6 Serre duality

Recall the famous Poincaré duality, which is a basic result on the structure of singular homology and cohomology groups of manifolds:

> Let X be an n-dimensional orientable closed manifold. Then for any integer i, we have $H^i(X, \mathbb{R}) \simeq H_{n-i}(X, \mathbb{R})$.

Thanks to the universal coefficient theorem, the right-hand-side is isomorphic to $H^{n-i}(X, \mathbb{R})^\vee$, the dual of the $(n-i)$-th cohomology group. Assume that X is a smooth manifold. Then one may interpret this duality as a perfect pairing as follows. Applying de Rham's theorem, one may identify $H^i(X, \mathbb{R}) = H^i_{dR}(X, \mathbb{R})$, and hence,

> the composition of the cup product map and the integration map $H^i(X, \mathbb{R}) \times H^{n-i}(X, \mathbb{R}) \to H^n(X, \mathbb{R}) \to \mathbb{R}$ defined by $(\eta, \xi) \mapsto \int_X (\eta \wedge \xi)$ gives a perfect pairing.

Since the sheaf cohomology generalizes the singular cohomology, we may expect there is an analogous "duality" theorem for varieties/schemes – at least, under mild assumptions on the underlying space.
Serre duality is a special case of the duality called the *coherent duality* in a much general setting. It is based on earlier works in several complex variables, however, we will observe it algebraically. One difference between the Poincaré duality and the Serre duality is the role of the "dualizing sheaf". We will see that the dualizing sheaf coincides with the canonical sheaf when the underlying space X is a nonsingular variety over an algebraically closed field.
First, we begin with the duality on the projective space. Let k be a field, let $X = \mathbb{P}^n_k$ be the projective n-space, and let $\omega_X = \wedge^n \Omega_{X/k} = \mathcal{O}_X(-n-1)$ be the canonical sheaf. We have the duality theorem for X as follows.

Theorem 185 (Duality for $\mathbb{P}^n_k$)**.**

(1) $H^n(X, \omega_X) \simeq k$.

(2) Fix an isomorphism $H^n(X, \omega_X) \simeq k$. *For any coherent sheaf* $\mathcal{F}$ *on* X*, the natural pairing*

$$\operatorname{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X) \simeq k$$

is a perfect pairing of finite-dimensional vector spaces over k*.*

(3) For every $i \geq 0$*, there is a natural functorial isomorphism*

$$\operatorname{Ext}^i(\mathcal{F}, \omega_X) \xrightarrow{\sim} H^{n-i}(X, \mathcal{F})^\vee.$$

Proof. (1) is clear from the computations we already made. Note that the pairing in (2) is natural, since any morphism $\mathcal{F} \to \omega_X$ induces a natural map on cohomology groups $H^i(X, \mathcal{F}) \to H^i(X, \omega_X)$ for each i. If $\mathcal{F} \simeq \mathcal{O}(q)$ for some $q \in \mathbb{Z}$, then $\operatorname{Hom}(\mathcal{F}, \omega_X) \simeq H^0(X, \omega_X(-q))$, so this pairing comes from the product map between

Laurent polynomials. In particular, the natural pairing is a perfect pairing if $\mathcal{F}$ is a finite direct sum of sheaves of the form $\mathcal{O}(q_i)$. Since any coherent sheaf $\mathcal{F}$ can be presented as a cokernel $\mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$ of a map of sheaves $\mathcal{E}_1 \to \mathcal{E}_0$, where each $\mathcal{E}_j$ is a finite direct sum of sheaves $\mathcal{O}(q_i)$. Now both $\mathrm{Hom}(-, \omega_X)$ and $H^n(X, -)^\vee$ are left-exact contravariant functors, so we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}(\mathcal{F}, \omega_X) & \longrightarrow & \mathrm{Hom}(\mathcal{E}_0, \omega_X) & \longrightarrow & \mathrm{Hom}(\mathcal{E}_1, \omega_X) \\ \| & & \| & & \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & 0 & \longrightarrow & H^n(X, \mathcal{F})^\vee & \longrightarrow & H^n(X, \mathcal{E}_0)^\vee & \longrightarrow & H^n(X, \mathcal{E}_1)^\vee. \end{array}$$

