ON THE GLOBAL GROSS-PRASAD CONJECTURE FOR YOSHIDA LIFTINGS

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Prof. J. Shalika to his 60th birthday

Introduction

In two articles in the Canadian Journal [16, 17], B. Gross and D. Prasad proclaimed a global conjecture concerning the decomposition of an automorphic representation of an adelic special orthogonal group G_1 upon restriction to an embedded orthogonal group G_2 of a quadratic space in smaller dimension and also its local counterpart. In the local situation, one can summarize the conjecture by saying that the occurrence of π_2 in the restriction of π_1 depends on the ϵ-factor attached to the representation $\pi_1 \otimes \pi_2$; in the global situation, assuming the existence of the local nontrivial invariant functional at all places and its nonvanishing on the spherical vector at almost all unramified places, one considers a specific linear functional given by a period integral. This period integral is then conjectured to give a nontrivial functional if and only if the central critical value of the L-function attached to $\pi_1 \otimes \pi_2$ is nonzero. In particular in the case when G_1 is the group of an n-dimensional nondegenerate quadratic space V and G_2 is the group of an $(n-1)$-dimensional subspace W of V, they showed that in low dimensions ($n \leq 4$) known results can be interpreted as evidence for this conjecture, using the well known isomorphisms for orthogonal groups in low dimensions.

The case $n = 5$ has been treated in the local situation by Prasad [27]; it can also be reinterpreted using these isomorphisms: The split special orthogonal group in dimension 5 is isomorphic to the projective symplectic similitude group PGSp_2, and the spin group of the 4-dimensional

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split orthogonal group is $\mathrm{SL}_2 \times \mathrm{SL}_2$. Prasad then showed that for forms on PGSp_2 that are lifts from the orthogonal group of a 4-dimensional space, the situation can be understood in terms of the seesaw dual reductive pair (in Kudla's sense)

$$\begin{array}{ccc} GSp_2 & & GO(4) \times GO(4)\ . \\ | & \times & | \\ G(SL_2 \times SL_2) & & GO(4) \end{array}$$

In classical terms, the analogous global question leads to the problem to determine those pairs of cuspidal elliptic modular forms (eigenforms of almost all Hecke operators) that can occur as summands if one decomposes the restriction of a cuspidal Siegel modular form F of degree 2 (that is an eigenform of almost all Hecke operators) to the diagonally embedded product of two upper half planes into a sum of (products of) eigenforms of almost all Hecke operators (for a discussion of the problems that arise in translating the representation theoretic statement into a classical statement see below and Remark 2.13. One can also rephrase this as the problem of calculating the period integral

$$\int\limits_{(\Gamma\backslash\mathbf{H})\times(\Gamma\backslash\mathbf{H})} F\left(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}\right)\overline{f_1(z_1)f_2(z_2)}d^*z_1d^*z_2$$

for two elliptic Hecke eigenforms f_1, f_2. The L-function that should occur then according to the conjecture of Gross and Prasad is the degree 16 L-function associated to the tensor product of the 4-dimensional representation of the L-group $\mathrm{Spin}(4)$ of $\mathrm{SO}(5)$ with the two 2-dimensional representation associated to two copies of $\mathrm{SO}(3)$ (due to the decomposition of $\mathrm{SO}(4)$ or rather its covering group mentioned above). We denote this L-function as $L(\mathrm{S}pin(F), f_1, f_2, s)$.
In this reformulation it is natural to go beyond the original question of nonvanishing and to try and get an explicit formula connecting the L-value in question with the period integral. In general, it seems rather difficult to calculate the period integral as above, since little is known about the restriction of Siegel modular forms to the diagonally embedded product of two upper half planes. One should also point out that an integral representation for the degree 16 L-function in question is not known yet. However, for theta series of quadratic forms the restriction to the diagonal of a degree two theta series becomes simply the product of the degree one theta series in the variables z_1, z_2, which allows one to get a calculation started. Thus here we only consider

Siegel modular forms (of trivial character) that arise as linear combinations of theta series of quaternary quadratic forms. Such Siegel modular forms, if they are eigenforms, are called Yoshida liftings, attached to a pair of elliptic cusp forms or, equivalently, to a pair of automorphic forms on the multiplicative group of an adelic quaternion algebra. These liftings have been investigated in [32, 4, 8], and the connection between the trilinear forms on the spaces of automorphic forms on the multiplicative group of an adelic quaternion algebra and the triple product L-function has been investigated in [20, 15, 7]. If one combines these results and applies them to the present situation, it turns out that the period integral in question can indeed be explicitly calculated in terms of the central critical value of the L-function mentioned; in the case of a Yoshida-lifting attached to the pair h_1, h_2 of elliptic cusp forms, this L-function is seen to split into the product $L(h_1, f_1, f_2, s)L(h_2, f_1, f_2, s)$, so that the central critical value becomes the product of the central critical values of these two triple product L-functions. We prove a formula that expresses the square of the period integral explicitly as the product of these two central critical values, multiplied by an explicitly known non-zero factor. We reformulate the obtained identity in a way which makes sense as well for an arbitrary Siegel modular form F in terms of the original $L(\mathrm{S}pin(F), f_1, f_2, s)$, in place of $L(h_1, f_1, f_2, s)L(h_2, f_1, f_2, s)$, hoping that such an identity indeed holds for any Siegel modular form F. At present we cannot prove it except for the case when F is a Siegel or Klingen Eisenstein series of level 1. In the case when F is the Saito-Kurokawa lifting of an elliptic Hecke eigenform h, one sees easily that the period integral is zero unless one has $f_1 = f_2 = f$; in this case the period integral can be transformed into the Petersson inner product of the restriction of the first Fourier Jacobi coefficient of F to the upper half plane with f and then leads us to a conjectural identity for the square of this Petersson inner product with the central critical value of $L(h, f, f; s)$.

Our calculation leaves the question open whether it can happen that the period integral vanishes for the classical modular forms considered but is non-zero for other functions in the same adelic representation space. Viewed locally, this amounts to the question whether an invariant nontrivial linear functional on the local representation space is necessarily non-zero at the given vector. At the infinite place we can exhibit such a vector (depending on the weights given) by applying a suitable differential operator to the Siegel modular form considered. At the finite places not dividing the level, it comes down to the question whether (for an unramified representation) a nontrivial invariant linear functional is necessarily non-zero at the spherical (or class 1) vector invariant under the maximal compact subgroup. This is generally expected, at least for generic representations. We intend to come back to this question in future work.

We also investigate the situation where the pair f_1, f_2 and the product $\mathbf{H} \times \mathbf{H}$ are replaced by a Hilbert modular form and the modular embedding of a Hilbert modular surface; in terms of the Gross-Prasad conjecture this amounts to replacing the split orthogonal group of a 4-dimensional space from above by a non split (but quasisplit) orthogonal group that is split at infinity. It turns out that one gets an analogous result; we prove this only in the simplest case when all modular forms involved have weight 2, the class number of the quadratic field involved is 1 and the order in a quaternion algebra belonging to the situation is a maximal order. The proof for the general case should be possible in an analogous manner.

