Fakultät MI, Fachrichtung Mathematik
Prof. Dr. Frank-Olaf Schreyer

UNIVERSITÄT DES SAARLANDES

Computer Algebra and Gröbner Bases

Winterterm 2020/21

All exercise sheets and course information can be found at: www.math.uni-sb.de/ag/schreyer/

Sheet 6 17. December 2020

Exercise 1. If $\psi : L \to N$ and $\varphi : M \to N$ are two R-module homomorphism with $\varphi \circ \psi$ then

$$H = \frac{\ker(\varphi)}{\operatorname{im}(\psi)}$$

is called the homology of the complex

$$L \xrightarrow{\psi} M \xrightarrow{\varphi} N$$

at M. Let

$$\begin{tikzcd}
E_1 \arrow[r, "a"] \arrow[d, "\psi_1"] & E_0 \arrow[r] \arrow[d, "\psi_0"] & L \arrow[r] \arrow[d, "\psi"] & 0 \\
F_1 \arrow[r, "b"] \arrow[d, "\varphi_1"] & F_0 \arrow[r] \arrow[d, "\varphi_0"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
G_1 \arrow[r, "c"] & G_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

be free presentations of ψ and φ. Prove the correctness of the following algorithm.

1. Compute the syzygy matrix of $(c|\varphi_0)$
$$H_0 \xrightarrow{\begin{pmatrix} g_0 \\ h_0 \end{pmatrix}} G_1 \oplus F_0 \xrightarrow{(c|\varphi_0)} G_0\ .$$
2. Compute the syzygy matrix of $(h_0|b|\psi_0)$
$$H_1 \xrightarrow{\begin{pmatrix} h_1 \\ g_1 \\ f_1 \end{pmatrix}} H_0 \oplus F_1 \oplus E_0 \xrightarrow{(h_0|b|\psi_0)} F_0\ .$$
3. Then
$$H_1 \xrightarrow{h_1} H_0 \longrightarrow H \longrightarrow 0$$
is a presentation of H.

Exercise 2. Draw the real points of the projective closure of the plane affine curves $A = V(y^2 - z^3)$ and $B = V(1/4y^2 + z^2 - 1)$ in all three charts of $\mathbb{P}^2$.

Exercise 3. Compute the homogeneous ideal of the projective closure of the affine curve parametrised by

$$\mathbb{A}^1 \to \mathbb{A}^3, t \mapsto (t, t^3, t^4).$$

Exercise 4. Consider the map

$$S^2 \to \mathbb{R}^3 = \mathbb{A}^3(\mathbb{R}), (x, y, z) \mapsto (yz, xz, xy).$$

Prove that the map factors over $\mathbb{P}^2(\mathbb{R})$ and compute equations of the algebraic closure in $\mathbb{A}^3$. Draw the image using for example the program Maple or
surfer https://imaginary.org/de/program/surfer.
The image is known under name Steiner surface or roman surface.