The statement just follows from the 5-lemma.
For (3), one may check that both sides are contravariant δ-functors for $\mathcal{F} \in \mathfrak{Coh}(X)$, indexed by $i \geq 0$. For $i = 0$ we have an isomorphism. Since any quasi-coherent sheaf $\mathcal{F}$ can be written as a quotient of a sheaf $\mathcal{E} = \bigoplus \mathcal{O}(-q)$ with $q \gg 0$, we have

$$\mathrm{Ext}^i(\mathcal{E}, \omega_X) = \bigoplus H^i(X, \omega_X(q)) = 0$$

for $i > 0$ by Serre vanishing. Similarly, we have

$$H^{n-i}(X, \mathcal{E})^\vee = \bigoplus H^{n-i}(X, \mathcal{O}(-q)) = 0$$

for $i > 0$ and $q > 0$. Hence, both sides are *coeffacable* (= any object $\mathcal{F}$ can be written as a quotient of an object $\mathcal{E}$ so that the image of $\mathcal{E}$ under the given functor is 0) for $i > 0$. By Grothendieck, such δ-functors are universal, hence isomorphic.

□

In particular, when $\mathcal{F}$ is locally free of finite rank on a projective n-space X, we have a natural isomorphism

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \simeq H^i(X, \mathcal{F}^\vee \otimes \omega_X) \xrightarrow{\sim} H^{n-i}(X, \mathcal{F})^\vee.$$

Remark 186. If we choose the coordinates $x_0, \cdots, x_n$ of $X = \mathbb{P}^n_k$, we may pick up a Čech cocycle

$$\alpha = \frac{x_0^n}{x_1 \cdots x_n} d\left(\frac{x_1}{x_0}\right) \wedge \cdots \wedge d\left(\frac{x_n}{x_0}\right)$$

which lies in $C^n(\mathfrak{U}, \omega_X)$, where $\mathfrak{U} = \{D_+(x_i)\}$. One can check that α gives a generator of $H^n(X, \omega_X)$, and is stable under the linear change of coordinates.

To generalize this, we take (1) and (2) as our guide. The key point was indeed: to fix an isomorphism $t : H^n(X, \omega_X) \xrightarrow{\sim} k$.

Definition 187. Let X be a projective scheme of dimension n over a field k. A *dualizing sheaf* for X is a coherent sheaf ω_X° on X, together with a *trace morphism* $t : H^n(X, \omega_X^\circ) \to k$, such that for all coherent sheaves $\mathcal{F}$ on X, the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{t} k$$

gives an isomorphism $\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \xrightarrow{\sim} H^n(X, \mathcal{F})^\vee$.

Proposition 188. *Let X be a projective scheme over k. Then the pair (ω_X°, t) of a dualizing sheaf and a trace morphism, if it exists, is unique up to a unique isomorphism. More precisely, if we have two pairs (ω°, t) and (ω', t'), then there is a unique isomorphism $\varphi : \omega^\circ \xrightarrow{\sim} \omega'$ such that $t = t' \circ H^n(\varphi)$.*

Proof. Since ω' is a dualizing sheaf, there is an isomorphism $\mathrm{Hom}(\omega^\circ, \omega') \simeq H^n(X, \omega^\circ)^\vee$. In particular, there is a unique morphism $\varphi : \omega^\circ \to \omega'$ corresponding to the element $t \in H^n(X, \omega^\circ)^\vee$. Note that the correspondence is given by the natural induced map on cohomology, that is, $t' \circ H^n(\varphi) = t$.
Similarly, using the fact that ω° is a dualizing sheaf, there is an isomorphism $\psi : \omega' \to \omega^\circ$ such that $t \circ H^n(\psi) = t'$. It follows that $t \circ H^n(\psi) \circ H^n(\varphi) = t \circ H^n(\psi \circ \varphi) = t$.
Since ω° is a dualizing sheaf, the morphism $\psi \circ \varphi : \omega^\circ \to \omega^\circ$ corresponds to the identity map; since there is a unique morphism $\omega^\circ \to \omega^\circ$ which preserves t. Similarly, we conclude that $\varphi \circ \psi$ is the identity map on ω'. □

In particular, the pair (ω°, t) represents the functor $\mathcal{F} \mapsto H^n(X, \mathcal{F})^\vee$ from $\mathfrak{Coh}(X)$ to $\mathfrak{Mod}(k)$.