1. Yoshida liftings and their restriction to the diagonal

For generalities on Siegel modular forms we refer to [Fre1]. For a symplectic matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n(\mathbf{R})$ (with $n \times n$-blocks A, B, C, D) we denote by $(M, Z) \mapsto M < Z >= (AZ + B)(CZ + D)^{-1}$ the usual action of the group $G^+Sp(n, \mathbf{R})$ of proper symplectic similitudes on Siegel's upper half space $\mathbf{H}_n$.

We shall mainly be concerned with Siegel modular forms for congruence subgroups of type

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbf{Z}) \mid C \equiv 0 \bmod N \right\}.$$

The space of Siegel modular forms (and cusp forms respectively) of degree n and weight k for $\Gamma_0^{(n)}(N)$ will be denoted by $M_n^k(N)$ $(S_n^k(N))$, for a vector valued modular form transforming according to the representation ρ the weight k above should be replaced by ρ. By $< \, , \, >$ we denote the Petersson scalar product.

We recall from [4, 8, 5] some notations concerning the Yoshida-liftings whose restrictions we are going to study in this article. For details we refer to the cited articles. We consider a definite quaternion algebra D over $\mathbf{Q}$ and an Eichler order R of square free level N in it and decompose N as $N = N_1 N_2$ where N_1 is the product of the primes that are ramified in D. On D we have the involution $x \mapsto \overline{x}$, the (reduced) trace $\mathrm{tr}(x) = x + \overline{x}$ and the (reduced) norm $n(x) = x\overline{x}$.

The group of proper similitudes of the quadratic form $q(x) = n(x)$ on D is isomorphic to $(D^\times \times D^\times)/Z(D^\times)$ (as algebraic group) via

$$(x_1, x_2) \mapsto \sigma_{x_1,x_2} \text{ with } \sigma_{x_1,x_2}(y) = x_1 y x_2^{-1},$$

the special orthogonal group is then the image of

$$\{(x_1, x_2) \in D^\times \times D^\times \mid n(x_1) = n(x_2)\}.$$

We denote by H the orthogonal group of (D, n) and by H^+ the special orthogonal group.

For $\nu \in \mathbf{N}$ let $U_\nu^{(0)}$ be the space of homogeneous harmonic polynomials of degree ν on $\mathbf{R}^3$ and view $P \in U_\nu^{(0)}$ as a polynomial on

$$D_\infty^{(0)} = \{x \in D_\infty | \mathrm{tr}(x) = 0\}$$

by putting

$$P(\sum_{i=1}^{3} x_i e_i) = P(x_1, x_2, x_3)$$

for an orthonormal basis $\{e_i\}$ of $D_\infty^{(0)}$ with respect to the norm form n. The space $U_\nu^{(0)}$ is known to have a basis of rational polynomials (i.e., polynomials that take rational values on vectors in $D^{(0)} = D_\infty^{(0)} \cap D$). The group $D_\infty^\times/\mathbf{R}^\times$ acts on $U_\nu^{(0)}$ through the representation τ_ν (of highest weight (ν)) given by

$$(\tau_\nu(y))(P)(x) = P(y^{-1}xy).$$

Changing the orthonormal basis above amounts to replacing P by $(\tau_\nu(y))(P)$ for some $y \in D_\infty^\times$.

By $\langle\langle\ \ ,\ \ \rangle\rangle_0$ we denote the suitably normalized invariant scalar product in the representation space $U_\nu^{(0)}$.

For $\nu_1 \geq \nu_2$ the $H^+(\mathbf{R})$-space

$$U_{\nu_1}^{(0)} \otimes U_{\nu_2}^{(0)}$$

(irreducible of highest weight $(\nu_1 + \nu_2, \nu_1 - \nu_2)$) is isomorphic to the $H^+(\mathbf{R})$-space U_{ν_1,ν_2} of $\mathbf{C}[X_1, X_2]$-valued harmonic forms on D_∞^2 transforming according to the representation of $GL_2(\mathbf{R})$ of highest weight $(\nu_1 + \nu_2, \nu_1 - \nu_2)$.

An intertwining map Ψ has been given explicitly in [5, Section 3]; for $\mathbf{x} = (x_1, x_2) \in D_\infty^2$ the polynomial $\Psi(Q)(\mathbf{x}) \in \mathbf{C}[X_1, X_2]$ is homogeneous of degree $2\nu_2$. We write now for $Q \in U_{\nu_1}^{(0)} \otimes U_{\nu_2}^{(0)}$

$$\Psi(Q)(\mathbf{x}) = \sum_{\alpha_1+\alpha_2=2\nu_2} c_{\alpha_1\alpha_2}(\mathbf{x}, Q) X_1^{\alpha_1} X_2^{\alpha_2}. \tag{1.1}$$

The map $\mathbf{x} \mapsto c_{\alpha_1\alpha_2}(\mathbf{x}, Q)$ is (for fixed Q) a polynomial in x_1, x_2 that is harmonic of degree $\alpha_1' = \alpha_1 + \nu_1 - \nu_2$ in x_1 and harmonic of degree $\alpha_2' = \alpha_2 + \nu_1 - \nu_2$ in x_2, and for $h \in H^+(\mathbf{R})$ we have

$$c_{\alpha_1\alpha_2}(h\mathbf{x}, Q) = c_{\alpha_1\alpha_2}(\mathbf{x}, h^{-1}Q).$$

The irreducibility of the space $U_{\nu_1}^{(0)} \otimes U_{\nu_2}^{(0)}$ implies that this map is nonzero for some Q. We denote by U_α the space of harmonic polynomials of degree α on D_∞ with invariant scalar product $\langle\langle\ ,\ \rangle\rangle$. If α is even, the $H(\mathbf{R})$-spaces U_α and $U_{\alpha/2}^{(0)} \otimes U_{\alpha/2}^{(0)}$ are isomorphic and will be identified.

The map

$$(Q, R_1, R_2) \mapsto \langle\langle c_{\alpha_1\alpha_2}(\cdot, Q), R_1 \otimes R_2 \rangle\rangle \tag{1.2}$$

for $Q \in U_{\nu_1}^{(0)} \otimes U_{\nu_2}^{(0)}, R_1 \in U_{\alpha_1'}, R_2 \in U_{\alpha_2'}$ defines then a nontrivial invariant trilinear form for the triple of $H(\mathbf{R})$-spaces $((U_{\nu_1}^{(0)} \otimes U_{\nu_2}^{(0)}), U_{\alpha_1'}, U_{\alpha_2'})$.

