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Computer Algebra and Gröbner Bases

Winterterm 2020/21

All exercise sheets and course information can be found at: www.math.uni-sb.de/ag/schreyer/

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Exercise 1.
Let $\varphi : X \to Y$ be a projective morphism. Prove that

1)
$$A_r = \{q \in Y \mid \dim X_q \geq r\} \subset Y$$
is a Zariski-closed subset of Y.
2) Suppose that X and Y are varieties and that φ is surjective. Prove that
$$\dim A_r + r < \dim X$$
for r with $r > \dim X - \dim Y$.

Exercise 2. Consider the algebraic set $S(e,d) \subset \mathbb{P}^{d+e-1}$ for $d, e \geq 1$ defined by the 2×2 minors of the matrix
$$\begin{pmatrix} x_0 & x_1 & \dots & x_{d-1} & y_0 & y_1 & \dots & y_{e-1} \\ x_1 & x_2 & \dots & x_d & y_1 & y_2 & \dots & y_e \end{pmatrix}$$
where $x_0 \dots y_e$ are the homogeneous coordinates on $\mathbb{P}^{d+e+1}$.

1) Prove that there exists a morphism $\pi : S(d,e) \to \mathbb{P}^1$ whose fibers are lines.
2) Let $\phi_1 : \mathbb{P}^1 \to \mathbb{P}^d = V(y_0, \dots, y_e)$ and $\phi_2 : \mathbb{P}^1 \to \mathbb{P}^2 = V(x_0, \dots, x_n)$ be the parametrisation of the rational normal curve of degree d and e in disjoint linear subspaces $\mathbb{P}^d \cup \mathbb{P}^e \subset \mathbb{P}^{d+e+1}$. Prove
$$S(e,d) = \bigcup_{p \in \mathbb{P}^1} \overline{\phi_1(p)\phi_2(p)}$$
where $\overline{\phi_1(p)\phi_2(p)}$ denotes the line joining $\phi_1(p)$ and $\phi_2(p)$.

Exercise 3. Let $L_1 \cup L_2 \cup L_3 \cup L_4 \subset \mathbb{P}^3$ be four general lines. Prove: Counted with multiplicities there are exactly two lines $L \subset \mathbb{P}^3$ which intersects all four lines.

Hint: Take the special case $L_1 = V(w, x), L_2 = V(y, z)$ and $L_3 = V(w + y, x + z)$ and prove that $L_1 \cup L_2 \cup L_3$ lies in a unique quadric hypersurface $Q \subset \mathbb{P}^3$ isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Exercise 4. Conider a conic $C \subset \mathbb{P}^2$ and six different points $p_1, \ldots, p_6$ on C. Prove Pacal's theorem: *The opposite sides of the hexagon $L_{12} = \overline{p_1 p_2}, L_{23} = \overline{p_2 p_3}, \ldots, L_{56} = \overline{p_5 p_6}, L_{61} = \overline{p_5 p_6}$ intersect in three points $q_1 = L_{12} \cap L_{45}, q_2 = L_{23} \cap L_{56}, q_3 = L_{34} \cap L_{61}$ which lie on a line.*

Hint: Consider the pencil of cubics

$$V(t_0 f + t_1 g) \subset \mathbb{P}^2 \text{ with } [t_0 : t_1] \in \mathbb{P}^1$$

where $f = \ell_{12}\ell_{34}\ell_{56}$ and $g = \ell_{23}\ell_{45}\ell_{61}$ are products of the equation ℓ_{ij} of L_{ij}.