The existence of a dualizing sheaf is much more tricky. A standard way of approaches is passing through Grothendieck's six operations. Suppose we have a morphism $f : X \to Y$ of schemes. Then we have an adjoint property

$$\mathrm{Hom}_X(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$$

for any $\mathcal{F} \in \mathfrak{Mod}(X)$ and $\mathcal{G} \in \mathfrak{Mod}(Y)$. In other words, the pair (f^*, f_*) is an adjoint pair; the pullback f^* is a left adjoint of f_*, and the pushforward f_* is a right adjoint of f^*. Similarly, we have a bijection of sets

$$\mathrm{Hom}(\mathcal{H}om(\mathcal{E}, \mathcal{F}), \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \mathcal{E} \otimes \mathcal{G})$$

for any $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ and a locally free sheaf of finite rank $\mathcal{E}$ on X. Of course, this is "weak", since it makes sense only if $\mathcal{E}$ is too "good" (we want to extend it at least for coherent sheaves). But anyway, the above bijection claims that the pair of functors $(\mathcal{H}om(\mathcal{E}, -), \mathcal{E} \otimes -)$ plays a role of an adjoint pair. This pair has a nicer property; we have

$$\mathrm{Hom}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \mathcal{H}om(\mathcal{E}, \mathcal{G})),$$

under the same assumption. In particular, $(\mathcal{E} \otimes -, \mathcal{H}om(\mathcal{E}, -))$ is also an adjoint pair, whereas f_* is NOT a left adjoint of f^*.
Nevertheless, it is worthwhile to consider a "right adjoint" of f_*. Let us have a look what happens locally. Let $B \to A$ be a ring homomorphism, M be an A-module, and let N be a B-module. Note that the pushforward $f_*(\widetilde{M})$ of a quasi-coherent sheaf $\widetilde{M}$ is $({}_BM)^\sim$, where ${}_BM$ means M considered as a B-module. We have a map

$$\mathrm{Hom}_A(M, \mathrm{Hom}_B(A, N)) \to \mathrm{Hom}_B({}_BM, N)$$

defined by: given $m \in M$ and an element $\phi \in \mathrm{Hom}_A(M, \mathrm{Hom}_B(A, N))$, we send $m \mapsto \phi(m)(1_A)$. One can show that this is a bijection. The functor ${}_A\mathrm{Hom}_B(A, -)$ is a "right adjoint" of ${}_B(-)$, which corresponds to the pushforward functor for ring spectra. One can also show that this bijection behaves nicely with respect to the localization at an element in B. In particular, this correspondence can be naturally "sheafify", at least for a morphism which looks like a morphism of ring spectra and modules over them – namely, an affine morphism and quasi-coherent sheaves.
Suppose that $f : X \to Y$ be an affine morphism between two schemes. The above discussion claims that the functor ${}_A\mathrm{Hom}_B(A, -)$ globalizes to a functor $\mathcal{G} \mapsto f^{-1}\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{G})$, $\mathcal{G} \in \mathfrak{Qco}(Y)$. The image of $\mathcal{G}$ is also quasi-coherent, this gives a functor $f^!_{ps} : \mathfrak{Qco}(Y) \to \mathfrak{Qco}(X)$. Moreover, we have a local isomorphism

$$f_*\mathcal{H}om_X(\mathcal{F}, f^!_{ps}\mathcal{G}) \xrightarrow{\sim} \mathcal{H}om_Y(f_*\mathcal{F}, \mathcal{G})$$

and a global isomorphism

$$\mathrm{Hom}_X(\mathcal{F}, f^!_{ps}\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_Y(f_*\mathcal{F}, \mathcal{G})$$

for any quasi-coherent sheaves $\mathcal{F} \in \mathfrak{Qco}(X)$ and $\mathcal{G} \in \mathfrak{Qco}(Y)$. We may regard $f^!_{ps}$ as a right adjoint of f_* in this manner.