Lemma 1.1. *Let integers $\nu_1 \geq \nu_2$ and β_1, β_2 be given for which*

$$\beta_1' = \beta_1 + \nu_1 - \nu_2, \beta_2' = \beta_2 + \nu_1 - \nu_2$$

are even. Then there exists a nontrivial $H(\mathbf{R})$- invariant trilinear form T on the space $U_{\nu_1,\nu_2} \otimes U_{\beta_1'} \otimes U_{\beta_2'}$ if and only if there exist integers $\alpha_1, \alpha_2, \gamma$ such that $\beta_i = \alpha_i + \gamma$ and $\alpha_1 + \alpha_2 = 2\nu_2$ holds. This form is unique up to scalar multiples and can be decomposed as

$$T = T_1^{(0)} \otimes T_2^{(0)}$$

with (up to scalars) unique nontrivial invariant trilinear forms

$$T_i^{(0)} = T_{i,\beta_1',\beta_2'}^{(0)}$$

on

$$U_{\nu_i}^{(0)} \otimes U_{\beta_1'/2}^{(0)} \otimes U_{\beta_2'/2}^{(0)}.$$

In particular, for $\gamma = 0$ and T fixed, the trilinear form given in (1.2) is proportional to T (with a nonzero factor $\tilde{c}(\nu_1, \nu_2, \alpha_1, \alpha_2)$).

Proof. Decomposing

$$U_{\beta_1'} = U_{\beta_1'/2}^{(0)} \otimes U_{\beta_1'/2}^{(0)}, U_{\beta_2'} = U_{\beta_2'/2}^{(0)} \otimes U_{\beta_2'/2}^{(0)}$$

as a $D_\infty^\times/\mathbf{R}^\times \times D_\infty^\times/\mathbf{R}^\times$-space one sees that T as asserted exists if and only if there are nontrivial invariant trilinear forms

$$T_i^{(0)} = T_{i,\beta_1',\beta_2'}^{(0)}$$

on

$$U_{\nu_i}^{(0)} \otimes U_{\beta_1'/2}^{(0)} \otimes U_{\beta_2'/2}^{(0)}$$

for $i = 1, 2$. In this case T decomposes as

$$T = T_1^{(0)} \otimes T_2^{(0)}.$$

The $T_i^{(0)}$ are known to exist if and only if the triples

$$(\beta_1'/2, \beta_2'/2, \nu_1), (\beta_1'/2, \beta_2'/2, \nu_2)$$

are balanced, i.e, the numbers in either triple are the lengths of the sides of a triangle (and then the form is unique up to scalars); they are unique up to scalar multiplication. It is then easily checked that the numerical condition given above is equivalent to the existence of nonnegative integers $\alpha_1, \alpha_2, \gamma$ satisfying $\beta_i = \alpha_i + \gamma$ and $\alpha_1 + \alpha_2 = 2\nu_2$. Consider now the Gegenbauer polynomial $G^{(\alpha)}(x, x') =$ obtained from

$$G_1^{(\alpha)}(t) \;=\; 2^\alpha \sum_{j=0}^{[\frac{\alpha}{2}]} (-1)^j \, \frac{1}{j!(\alpha - 2j)!} \, \frac{(\alpha - j)!}{2^{2j}} \, t^{\alpha - 2j}$$

by

$$\tilde{G}^{(\alpha)}(x, x') = 2^\alpha (n(x)n(x'))^{\alpha/2} G_1^{(\alpha)}\left(\frac{\operatorname{tr}(x\overline{x'})}{2\sqrt{n(x)n(x')}}\right)$$

and normalize the scalar product on U_α such that $G^{(\alpha)}$ is a reproducing kernel, i. e.

$$\langle\langle G^{(\alpha)}(x, x'), Q(x)\rangle\rangle_\alpha = Q(x')$$

for all $Q \in U_\alpha$. Then for $\alpha_1, \alpha_2, \alpha_1', \alpha_2'$ as above and some fixed $Q \in U_{\nu_1, \nu_2}$ the map

$$(x_1, x_2) \mapsto T(Q, G^{(\alpha_1')}(x_1, \cdot), G^{(\alpha_2')}(x_2, \cdot))$$

defines a polynomial $R_Q(x_1, x_2)$ in x_1, x_2 that is harmonic of degree $\alpha_1' = \alpha_1 + \nu_1 - \nu_2$ in x_1 and harmonic of degree $\alpha_2' = \alpha_2 + \nu_1 - \nu_2$ in x_2, and for $h \in H^+(\mathbf{R})$ we have $R_Q(h\mathbf{x}) = R_{h^{-1}Q}(\mathbf{x})$.
As above we can therefore conclude that one has

$$c_{\alpha_1\alpha_2}(\mathbf{x}, Q) = \tilde{c}(\nu_1, \nu_2, \alpha_1, \alpha_2) T(Q, G^{(\alpha_1')}(x_1, \cdot), G^{(\alpha_2')}(x_2, \cdot), \tag{1.3}$$

where the factor of proportionality $\tilde{c}(\nu_1, \nu_2, \alpha_1, \alpha_2)$ is not zero.
We denote by

$$\mathcal{A}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times, \nu)$$

the space of functions $\varphi : D_{\mathbf{A}}^\times \to U_\nu^{(0)}$ satisfying $\varphi(\gamma x u) = \tau_\nu(u_\infty^{-1})\varphi(x)$ for $\gamma \in D_{\mathbf{Q}}^\times$ and $u = u_\infty u_f \in R_{\mathbf{A}}^\times$, where

$$R_{\mathbf{A}}^\times = D_\infty^\times \times \prod_p R_p^\times$$

is the adelic group of units of R. These functions are determined by their values on the representatives y_i of a double coset decomposition

$$D_{\mathbf{A}}^\times = \cup_{i=1}^r D^\times y_i R_{\mathbf{A}}^\times$$

(where we choose the y_i to satisfy $y_{i,\infty} = 1$ and $n(y_i) = 1$).
The natural inner product on the space $\mathcal{A}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times, \nu)$ is given by

$$\langle \varphi, \psi \rangle = \sum_{i=1}^r \frac{\langle\langle \varphi(y_i), \psi(y_i) \rangle\rangle_0}{e_i},$$

where $e_i = \mid (y_i R y_i^{-1})^\times \mid$ is the number of units of the order $R_i = y_i R y_i^{-1}$ of D.
On the space $\mathcal{A}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times, \nu)$ we have for $p \not| N$ (hermitian) Hecke operators $\tilde{T}(p)$ (given explicitly by the $\mathrm{End}(U_\nu^{(0)})$-valued Brandt matrices $(B_{ij}(p)))$ and for $p \mid N$ involutions $\widetilde{w_p}$ commuting with the Hecke operators and with each other.
For $i = 1, 2$ and $\nu_1 \geq \nu_2$ with $\nu_1 - \nu_2$ even we consider now functions φ_i in $\mathcal{A}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times, \nu_i)$.

