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UNIVERSITÄT DES SAARLANDES

Computer Algebra and Gröbner Bases

Winterterm 2020/21

All exercise sheets and course information can be found at: www.math.uni-sb.de/ag/schreyer/

Sheet 5 10. December 2020

Exercise 1 + 2. Consider the ring extension
$$R = \mathbb{R}[e_2, e_3] \hookrightarrow T = \mathbb{R}[t_1, t_2]$$
defined by $e_2 \mapsto t_1t_2 - (t_1 + t_2)^2, e_3 \mapsto t_1t_2(-t_1 - t_2)$.

(1) Prove that $S = R[t_1] \cong R[x]/(x^3 + e_2x + e_3)$ and conclude that
$$R \subset S \subset T$$
is a tower of finite extensions.

(2) Compute the degrees of the field extensions
$$Q(R) \subset Q(S) \subset Q(T).$$

(3) Let $(b_2, b_3) \in \mathbb{A}^2(\mathbb{R})$ be a point. How many maximal ideals $\mathfrak{P}$ in S can lie over the maximal ideal of $\mathfrak{p} = (e_2 - b_2, e_3 - b_3)$. How many maximal ideals $\mathfrak{P}'$ in T can lie over $\mathfrak{p}$.

(4) What residue fields $S/\mathfrak{P}$ and $T/\mathfrak{P}'$ occur?

Exercise 3. Consider the ideal $I \subset K[x_0, \dots, x_n]$ generated by the 2×2 minors of the matrix
$$\begin{pmatrix} x_0 & x_1 & \dots & x_{n-1} \\ x_1 & x_2 & \dots & x_n \end{pmatrix}$$
Compute $\dim V(I)$.

Exercise 4. Let $\mathfrak{p}_1 \subset \mathfrak{p}_2$ be prime ideals of $K[x_1, \dots, x_n]$ and let $A_j = V(\mathfrak{p}_j) \subset \mathbb{A}^n$ be the corresponding varieties. Prove that $\dim A_1 = \dim A_2$ implies $\mathfrak{p}_1 = \mathfrak{p}_2$.