak{A}$, there is a monomorphism $u : A \hookrightarrow M$ for some M such that $F(u) = 0$. The following lemma is useful during this and the next sections.

Lemma 169. *Let $T = (T^i)_{i \geq 0}$ be a covariant δ-functor from $\mathfrak{A} \to \mathfrak{B}$. If T^i is effaceable for every $i > 0$, then T is universal.*

Lemma 170. *Let $\mathcal{I}$ be an injective object in $\mathfrak{Mod}(X)$. Then for any open subset $U \subseteq X$, $\mathcal{I}|_U$ is an injective object in $\mathfrak{Mod}(U)$.*

Proof. Let $j : U \hookrightarrow X$ be the inclusion. Given any inclusion of $\mathcal{O}_U$-modules $\mathcal{F} \hookrightarrow \mathcal{G}$, and a given morphism $\mathcal{F} \to \mathcal{I}|_U$, we have an inclusion $j_!(\mathcal{F}) \hookrightarrow j_!(\mathcal{G})$ and a morphism $j_!(\mathcal{F}) \to j_!\mathcal{I}|_U$, where $j_!$ denotes the extension by zero outside of U. Since $j_!\mathcal{I}|_U$ is a subsheaf of $\mathcal{I}$, we have a morphism $j_!\mathcal{F} \to \mathcal{I}$ to an injective object. This extends to a map $j_!\mathcal{G} \to \mathcal{I}$. Taking the restriction onto U gives the required map $\mathcal{G} \to \mathcal{I}|_U$. □

In particular, when $\mathcal{G}$ is an injective object in $\mathfrak{Mod}(X)$, we have $\mathrm{Ext}^i_X(\mathcal{F}, \mathcal{G}) = 0$ and $\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G}) = 0$ for any $\mathcal{O}_X$-module $\mathcal{F}$ and $i > 0$.

Proposition 171. *For any open subset $U \subseteq X$, we have $\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G})|_U \simeq \mathcal{E}xt^i_X(\mathcal{F}|_U, \mathcal{G}|_U)$.*

Proof. Fix $\mathcal{F}$. Both sides are universal δ-functors in $\mathcal{G}$ from $\mathfrak{Mod}(X)$ to $\mathfrak{Mod}(U)$. Since they agree on $i = 0$, they must coincide. (To be precise, one has to show that the functor from the left-hand-side is "universal", which can be checked by showing that the functor is effaceable.) □

When we take $\mathcal{F} = \mathcal{O}_X$, both the functors Ext and $\mathcal{E}xt$ become very simple as follows.

Proposition 172. *For any $\mathcal{G} \in \mathfrak{Mod}(X)$, we have:*

(1) $\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}$;

(2) $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G}) = 0$ *for* $i > 0$;

(3) $\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{G}) \simeq H^i(X, \mathcal{G})$ *for all* $i \geq 0$;

Proof. Note that the functor $\mathcal{H}om(\mathcal{O}_X, -)$ is the identity functor, and hence, it does not have nontrivial derived functors for $i > 0$. The functor $\hom(\mathcal{O}_X, -) = \Gamma(X, -)$ is the global section functor, and its derived functors are the sheaf cohomology functors. □

If there is a short exact sequence $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ of $\mathcal{O}_X$-modules, then we immediately have a cohomology long exact sequence from the construction. The following proposition corresponds to a converse direction;

Proposition 173. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of $\mathcal{O}_X$-modules, then for any $\mathcal{O}_X$-module $\mathcal{G}$, we have a long exact sequence*

$$\begin{aligned} 0 &\to \operatorname{Hom}(\mathcal{F}'', \mathcal{G}) \to \operatorname{Hom}(\mathcal{F}, \mathcal{G}) \to \operatorname{Hom}(\mathcal{F}', \mathcal{G}) \\ &\to \operatorname{Ext}^1(\mathcal{F}'', \mathcal{G}) \to \operatorname{Ext}^1(\mathcal{F}, \mathcal{G}) \to \operatorname{Ext}^1(\mathcal{F}', \mathcal{G}) \\ &\to \operatorname{Ext}^2(\mathcal{F}'', \mathcal{G}) \to \cdots, \end{aligned}$$

and similarly for $\mathcal{E}xt$ sheaves.

Proof. Let $0 \to \mathcal{G} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{G}$. Since the functor $\operatorname{Hom}(-, \mathcal{I})$ is exact, we have a short exact sequence of complexes

$$0 \to \operatorname{Hom}(\mathcal{F}'', \mathcal{I}^\bullet) \to \operatorname{Hom}(\mathcal{F}, \mathcal{I}^\bullet) \to \operatorname{Hom}(\mathcal{F}', \mathcal{I}^\bullet) \to 0.$$

Now take the associated long exact sequence of cohomology groups h^i.
Using the $\mathcal{H}om(-, \mathcal{I})$ which is an exact functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Mod}(X)$, we get the analogous statement for $\mathcal{E}xt$'s. □

In particular, $\mathcal{E}xt$ sheaves can be computed from a locally free resolution of $\mathcal{F}$; we do not need to take an injective resolution. Since any coherent sheaf on a noetherian scheme is finitely generated and finitely presented; thus any coherent sheaf on X has a locally free resolution $\mathcal{L}^\bullet \to \mathcal{F} \to 0$ (may have infinite length).