The Yoshida lifting (of degree 2) of the pair (φ_1, φ_2) is then given as

$$\begin{aligned}
(1.4) \quad & Y^{(2)}(\varphi_1, \varphi_2)(Z)(X_1, X_2) = \\
&= \sum_{i,j=1}^r \frac{1}{e_i e_j} \sum_{(x_1, x_2) \in (y_i R y_j^{-1})^2} \Psi(\varphi_1(y_i) \otimes \varphi_2(y_j))(x_1, x_2)(X_1, X_2) \times \\
&\qquad \times \exp(2\pi i \mathrm{tr}\left(\begin{pmatrix} n(x_1) & tr(\overline{x_1} x_2) \\ tr(\overline{x_1} x_2) & n(x_2) \end{pmatrix} Z \right).
\end{aligned}$$

This is a vector valued holomorphic Siegel modular form for the group $\Gamma_0^{(2)}(N)$ with trivial character and with respect to the representation $\sigma_{2\nu_2} \otimes \det^{\nu_1 - \nu_2 + 2}$ (where $\sigma_{2\nu_2}$ denotes the $2\nu_2$-th symmetric power representation of GL_2).

If we consider the restriction of such a modular form to the diagonal $\left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix}\right)$, the coefficient of $X_1^{\alpha_1} X_2^{\alpha_2}$ becomes a function $F^{(\alpha_1, \alpha_2)}(z_1, z_2)$ which is in both variables a scalar valued modular form for the group $\Gamma_0(N)$ with trivial character of weight

$$\alpha_1 + \nu_1 - \nu_2 + 2 \text{ in } z_1, \alpha_2 + \nu_1 - \nu_2 + 2 \text{ in } z_2.$$

In particular the weights in the variables z_1, z_2 add up to $2\nu_1 + 4$ for each pair (α_1, α_2) with $\alpha_1 + \alpha_2 = 2\nu_2$ and the coefficient of $X_1^{\alpha_1} X_2^{\alpha_2}$ vanishes unless

$$\alpha_1' = \alpha_1 + \nu_1 - \nu_2, \alpha_2' = \alpha_2 + \nu_1 - \nu_2$$

are even.

If f_1, f_2 are elliptic modular forms of weights k_1, k_2 we define then

$$\langle F\left(\left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix}\right)\right), f_1(z_1)f_2(z_2)\rangle_{k_1,k_2}$$

to be the double Petersson product

$$\langle\langle F^{(\alpha_1,\alpha_2)}(z_1, z_2), f_1(z_1)\rangle_{k_1}, f_2(z_2)\rangle_{k_2}$$

if

$$k_1 = \alpha_1 + \nu_1 - \nu_2 + 2, k_2 = \alpha_2 + \nu_1 - \nu_2 + 2$$

for some α_1, α_2 with $\alpha_1 + \alpha_2 = 2\nu_2$ and to be zero otherwise (this definition coincides with the Petersson product of the corresponding automorphic forms on the groups $Sp_2(\mathbf{A})$ or $Sp_2(\mathbf{R})$ (restricted to the naturally embedded $SL_2(\mathbf{A}) \times SL_2(\mathbf{A})$) and $SL_2(\mathbf{A})$ (or the respective real groups)).

We will mainly consider Yoshida liftings for pairs of forms that are eigenforms of all Hecke operators and of all the involutions. It is then easy to see that $Y^{(2)}(\varphi_1, \varphi_2)(Z)$ is identically zero unless φ_1, φ_2 have the same eigenvalue under the involution $\widetilde{w_p}$ for all $p \mid N$. The precise conditions under which the lifting is nonzero have been stated in [8].

We will finally need some facts about the correspondence studied e.g. in [10, 22, 29, 24] between modular forms for $\Gamma_0(N)$ (with trivial character) and automorphic forms on the adelic quaternion algebra $D^\times_{\mathbf{A}}$. We consider the essential part

$$\mathcal{A}_{\text{ess}}(D^\times_{\mathbf{A}}, R^\times_{\mathbf{A}}, \nu)$$

consisting of functions φ that are orthogonal to all $\psi \in \mathcal{A}(D^\times_{\mathbf{A}}, (R'_{\mathbf{A}})^\times, \nu)$ for orders R' strictly containing R; this space is invariant under the $\tilde{T}(p)$ for $p \nmid N$ and the $\tilde{w}_p$ for $p \mid N$ and hence has a basis of common eigenfunctions of all the $\tilde{T}(p)$ for $p \nmid N$ and all the involutions $\tilde{w}_p$ for $p \mid N$. Being the components of eigenvectors of a rational matrix with real eigenvalues the values of these eigenfunctions are real, (i.e., polynomials with real coefficients in the vector values case) when suitably normalized.

Moreover the eigenfunctions are in one to one correspondence with the newforms in the space

$$S^{2+2\nu}(N)$$

of elliptic cusp forms of weight $2 + 2\nu$ for the group $\Gamma_0(N)$ that are eigenfunctions of all Hecke operators (if τ is the trivial representation and R is a maximal order one has to restrict here to functions orthogonal to the constant function 1 on the quaternion side in order to obtain cusp forms on the modular forms side). This correspondence

(Eichler's correspondence) preserves Hecke eigenvalues for $p \nmid N$, and if φ corresponds to $f \in S^{2+2\nu}(N)$ then the eigenvalue of f under the Atkin-Lehner involution w_p is equal to that of φ under $\tilde{w}_p$ if D splits at p and equal to minus that of φ under $\tilde{w}_p$ if D_p is a skew field. The correspondence can be explicitly described by associating to φ the modular form

$$h(z) = \sum_{i,j=1}^{r} \frac{1}{e_i e_j} \sum_{x \in (y_i R y_j^{-1})} (\varphi(y_i) \otimes \varphi(y_j))(x) \exp(2\pi i n(x) z)$$

(where as above $\varphi(y_i) \otimes \varphi(y_j)$ denotes the harmonic polynomial in $U_{2\nu}$ obtained by identifying $U_\nu^{(0)} \otimes U_\nu^{(0)}$ with $U_{2\nu}$.)
An extension of Eichler's correspondence to forms φ as above that are not essential but eigenfunctions of all the involutions $\widetilde{w_p}$ has been given in [21, 7].

2. Computation of periods

Our goal is the computation of the periods

$$\langle Y^{(2)}(\varphi_1, \varphi_2) \left(\left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix} \right) \right), f_1(z_1) f_2(z_2) \rangle_{k_1,k_2}$$

defined above for elliptic modular forms f_1, f_2 for the group $\Gamma_0(N)$. For this we study first how the vanishing of this period integral depends on the eigenvalues of the functions involved under the Atkin-Lehner involutions or their quaternionic and Siegel modular forms counterparts.
For a Siegel modular form F for the group $\Gamma_0^{(2)}(N)$ we let the Atkin-Lehner involutions with respect to the variables z_1, z_2 act on the restriction of F to the diagonal matrices $\left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix} \right)$ and denote by $F\left(\left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix} \right) \right) |\widetilde{W}_p$ the result of this action.