Caution. This is not a standard way of description; to provide a natural description for the exceptional inverse image functor $f^!$, we need to consider the derived categories. However, both notions coincide when f is finite and flat.

We will briefly have a look how this construction helps us to find a dualizing sheaf. Let $X \subseteq \mathbb{P}^N_k$ be a locally Cohen-Macaulay (= all the local rings are Cohen-Macaulay) equidimensional projective scheme of dimension n. By taking a linear projection at a linear subspace Λ of codimension $(n+1)$ with $X \cap \Lambda = \emptyset$, we have a finite and flat morphism $\pi : X \to \mathbb{P}^n$. We have the duality theorem for such an X and a locally free sheaf $\mathcal{F}$ on X of finite rank:

Proposition 189. *There is a dualizing sheaf ω_X which gives an isomorphism $\mathrm{Ext}^i(\mathcal{F}, \omega_X) \simeq H^{n-i}(X, \mathcal{F})^\vee$ for every $i \geq 0$, $\mathcal{F}$ a locally free sheaf of finite rank.*

Proof. Since π is finite, it is affine. We take $\omega_X := \pi^!_{ps}\omega_{\mathbb{P}^n}$. We have isomorphisms

$$\begin{aligned}
\mathrm{Ext}^i_X(\mathcal{F}, \omega_X) &\simeq H^i(X, \mathcal{F}^\vee \otimes \omega_X) \\
&\simeq H^i(\mathbb{P}^n, \pi_*(\mathcal{F}^\vee \otimes \omega_X)) && \text{(an analogue of Leray spectral sequence)} \\
&\simeq H^i(\mathbb{P}^n, \pi_*(\mathcal{H}om_X(\mathcal{F}, \omega_X))) \\
&\simeq H^i(\mathbb{P}^n, \mathcal{H}om_{\mathbb{P}^n}(\pi_*\mathcal{F}, \omega_{\mathbb{P}^n})) && \text{(the local isomorphism above)} \\
&\simeq H^i(\mathbb{P}^n, (\pi_*\mathcal{F})^\vee \otimes \omega_{\mathbb{P}^n}) && \text{((}\pi_*\mathcal{F}\text{) is locally free)} \\
&\simeq \mathrm{Ext}^i_{\mathbb{P}^n}(\pi_*\mathcal{F}, \omega_{\mathbb{P}^n}) \\
&\simeq H^{n-i}(X, \mathcal{F})^\vee && \text{(Serre duality for } \mathbb{P}^n\text{)}
\end{aligned}$$

□

Now we go back to the arguments in Hartshorne's book. At the moment, it is still quite mysterious that how the dualizing sheaf on X looks like unless X is a projective space. We begin with some preliminaries.

Lemma 190. *Let X be a closed subscheme of codimension r of $\mathbb{P}^N_k$. Then $\mathcal{E}xt^i_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N}) = 0$ for all $i < r$.*

Proof. Notice that $\mathcal{F}^i := \mathcal{E}xt^i_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$ is a coherent sheaf on $\mathbb{P}^N$ for each i. By Serre vanishing, it will be globally generated after twisting by a sufficiently large number q. Hence, to conclude that $\mathcal{F}^i = 0$, it is sufficient to show that $\mathcal{F}^i(q)$ does not have a nonzero global section for all $q \gg 0$. Since

$$H^0(X, \mathcal{F}^i(q)) = \mathrm{Ext}^i_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N}(q)) = \mathrm{Ext}^i_{\mathbb{P}^N}(\mathcal{O}_X(-q), \omega_{\mathbb{P}^N})$$

for $q \gg 0$, it is isomorphic to $H^{N-i}(\mathbb{P}^N, \mathcal{O}_X(-q))^\vee$ by Serre duality for $\mathbb{P}^N$. For $i < r$, we have $N - i > \dim X$, so it vanishes thanks to the dimensional cohomology vanishing. $\square$