Proposition 174. *Suppose that there is an exact sequence*

$$\cdots \to \mathcal{L}^{-2} \to \mathcal{L}^{-1} \to \mathcal{L}^0 \to \mathcal{F} \to 0$$

in $\mathfrak{Mod}(X)$, where $\mathcal{L}^i$ are locally free sheaves of finite rank. Then for any $\mathcal{G} \in \mathfrak{Mod}(X)$, we have

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \simeq h^{-i}(\mathcal{H}om(\mathcal{L}^\bullet, \mathcal{G})).$$

Proof. Note that both sides vanish for $i > 0$ and $\mathcal{G}$ injective; since the functor $\mathcal{H}om(-, \mathcal{G})$ is exact. This implies that they are effaceable δ-functors, hence universal. They are equal for $i = 0$, we have the natural isomorphism. □

Caution. It does not mean that Ext or $\mathcal{E}xt$ can be constructed as a derived functor in its first variable, since the category $\mathfrak{Mod}(X)$ does not have enough projectives in general.

Recall that we have a natural isomorphism $\operatorname{Hom}(\mathcal{F}, \mathcal{G} \otimes \mathcal{L}^\vee) \simeq \operatorname{Hom}(\mathcal{F} \otimes \mathcal{L}, \mathcal{G})$ where $\mathcal{L}$ is a locally free sheaf of finite rank. We will show the analogous isomorphism for Ext groups and $\mathcal{E}xt$ sheaves.

Lemma 175. *Let $\mathcal{L}$ be a locally free sheaf on X of finite rank, and let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then $\mathcal{L} \otimes \mathcal{I}$ is injective.*

Proof. It is enough to show that the functor $\mathrm{Hom}(-, \mathcal{L} \otimes \mathcal{I})$ is exact, which is the same as the functor $\mathrm{Hom}(- \otimes \mathcal{L}^\vee, \mathcal{I})$. Since it is a composition of two exact functors $(- \otimes \mathcal{L}^\vee)$ and $\mathrm{Hom}(-, \mathcal{I})$, we are done. □

Proposition 176. *Let $\mathcal{L}$ be a locally free sheaf of finite rank, and let $\mathcal{L}^\vee$ be its dual. For any $\mathcal{F}, \mathcal{G} \in \mathfrak{Mod}(X)$, we have*

$$\mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \simeq \mathrm{Ext}^i(\mathcal{F}, \mathcal{G} \otimes \mathcal{L}^\vee)$$

for each i, and similarly

$$\mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \simeq \mathcal{E}xt^i(\mathcal{F}, \mathcal{G} \otimes \mathcal{L}^\vee) \simeq \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{L}^\vee.$$

Proof. We already know the case $i = 0$. For the general case, one can show that all of them are effaceable δ-functors, hence, universal. □

Proposition 177. *Let X be a variety, $\mathcal{F}$ be a coherent sheaf on X, and let $\mathcal{G}$ be any $\mathcal{O}_X$-module. For any point $x \in X$, we have*

$$\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G})_x \simeq \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

for any i, where the right-hand-side is the Ext group over the (local) ring $\mathcal{O}_{X,x}$.

Proof. Since the question is local, we may assume that X is affine. Then $\mathcal{F}$ has a (locally) free resolution $\mathcal{L}^\bullet \to \mathcal{F} \to 0$ (from a finitely generated $A = \Gamma(X, \mathcal{O}_X)$-module $\Gamma(X, \mathcal{F})$). On the stalks at x, it gives a free resolution of modules over the local ring $\mathcal{O}_{X,x}$. We may compute the both sides by using these locally free resolutions, which coincide when $i = 0$. Note that without the assumption $\mathcal{F}$ being coherent, even the case $i = 0$ is not true in general. □

Proposition 178. *Let X be a projective variety over a field k, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X, and let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on X. There is an integer $n_0 > 0$ (depending on $\mathcal{F}, \mathcal{G}, i$) such that for every $n \geq n_0$ we have*

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}(n)) \simeq \Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n))).$$

Proof. Fix $\mathcal{F}$. Note that the functor $\mathrm{Ext}^i(\mathcal{F}, -)$ is the composition of two left exact functors $\Gamma(X, -)$ and $\mathcal{E}xt^i(\mathcal{F}, -)$. As a special case of the Grothendieck spectral sequence, there is a convergent spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G})) \Longrightarrow \mathrm{Ext}^{p+q}(\mathcal{F}, \mathcal{G})$$

for any coherent sheaf $\mathcal{G}$ on X. Since $\mathcal{O}_X(1)$ is ample and $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n)) \simeq \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{O}_X(n)$, it does not have nonvanishing higher cohomology groups $H^p(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n))) = 0$ for all $p > 0$ and sufficiently large $n \gg 0$ by Serre vanishing. When it happens, $H^0(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n))) = E_2^{0,q} \simeq E_\infty^{0,q} = \mathrm{Ext}^q(\mathcal{F}, \mathcal{G}(n))$ as desired. □