Lemma 2.1. *Let N be squarefree and let F be a vector valued Siegel modular form of degree 2 for the representation ρ of $GL_2(\mathbf{C})$ of highest weight (λ_1, λ_2) in the space $\mathbf{C}[X_1, X_2]_{\lambda_1 - \lambda_2}$ of homogeneous polynomials of degree $\lambda_1 - \lambda_2$ in X_1, X_2 with respect to $\Gamma_0^{(2)}(N)$ and assume for $p \mid N$ that the restriction of F to the diagonal is an eigenform of $\widetilde{W}_p$ with eigenvalue $\tilde{\epsilon}_p^{(0)}$. Let f_1, f_2 be elliptic cusp forms of weights k_1, k_2 for $\Gamma_0(N)$ that are eigenforms of the Atkin-Lehner involution w_p with eigenvalues $\epsilon_p^{(1)}, \epsilon_p^{(2)}$. Then the period integral*

$$\langle F \left(\left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix} \right) \right), f_1(z_1) f_2(z_2) \rangle_{k_1,k_2} \tag{2.1}$$

is zero unless one has $\tilde{\epsilon_p^{(0)}} \epsilon_p^{(1)} \epsilon_p^{(2)} = 1$.

Proof. Applying the Atkin-Lehner involution w_p to both variables z_1, z_2 one sees that this is obvious.

We can view the condition of Lemma 2.1 as a (necessary) local condition for the nonvanishing of the period integral at the finite primes dividing the level, with a similar role being played at the infinite primes by the condition that modular forms of the weights k_1, k_2 of f_1, f_2 appear in the decomposition of the restriction of the vector valued modular form F to the diagonal (or of a suitable form in the representation space of F, see below).

Lemma 2.2. *Let f_1, f_2, h_1, h_2 be modular forms for $\Gamma_0(N)$ that are eigenfunctions of all Atkin-Lehner involutions for the $p \mid N$ with f_1, f_2 cuspidal. Let h_1, h_2 have the same eigenvalue ϵ'_p for all the w_p for the $p \mid N$ and denote by $\epsilon_p^{(1)}, \epsilon_p^{(2)}$ the Atkin-Lehner eigenvalues at $p \mid N$ of f_1, f_2.*
For a factorization $N = N_1 N_2$ where N_1 has an odd number of prime factors let D_{N_1} be the quaternion algebra over $\mathbf{Q}$ that is ramified precisely at ∞ and the primes $p \mid N_1$ and R_{N_1} an Eichler order of level N in D_{N_1}.
Let $\varphi_1^{(N_1)}, \varphi_2^{(N_1)}$ be the forms in $\mathcal{A}((D_{N_1})^\times_{\mathbf{A}}, (R_{N_1})^\times_{\mathbf{A}}, \tau_i)(i = 1, 2)$ corresponding to h_1, h_2 under Eichler's correspondence.
Then the period integral

$$\langle Y^{(2)}(\varphi_1^{(N_1)}, \varphi_2^{(N_1)}) \left(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \right), f_1(z_1) f_2(z_2) \rangle_{k_1,k_2} \tag{2.2}$$

is zero unless $\epsilon'_p \epsilon_p^{(1)} \epsilon_p^{(2)} = -1$ holds for precisely those p that divide N_1; in particular it is always zero unless $\prod_{p \mid N} \epsilon'_p \epsilon_p^{(1)} \epsilon_p^{(2)} = -1$ holds.

Proof. For each factorization of N as above we denote by $\widetilde{\epsilon_p}(N_1)$ the eigenvalue under $\widetilde{w_p}$ of $\varphi_1^{(N_1)}, \varphi_2^{(N_1)}$, we have $\widetilde{\epsilon_p}(N_1) = -\epsilon'_p$ for the p dividing N_1 and $\widetilde{\epsilon_p}(N_1) = \epsilon'_p$ for the p dividing N_2. Hence the product $\widetilde{\epsilon_p}(N_1)\epsilon_p^{(1)}\epsilon_p^{(2)}$ is 1 for all p dividing N if N_1 is the product of the primes $p \mid N$ such that $\epsilon'_p \epsilon_p^{(1)} \epsilon_p^{(2)} = -1$ and is -1 for at least one $p \mid N$ otherwise; in particular a decomposition for which $\widetilde{\epsilon_p}(N_1)\epsilon_p^{(1)}\epsilon_p^{(2)} = 1$ for all $p \mid N$ holds and N_1 has an odd number of prime factors exists if and only if we have $\prod_{p \mid N} \widetilde{\epsilon_p}(N_1)\epsilon_p^{(1)}\epsilon_p^{(2)} = -1$.

The $\widetilde{W_p}$-eigenvalue of the restriction of $Y^{(2)}(\varphi_1^{(N_1)}, \varphi_2^{(N_1)})$ to the diagonal is $\widetilde{\epsilon_p}(N_1)$ by the result of Lemma 9.1 of [4] on the eigenvalue of $Y^{(2)}(\varphi_1^{(N_1)}, \varphi_2^{(N_1)})$ under the analogue for Siegel modular forms of the Atkin-Lehner involution. The assertion then follows from the previous lemma.

For simplicity we will in the sequel assume that h_1, h_2, f_1, f_2 are all newforms of (square free) level N; essentially the same results can be obtained for more general quadruples of forms of square free level using the methods of [7].

Lemma 2.3. *Let $N \neq 1$ be squarefree, D, R as described in Section 1, let f_1, f_2 be normalized newforms of weights k_1, k_2 for the group $\Gamma_0(N)$. Let $\varphi_1, \varphi_2 \in \mathcal{A}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times, \tau_i)$ be as above. Assume that the (even) weights k_1, k_2 of f_1, f_2 can be written as $k_i = \alpha_i + \nu_1 - \nu_2 + 2 = \alpha_i' + 2$ with nonnegative integers α_i satisfying $\alpha_1 + \alpha_2 = 2\nu_2$ and denote for $i = 1, 2$ by ψ_i the $U_{\alpha_i'/2}$-valued form in $\mathcal{A}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times, \tau_{\alpha_i'})$ corresponding to f_i under Eichler's correspondence.*
Then the period integral

$$\langle Y^{(2)}(\varphi_1, \varphi_2)\left(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}\right), f_1(z_1) f_2(z_2)\rangle_{k_1,k_2} \tag{2.3}$$

has the (real) value

$$\begin{aligned} c\langle f_1, f_1\rangle\langle f_2, f_2\rangle(\sum_{j=1}^{r} T^{(0)}_{\nu_1,\alpha_1',\alpha_2'}(\varphi_1(y_i) \otimes \psi_1(y_i) \otimes \psi_2(y_i))) \\ \times (\sum_{j=1}^{r} T^{(0)}_{\nu_2,\alpha_1',\alpha_2'}(\varphi_2(y_i) \otimes \psi_1(y_i) \otimes \psi_2(y_i))), \end{aligned} \tag{2.4}$$

with a nonzero constant c depending only on ν_1, ν_2, k_1, k_2.