Lemma 191. *Let X be a closed subscheme of $\mathbb{P}^N_k$ of codimension r, and let $\omega^\circ_X := \mathcal{E}xt^r_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$. Then for any $\mathcal{O}_X$-module $\mathcal{F}$, there is a functorial isomorphism*

$$\mathrm{Hom}_X(\mathcal{F}, \omega^\circ_X) \simeq \mathrm{Ext}^r_{\mathbb{P}^N}(\mathcal{F}, \omega_{\mathbb{P}^N}).$$

Proof. Let $0 \to \omega_{\mathbb{P}^N} \to \mathcal{I}^\bullet$ be an injective resolution of $\omega_{\mathbb{P}^N}$. Note that the group $\mathrm{Ext}^i_{\mathbb{P}^N}(\mathcal{F}, \omega_{\mathbb{P}^N})$ is the i-th cohomology group $h^i(\mathrm{Hom}_{\mathbb{P}^N}(\mathcal{F}, \mathcal{I}^\bullet))$. Since $\mathcal{F}$ is an $\mathcal{O}_X$-module, any morphism $\mathcal{F} \to \mathcal{I}^i$ factors through $\mathcal{J}^i := \mathcal{H}om_{\mathbb{P}^N}(\mathcal{O}_X, \mathcal{I}^i)$. For any $\mathcal{F} \in \mathfrak{Mod}(X)$, we have $\mathrm{Hom}_X(\mathcal{F}, \mathcal{J}^i) = \mathrm{Hom}_{\mathbb{P}^N}(\mathcal{F}, \mathcal{I}^i)$. We have

$$\mathrm{Ext}^i_{\mathbb{P}^N}(\mathcal{F}, \omega_{\mathbb{P}^N}) = h^i(\mathrm{Hom}_X(\mathcal{F}, \mathcal{J}^\bullet)).$$

Also note that $\mathrm{Hom}_X(-, \mathcal{J}^i)$ is exact, and hence, $\mathcal{J}^i$ is an injective $\mathcal{O}_X$-module. Consider the complex $0 \to \mathcal{J}^0 \to \mathcal{J}^1 \to \cdots$ (this is not an injective resolution of an $\mathcal{O}_X$-module). Thanks to the above lemma, we have $h^i(\mathcal{J}^\bullet) = \mathcal{E}xt^i_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N}) = 0$ for $i < r$, so the complex $\mathcal{J}^\bullet$ is exact up to r-th step. In fact, this is split exact since each $\mathcal{J}^i$ is injective. Hence, we may write the complex as a direct sum of two injective complexes $\mathcal{J}^\bullet = \mathcal{J}^\bullet_1 \oplus \mathcal{J}^\bullet_2$, where $\mathcal{J}^\bullet_1$ is in degrees $0 \le i \le r$ and exact, and $\mathcal{J}^\bullet_2$ is in degrees $i \ge r$, in particular, $\mathcal{J}^r = \mathcal{J}^r_1 \oplus \mathcal{J}^r_2$. It follows that $\omega^\circ_X = ker[d^r : \mathcal{J}^r_2 \to \mathcal{J}^{r+1}_2 = \mathcal{J}^{r+1}]$, and that for any $\mathcal{O}_X$-module $\mathcal{F}$, we have

$$\begin{aligned}
\mathrm{Hom}_X(\mathcal{F}, \omega^\circ_X) &\simeq \mathrm{Hom}_X(\mathcal{F}, \mathcal{E}xt^r_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N})) \\
&\simeq \mathrm{Hom}_X(\mathcal{F}, \ker[d^r : \mathcal{J}^r_2 \to \mathcal{J}^{r+1}_2]) \\
&\simeq h^r(\mathrm{Hom}_X(\mathcal{F}, \mathcal{J}^\bullet)) \\
&\simeq h^r(\mathrm{Hom}_{\mathbb{P}^N}(\mathcal{F}, \mathcal{I}^\bullet)) \\
&\simeq \mathrm{Ext}^r_{\mathbb{P}^N}(\mathcal{F}, \omega_{\mathbb{P}^N})
\end{aligned}$$

as desired. $\square$

Proposition 192. *Let X be a projective scheme over a field k. Then X has a dualizing sheaf ω_X°.*