Exercise 179 (Yoneda extension)**.** Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}', \mathcal{F}'' \in \mathfrak{Mod}(X)$. An *extension* of $\mathcal{F}''$ by $\mathcal{F}'$ is a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0.$$

Two extensions are isomorphic if there is an isomorphism of short exact sequences, inducing the identity maps on $\mathcal{F}'$ and $\mathcal{F}''$. Given an extension as above, consider the long exact sequence arising from $\mathrm{Hom}(\mathcal{F}'', -)$. In particular, we have a connecting morphism

$$\delta : \mathrm{Hom}(\mathcal{F}'', \mathcal{F}'') \to \mathrm{Ext}^1(\mathcal{F}'', \mathcal{F}').$$

Let $\xi \in \mathrm{Ext}^1(\mathcal{F}'', \mathcal{F}')$ be the image of the identity morphism $\delta(id_{\mathcal{F}''})$. Show that this process gives a 1-1 correspondence between the isomorphism classes of extensions of $\mathcal{F}''$ by $\mathcal{F}'$, and elements of the group $\mathrm{Ext}^1(\mathcal{F}'', \mathcal{F}')$.

Higher direct images are useful when we deal with a relative case, that is, studying a morphism $f : X \to Y$ of varieties, or schemes. One may think this is a family of schemes over Y, where the members of the family are the fibers $X_y := X \times_Y \operatorname{Spec} k(y)$ for various points $y \in Y$. We need some form of "relative cohomology theory of X over Y", or equivalently, "cohomology theory along fibers of X over Y". This allows us to fulfill a philosophy: "geometry of X" is determined by the geometry of Y and the geometry of fibers.

Definition 180. Let $f : X \to Y$ be a continuous map of topological spaces. We define the *higher direct image* functors $R^i f_* : \mathfrak{Ab}(X) \to \mathfrak{Ab}(Y)$ to be the right derived functors of the direct image functor f_*.

The first and the most important property is that they give cohomology on fibers:

Proposition 181. *For each $i \geq 0$ and each $\mathcal{F} \in \mathfrak{Ab}(X)$, $R^i f_*(\mathcal{F})$ is a sheaf associated to the presheaf*

$$V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}).$$

on Y.

In the case of noetherian schemes and quasi-coherent sheaves, we have a better situation: affine open subschemes of Y are good enough.

Proposition 182. *Let X, Y be noetherian schemes, and let $f : X \to Y$ be a morphism. For any quasi-coherent sheaf $\mathcal{F}$ on X, the sheaves $R^i f_*(\mathcal{F})$ are quasi-coherent on Y. When $Y = \operatorname{Spec} A$ is affine, we have*

$$R^i f_* \mathcal{F} \simeq H^i(X, \mathcal{F})^{\sim}$$

for any quasi-coherent sheaf $\mathcal{F}$ on X.

Proof. We address an idea only, and skip the details. Both functors are universal δ-functors which coincide when $i = 0$. $\square$

In general, the cohomology of $\mathcal{F}$ on X is determined by the cohomology of all the higher direct images $R^i f_* \mathcal{F}$ on Y.

Theorem 183 (Leray spectral sequence)**.** *Let X, Y be topological spaces, $f : X \to Y$ be a continuous map, and let $\mathcal{F}$ be a sheaf of abelian groups on X. Then there is a spectral sequence $E_2^{p,q} = H^p(Y, R^q f_*(\mathcal{F}))$ which converges to $E_\infty^{p,q} = H^{p+q}(X, \mathcal{F})$.*

Corollary 184. *Let X, Y be noetherian scheme, and let $f : X \to Y$ be a morphism. Assume that f is finite, or f is affine, that is, $X \simeq \mathbf{Spec}\mathscr{A}$ for some quasi-coherent $\mathcal{O}_Y$-algebra $\mathscr{A}$. Then $H^i(X, \mathcal{F}) \simeq H^i(Y, f_*\mathcal{F})$.*

Proof. We have $R^i f_*(\mathcal{F}) = 0$ for all $i > 0$ in such cases. Let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. Note that $H^i(X, \mathcal{F}) = h^i(\Gamma(X, \mathcal{I}^\bullet)) = h^i(\Gamma(Y, f_*\mathcal{I}^\bullet))$. Note that $f_*\mathcal{I}^\bullet$ is a resolution of $f_*\mathcal{F}$ since it does not have nonzero cohomology $R^i f_*$ for $i > 0$. Since $\mathcal{I}^i$ is injective for each i, we observe that both $\mathcal{I}$ and $f_*\mathcal{I}^i$ are flasque. Therefore, $f_*\mathcal{I}^\bullet$ is indeed a flasque resolution of $f_*\mathcal{F}$. Taking the global section functor $\Gamma(Y, -)$ and then the cohomology, we have $H^i(Y, f_*\mathcal{F})$ as results. $\square$