Proof. The coefficient of $X_1^{\alpha_1} X_2^{\alpha_2}$ of the (i,j)-term in $F(z_1, z_2) = Y^{(2)}(\varphi_1, \varphi_2)\left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix}\right)$ is (with $Q_{ij} := \varphi_1(y_i) \otimes \varphi_2(y_j)$) equal to

$$\begin{aligned} \tilde{c}(\nu_1, \nu_2, \alpha_1, \alpha_2) \sum_{(x_1,x_2)\in I_{ij}} T(Q_{ij}, G^{(\alpha_1')}(x_1, \cdot), G^{(\alpha_2')}(x_2, \cdot)) \\ \times \exp(2\pi i n(x_1) z_1) \exp(2\pi i n(x_2) z_2) \end{aligned} \tag{2.5}$$

by (1.3). We write

$$\Theta_{ij}^{(\alpha')}(z)(x') = \sum_{x \in I_{ij}} G^{(\alpha')}(x, x') \exp(2\pi i n(x) z) \tag{2.6}$$

for the $U^{\alpha'}$-valued theta series attached to I_{ij} and the Gegenbauer polynomial $G^{(\alpha)}(x, x')$ and rewrite (2.5) as

$$\tilde{c}(\nu_1, \nu_2, \alpha_1, \alpha_2) T(Q_{ij}, \Theta_{ij}^{(\alpha_1')}(z_1), \Theta_{ij}^{(\alpha_2')}(z_2)). \tag{2.7}$$

The ij-term of the period integral (2.3) becomes then

$$\tilde{c}(\nu_1, \nu_2, \alpha_1, \alpha_2) T(Q_{ij}, \langle\Theta_{ij}^{(\alpha_1')}(z_1), f_1(z_1)\rangle, \langle\Theta_{ij}^{(\alpha_2')}(z_2), f_2(z_2)\rangle, \tag{2.8}$$

which by (3.13) of [6] and the factorization

$$T = T_{\nu_1,\nu_2,\alpha'_1,\alpha'_2} = T^{(0)}_{\nu_1,\alpha'_1,\alpha'_2} \otimes T^{(0)}_{\nu_2,\alpha'_1,\alpha'_2} = T_1^{(0)} \otimes T_2^{(0)} \tag{2.9}$$

is equal to

$$\tilde{c}(\nu_1,\nu_2,\alpha_1,\alpha_2)\langle f_1,f_1\rangle\langle f_2,f_2\rangle T_1^{(0)}(\varphi_1(y_i),\psi_1(y_i),\psi_2(y_i)) \\ \times T_2^{(0)}(\varphi_2(y_j),\psi_1(y_j),\psi_2(y_j)). \tag{2.10}$$

Summation over i,j proves the assertion. The value computed is real since the values of the φ_i,ψ_i are so and since T_0 is known to be real.

Theorem 2.4. *Let $h_1,h_2,f_1,f_2,\psi_1,\psi_2$ be as in Lemma 2.2, Lemma 2.3 with Atkin-Lehner eigenvalues ϵ'_p for h_1,h_2 and $\epsilon_p^{(1)},\epsilon_p^{(2)}$ for f_1,f_2; assume $\prod_{p\mid N}\epsilon'_p\epsilon_p^{(1)}\epsilon_p^{(2)} = -1$. Let D be the quaternion algebra over $\mathbf{Q}$ which is ramified precisely at the primes $p \mid N$ for which $\epsilon'_p\epsilon_p^{(1)}\epsilon_p^{(2)} = -1$ holds and R an Eichler order of level N in D, let φ_1,φ_2 be the forms in $\mathcal{A}(D_{\mathbf{A}}^\times,R_{\mathbf{A}}^\times,\tau_{1,2})$ corresponding to h_1,h_2 under Eichler's correspondence.*
Then the square of the period integral

$$\langle Y^{(2)}(\varphi_1,\varphi_2)\left(\begin{pmatrix} z_1 & 0\\ 0 & z_2\end{pmatrix}\right), f_1(z_1)f_2(z_2)\rangle_{k_1,k_2} \tag{2.11}$$

is equal to

$$\frac{c}{\langle h_1,h_1\rangle\langle h_2,h_2\rangle}L(h_1,f_1,f_2;\frac{1}{2})L(h_2,f_1,f_2;\frac{1}{2}), \tag{2.12}$$

where c is an explicitly computable nonzero number depending only on ν_1,ν_2,k_1,k_2,N and the triple product L-function $L(h,g,f;s)$ is normalized to have its functional equation under $s \mapsto 1-s$.
In particular the period integral is nonzero if and only if the central critical value of $L(h_1,f_1,f_2;s)L(h_2,f_1,f_2;s)$ is nonzero.

Proof. The choice of the decomposition $N = N_1N_2$ made above implies that we can use Theorem 5.7 of [7] to express the right hand side of (2.4) by the product of central critical values of the triple product L-functions associated to $(h_1,f_1,f_2),(h_2,f_1,f_2)$. The Petersson norms of f_1,f_2 appearing in Theorem 5.7 of [7] cancel against those appearing in the proof of Lemma 2.3.

Remark 2.5. a) *If the product $\prod_{p\mid N}\epsilon'_p\epsilon_p^{(1)}\epsilon_p^{(2)}$ is $+1$ we know from* [7] *that the sign in the functional equation of the triple product L-functions $L(h_1,f_1,f_2;s),L(h_2,f_1,f_2;s)$ is -1 and hence the* *central critical values are zero; from Lemma 2.2 we know that for any Yoshida lifting F associated to h_1, h_2 as in Lemma 2.2 the Petersson product of the restriction of F to the diagonal and $f_1(z_1), f_2(z_2)$ is zero as well.*

b) *It should be noticed that given h_1, h_2 there are $2^{\omega(N)-1}$ possible choices of the quaternion algebra with respect to which one considers the Yoshida lifting associated to h_1, h_2. All these Yoshida liftings are different, but have the same Satake parameters for all $p \nmid N$. Given f_1, f_2 with $\prod_{p \mid N} \epsilon'_p \epsilon_p^{(1)} \epsilon_p^{(2)} = -1$ there is then precisely one choice of quaternion algebra that leads to a nontrivial result for the period integral, all the others give automatically zero by Lemma 2.2. The choice of this quaternion algebra should be seen as variation of the Vogan L-packet of the p-adic component of the adelic representation generated by the Siegel modular form for the $p \mid N$ in such a way that the resulting L-packet satisfies the local Gross-Prasad condition for the split 5-dimensional and 4-dimensional orthogonal groups.*

We want to rephrase the result of Theorem 2.4 in order to replace the factor of comparison $\langle h_1, h_1\rangle\langle h_2, h_2\rangle$ occurring by a factor depending only on $F = Y^{(2)}(\varphi_1, \varphi_2)$ instead of h_1, h_2. Concerning the symmetric square L-function of F occurring in the following corollary we remind the reader that we view F as as an automorphic form on the adelic orthogonal group of the 5-dimensional quadratic space V of discriminant 1 over $\mathbf{Q}$ that contains a 2-dimensional totally isotropic subspace.

Corollary 2.6. *Under the assumptions of Theorem 2.4 and the additional assumption that h_1, h_2 are not proportional, the value of (2.12) is equal to:*

$$\frac{c\langle F, F\rangle}{L^{(N)}(F, Sym^2, 1)} L(h_1, f_1, f_2; \frac{1}{2}) L(h_2, f_1, f_2; \frac{1}{2}), \tag{2.13}$$

where again c is an explicitly computable nonzero constant depending only on the levels and weights involved and $L^{(N)}(F, Sym^2, s)$ is the N-free part of the L function of F with respect to the symmetric square of the 4 dimensional representation of the L-group of the group $SO(V)$ (V as above).