Proof. Embed X as a closed subscheme of $\mathbb{P}_k^N$ for some N, let r be its codimension, and let $\omega_X^\circ := \mathcal{E}xt^r_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$ as above. For any $\mathcal{O}_X$-module $\mathcal{F}$, we have an isomorphism

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ) \simeq \mathrm{Ext}^r_{\mathbb{P}^N}(\mathcal{F}, \omega_{\mathbb{P}^N}).$$

When $\mathcal{F}$ is coherent, the duality theorem for $\mathbb{P}^N$ gives an isomorphism

$$\mathrm{Ext}^r_{\mathbb{P}^N}(\mathcal{F}, \omega_{\mathbb{P}^N}) \simeq H^{N-r}(\mathbb{P}^N, \mathcal{F})^\vee.$$

Since $N - r = n = \dim X$, and $\mathcal{F}$ is a sheaf on X, we have a functorial isomorphism

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ) \simeq H^n(X, \mathcal{F})^\vee$$

for any coherent sheaf $\mathcal{F} \in \mathfrak{Coh}(X)$. In particular, if we take $\mathcal{F} = \omega_X^\circ$, the identity element $id \in \mathrm{Hom}_X(\omega_X^\circ, \omega_X^\circ)$ gives a trace homomorphism $t : H^n(X, \omega_X^\circ) \to k$. By its functoriality, the pair (ω_X°, t) is a dualizing sheaf for X. □

This leads to the following duality theorem for a projective scheme.

Theorem 193 (Duality for a projective scheme)**.** *Let X be a projective scheme of dimension n over an algebraically closed field k, let ω_X° be a dualizing sheaf on X, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Then:*

(1) for each $i \geq 0$ and coherent sheaf $\mathcal{F}$ on X, there is a natural functorial map

$$\theta^i : \mathrm{Ext}^i_X(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})^\vee$$

such that θ^0 is the map given in the definition of the dualizing sheaf above;

(2) the following are equivalent:

- *(i) X is locally Cohen-Macaulay and equidimensional;*
- *(ii) for any locally free sheaf $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}(-q)) = 0$ for $i < n$ and $q \gg 0$;*
- *(iii) the map θ^i above is an isomorphism for every $i \geq 0$ and for every coherent sheaf $\mathcal{F}$ on X.*

Proof. First of all, we write a given coherent sheaf $\mathcal{F}$ as a quotient of a locally free sheaf $\mathcal{E} = \bigoplus \mathcal{O}_X(-q)$ of finite rank with $q \gg 0$. Then $\mathrm{Ext}^i(\mathcal{E}, \omega_X^\circ) \simeq \bigoplus H^i(X, \omega_X^\circ(q))$, which is 0 for $i > 0$ by Serre vanishing. Thus, the functor $\mathrm{Ext}^i(-, \omega_X^\circ)$ is coeffaceable for $i > 0$, in particular, they form a universal contravariant δ-functor. On the right hand side, we have a contravariant δ-functor, also indexed by $i \geq 0$ – which we do not know it is universal at the moment. Anyway, there is a unique morphism of δ-functors (θ^i) reducing to the given θ^0 for $i = 0$.

(i) $\Rightarrow$ (ii) Embed X as a closed subscheme of $\mathbb{P}^N_k$. For any locally free sheaf $\mathcal{F}$ on X, and any closed point $x \in X$, we have $\operatorname{depth} \mathcal{F}_x = \dim \mathcal{O}_{X,x} = n$, since X is locally Cohen-Macaulay and equidimensional of dimension n. Let $A = \mathcal{O}_{\mathbb{P}^N,x}$ be the local ring of $\mathbb{P}^N$ at x. Since k is an algebraically closed field, it is a regular local ring of dimension N. The depth of $\mathcal{F}_x$ computed over $\mathcal{O}_{X,x}$ is same as the one over A. Thanks to Auslander-Buchsbaum formula

$$pd_A(\mathcal{F}_x) + \operatorname{depth} \mathcal{F}_x = \operatorname{depth} A = N,$$

the projective dimension of $\mathcal{F}_x$ (= the least length of a projective A-resolution of the module $\mathcal{F}_x$) is $N - n$. Since the stalk $\mathcal{E}xt^i_{\mathbb{P}^N}(\mathcal{F}, \mathcal{G})_x$ coincides with the module $\operatorname{Ext}^i_A(\mathcal{F}_x, \mathcal{G}_x)$ for any coherent sheaf $\mathcal{F}$, the stalk becomes zero for $i > N - n$ and for any closed point $x \in X$. We conclude that

$$\mathcal{E}xt^i_{\mathbb{P}^N}(\mathcal{F}, -) = 0$$

for $i > N - n$. On the other hand, by Serre duality for $\mathbb{P}^N$, the cohomology group $H^i(X, \mathcal{F}(-q))$ is isomorphic to $\operatorname{Ext}^{N-i}_{\mathbb{P}^N}(\mathcal{F}, \omega_{\mathbb{P}^N}(q))^\vee$. For a sufficiently large $q \gg 0$, this Ext is isomorphic to $\Gamma(\mathbb{P}^N, \mathcal{E}xt^{N-i}_{\mathbb{P}^N}(\mathcal{F}, \omega_{\mathbb{P}^N}(q)))$, which is 0 when $N - i > N - n$. In other words, $H^i(X, \mathcal{F}(-q)) = 0$ for $i < n$ and $q \gg 0$.
(ii) $\Rightarrow$ (i) Let $\mathcal{F} = \mathcal{O}_X$. We take a running the argument backwards. We find that $\mathcal{E}xt^i(\mathcal{O}_X, \omega_{\mathbb{P}^N}) = \mathcal{E}xt^i(\mathcal{O}_X, \omega_{\mathbb{P}^N}(q)) \otimes \mathcal{O}_{\mathbb{P}^N}(-q) = 0$ for $i > N - n$. In particular, over the local ring $A = \mathcal{O}_{\mathbb{P}^N,x}$, the module

$$\operatorname{Ext}^i_A(\mathcal{O}_{X,x}, A) = 0$$

vanishes for every $i > N - n$. Hence, the projective dimension of an A-module $\mathcal{O}_{X,x}$ is at most $N - n$ (for instance, one can understand this by using Yoneda extensions). Again by Auslander-Buchsbaum formula, we have $\operatorname{depth} \mathcal{O}_{X,x} \geq n$. Since $\dim X = n$ and the depth cannot exceed the dimension (of a local ring), we must have an equality for every closed point x of X. This implies that X is locally Cohen-Macaulay and of pure dimension n.
(ii) $\Rightarrow$ (iii) To show that θ^i is an isomorphism, it is sufficient to check that the contravariant δ-functor $(H^{n-i}(X, \mathcal{F})^\vee)$ is universal. Given a coherent sheaf $\mathcal{F}$, write $\mathcal{F}$ as a quotient of a locally free sheaf $\mathcal{E} = \bigoplus \mathcal{O}_X(-q)$ of finite rank with $q \gg 0$. Since $H^{n-i}(X, \mathcal{E})^\vee = 0$ for $i > 0$, the functor is coeffaceable for each $i > 0$, and hence universal.
(iii) $\Rightarrow$ (ii) If θ^i is an isomorphism, then for any locally free $\mathcal{E}$, we have

$$H^i(X, \mathcal{E}(-q)) \simeq \operatorname{Ext}^{n-i}_X(\mathcal{E}(-q), \omega^\circ_X)^\vee \simeq \operatorname{Ext}^{n-i}_X(\mathcal{O}_X, \mathcal{E}^\vee \otimes \omega^\circ_X(q))^\vee = H^{n-i}(X, \mathcal{E}^\vee \otimes \omega^\circ_X(q))^\vee.$$

Hence, it is 0 for every $n - i > 0$ and $q \gg 0$ by Serre vanishing. □

Remark 194. When X is nonsingular over k, or a local complete intersection, then X is locally Cohen-Macaulay.

Corollary 195. *Let X be a projective Cohen-Macaulay scheme of pure dimension n over k. Then for any locally free sheaf $\mathcal{E}$, we have natural isomorphisms*

$$H^i(X, \mathcal{E}) \simeq H^{n-i}(X, \mathcal{E}^\vee \otimes \omega^\circ_X)^\vee.$$