Proof. Since h_1, h_2 are not proportional, the Siegel modular form F is cuspidal and the Petersson product $\langle F, F\rangle$ is well defined. From [4, Proposition 10.2] we recall that $\langle F, F\rangle$ is (up to a nonzero constant) equal to the the residue at $s = 1$ of the N-free part $D_F^{(N)}(s)$ of the degree 5 L-function associated to F (normalizing the φ_i to $\langle \varphi_i, \varphi_i\rangle = 1$; the formulas given in [4] generalize easily to the situation where the φ_i take values in harmonic polynomials). It is also well known that $\langle h_i, h_i\rangle$ is equal (up to a nonzero constant depending only on weights and levels) to $D_{h_i}^{(N)}(1)$ where $D_{h_i}^{(N)}(s)$ is the symmetric square L-function associated to h_i. Comparing the parameters of the L-functions $L^{(N)}(F, \mathrm{Sym}^2, s)$ and $D_F^{(N)}(s)D_{h_1}^{(N)}(s)D_{h_2}^{(N)}(s)$ we see that the value of $L^{(N)}(F, \mathrm{Sym}^2, s)$ at $s = 1$ is equal to the residue at $s = 1$ of $D_F^{(N)}(s)D_{h_1}^{(N)}(s)D_{h_2}^{(N)}(s)$, which gives the assertion.

Remark. We can as well view $L^{(N)}(F, \mathrm{Sym}^2, s)$ as the exterior square of the degree 5 L-function associated to F.

Let us discuss now two degenerate cases:

Corollary 2.7. a) *Under the assumptions of Theorem 2.4 replace φ_2 by the constant function $(\sum_i \frac{1}{e_i})^{-1}$ (and hence $h_2(z)$ by the Eisenstein series $E(z) = (\sum_{i,j} \frac{1}{e_i e_j})^{-1} \sum_{i,j} \frac{1}{e_i e_j}\Theta_{ij}^{(0)}(z)$). Then the period integral*

$$\langle Y^{(2)}(\varphi_1, \varphi_2)\left(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}\right), f_1(z_1)f_2(z_2)\rangle_{k_1,k_2} \tag{2.14}$$

is zero unless $f_1 = f_2 =: f$, in which case its square is equal to the value at $s = 1$ of

$$\frac{c\langle F, F\rangle}{L^{(N)}(F, Sym^2, s)} L(h_1, f, f; s - \frac{1}{2})L(E, f, f; s - \frac{1}{2}), \tag{2.15}$$

where again c is an explicitly computable nonzero constant depending only on the levels and weights involved and $L(E, f_1, f_2; s)$ is defined in the same way as the triple product L-function for a triple of cusp forms, setting the p-parameters of E equal to $p^{1/2}, p^{-1/2}$ for $p \nmid N$.

b) *Under the assumptions of Theorem 2.4 let $h = h_1 = h_2$. Then the period integral (2.11) is equal to*

$$\frac{c}{\langle h, h\rangle} L(h, f_1, f_2; \frac{1}{2}), \tag{2.16}$$

where c is a nonzero constant depending only on the levels and weights involved.

Proof. Both assertions are obtained in the same way as Theorem 2.4 and Corollary 2.6; notice that in case a) (with $\omega(N)$ denoting the number of prime factors of N) both $L(E, f, f; s - \frac{1}{2})$ and $L^{(N)}(F, \mathrm{Sym}^2, s)$ are of order $\omega(N) - 1$ at $s = 1$.

Remark

a) The form of our result given in Corollary 2.6 could in principle be true for any Siegel modular form F instead of a Yoshida lifting if one replaces $L(h_1, f_1, f_2; \frac{1}{2})L(h_2, f_1, f_2; \frac{1}{2})$ by the value $L(\mathrm{S}pin(F), f_1, f_2, \frac{1}{2})$ of the spin L-function mentioned in the introduction. There is, however, not much known about the analytic properties of $L^{(N)}(F, \mathrm{Sym}^2, s)$, in particular this L- function might have a zero or a pole at $s = 1$.

b) In the degenerate case of Corollary 2.7 a) the Yoshida lifting F is the Saito-Kurokawa lifting associated to h_1. The result of that case could also be true in the case that h_1, f_1, f_2 are of level 1 and F is the Saito-Kurokawa lifting of h_1, but we can not prove this at present (except for the vanishing of the period integral in the case that $f_1 \neq f_2$, which is easily proved).

We notice that in that last case (as well as in the related case of a Yoshida lifting of Saito-Kurokawa type) the period integral is seen to be equal to the Petersson product $\langle \phi_1(\tau, 0), f(\tau)\rangle$, where $\phi_1(\tau, z)$ is the first Fourier Jacobi coefficient of F.

In the case of Corollary 2.7 b) the Yoshida lift F can be viewed as an Eisenstein series of Klingen type associated to h, in particular its image under Siegel's Φ-operator is equal to h (more precisely, the Klingen Eisenstein series in question is a sum of Yoshida liftings associated to various quaternion algebras of level dividing N (see [6]), where the other contributions yield a vanishing period integral). One checks that the result of Corollary 2.7 b) is valid also for F denoting the Klingen Eisenstein series attached to a cuspidal normalized Hecke eigenform h of level 1 and f_1, f_2 two cuspidal normalized Hecke eigenforms of level 1.

We can obtain a result similar to that of Theorem 2.4 for more general weights k_1, k_2 of the modular forms f_1, f_2. For this, remember that according to Lemma 1.1 the value given in (2.4) for the period integral in question also makes sense if one replaces throughout α_1', α_2' by $\beta_1' = \alpha_1' + \gamma, \beta_2' = \alpha_2' + \gamma$ for some fixed $\gamma > 0$; the forms f_1, f_2 then having weights $k_i = \alpha_i' + 2 + \gamma$ for $i = 1, 2$. As noticed above, our period integral becomes 0 in this situation. We can, however, modify the function $Y^{(2)}(\varphi_1, \varphi_2)(Z)$ by a differential operator $\tilde{\mathcal{D}}^{\gamma}_{2,\alpha_1,\alpha_2}$ in such a way that $\tilde{\mathcal{D}}^{\gamma}_{2,\alpha_1,\alpha_2} Y^{(2)}(\varphi_1, \varphi_2)$ is a function on $\mathbf{H} \times \mathbf{H}$ that is a modular form of weights k_1, k_2 of z_1, z_2 as described above and yields a value for the period integral of the same form as the one given in form (2.4). More precisely, we have:

Proposition 2.8. *For nonnegative integers k, r and l with $k \geq 2$ and any partition $l = a+b$, there exists a (non-zero) holomorphic differential operator $\mathcal{D}^r_{k,a,b}$ (polynomial in $X_1^2 \frac{\partial}{\partial z_1}, X_1 X_2 \frac{\partial}{\partial z_{12}}, X_2^2 \frac{\partial}{\partial z_2}$, evaluated in $z_{12} = 0$) mapping $\mathbf{C}[X_1, X_2]_l$-valued functions on $\mathbf{H_2}$ to $\mathbf{C} \cdot X_1^{a+r} X_2^{b+r}$-valued functions on $\mathbf{H} \times \mathbf{H}$ and satisfying*

$$\mathcal{D}^r_{k,a,b} \left(F \mid_{k,l} M_1^{\uparrow} M_2^{\downarrow} \right) = (\mathcal{D}^r_{k,a,b} F) \mid^{z_1}_{k+a+r} M_1 \mid^{z_2}_{k+b+r} M_2$$

for all M_1, $M_2 \in SL(2, \mathbf{R})$; here the upper indices z_1 and z_2 at the slash-operator indicate the variable, with respect to which one has to apply the elements of $SL(2, \mathbf{R})$ and $\uparrow\downarrow$ denote the standard embedding of $SL(2) \times SL(2)$ into $Sp(2)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\uparrow} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\downarrow} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & A & 0 & B \\ c & 0 & d & 0 \\ 0 & C & 0 & D \end{pmatrix}$$

Of course one can consider $\mathcal{D}^r_{k,a,b} F$ as a $\mathbf{C}$-valued function.

Remark 2.9. *One may indeed show, that there exists (up to multiplication by a constant) precisely one such nontrivial holomorphic differential operator.*

Corollary 2.10. *The differential operator $\mathcal{D}^r_{k,a,b}$ defined above gives rise to a map*

$$\tilde{\mathcal{D}}^r_{k,a,b} : M^2_{k,l}(\Gamma^2_0(N)) \longrightarrow M^1_{k+a+r}(\Gamma^1_0(N)) \otimes M^1_{k+b+r}(\Gamma^1_0(N))$$

of spaces of modular forms. (It is easy to see that for $r > 0$ this map actually goes into spaces of cusp foms).

Corollary 2.11. *Denoting by $Sym_2(\mathbf{C})$ the space of complex symmetric matrices of size 2 we define a polynomial function*

$$Q : Sym_2(\mathbf{C}) \longrightarrow \mathbf{C}[X_1, X_2]_{2r}$$

by

$$\mathcal{D}^r_{k,a,b} e^{tr(TZ)} = Q(T) e^{t_1 z_1 + t_2 z_2}$$

where $Z = \begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix} \in \mathbf{H}_2$.

Furthermore we assume (with $k = \frac{m}{2} + \nu$) that $P : \mathbf{C}^{(m,2)} \longrightarrow \mathbf{C}[X_1, X_2]_l$ is a polynomial function satisfying

a) *P is pluriharmonic*

b) *$P((X_1,X_2)A)=\rho_{\nu,l}(A)P(X_1,X_2)$ for all $A\in GL(2,\mathbf{C})$.*
Then, for $\mathbf{Y}_1,\mathbf{Y_2}\in\mathbf{C}^m$

$$(\mathbf{Y}_1,\mathbf{Y}_2)\longmapsto\left\{P(\mathbf{Y}_1,\mathbf{Y}_2)\cdot Q(\begin{pmatrix}\mathbf{Y}_1^t\mathbf{Y}_1 & \mathbf{Y}_1^t\mathbf{Y}_2\\ \mathbf{Y}_2^t\mathbf{Y}_1 & \mathbf{Y}_2^t\mathbf{Y}_2\end{pmatrix})\right\}_{a+r+\nu,b+r+\nu}$$

defines an element of $H_{a+r+\nu}(m)\otimes H_{b+r+\nu}(m)$, where $H_\mu(m)$ is the space of harmonic polynomials in m variables (for the standard quadratic form), homogeneous of degree μ and for any $R\in\mathbf{C}[X_1,X_2]_{l+2r}$ we denote by $\{R\}_{\alpha,\beta}$ the coefficient of $X_1^\alpha X_2^\beta$ in R, $\alpha+\beta=l+2r$.

The proof of Corollary 2.11 is a vector-valued variant of similar statements in [2] and [9, p.200], using the proposition above and the characterization of harmonic polynomials by the Gauß-transform; we leave the details of proof to the reader.

Proof of Proposition 2.8.
We start from a Maaß-type differential operator δ_{k+l} which maps $\mathbf{C}[X_1,X_2]_l$-valued functions on $\mathbf{H}_2$ to $\mathbf{C}[X_1,X_2]_{l+2}$-valued ones and satisfies

$$(\delta_{k+l}F)\mid_{k,l+2}M=\delta_{k+l}\left(F\mid_{k,l}M\right)$$

for all $M\in Sp(2,\mathbf{R})$.
It is well known, how such operators arise from elements of the universal enveloping algebra of the complexified Lie algebra of $Sp(2,\mathbf{R})$, see e.g. [19] . In our case (we refer to [3] for details) we can describe these operators quite explicitly in terms of the simple operators

$$DF:=\left(\frac{1}{2\pi i}\frac{\partial}{\partial Z}\right)[\mathbf{X}]$$

and

$$NF:=\left(-\frac{1}{4\pi}(ImZ)^{-1}F\right)[\mathbf{X}]$$

Here $\mathbf{X}$ stands for the column vector $\begin{pmatrix}X_1\\X_2\end{pmatrix}$. Then we define

$$\delta_kF=kNF+DF$$

It is remarkable (and already incorporated in our notation!) that δ_{k+l} depends only on $k+l$.
The iteration

$$\delta_{k+l}^r:=\delta_{k+l+2r-2}\circ\cdots\circ\delta_{k+l+2}\circ\delta_{k+l}$$

can also be described explicitly by

$$\delta_{k+l}^r=\sum_{i=0}^r\frac{\Gamma(k+l+r)}{\Gamma(k+l+r-i)}\binom{r}{i}N^iD^{r-i}$$

For a function $F : \mathbf{H}_2 \longrightarrow \mathbf{C}[X_1, X_2]_l$ and a decomposition $l = a + b$ we put

$$\nabla^r_{k+l}(a,b)F =: X_1^{a+r} X_2^{b+r} - \text{coefficient of } \left(\delta^r_{2,k+l} F\right)_{|\mathbf{H}\times\mathbf{H}}$$

Then ∇ has already the transfomation properties required in the proposition, i.e.

$$\nabla^r_{k+l}(a,b)\left(F \mid_{k,l} M_1^{\uparrow} M_2^{\downarrow}\right) = \left(\nabla^r_{k+l}(a,b)F\right) \mid^{z_1}_{k+a+r} M_1 \mid^{z_2}_{k+b+r} M_2$$

for $M_1,\ M_2 \in SL(2,\mathbf{R})$.

Moreover, if F is in addition a holomorphic function on $\mathbf{H}_2$, then $\nabla^r_{k+l}(a,b)F$ is a nearly holomorphic function in the sense of Shimura (with respect to both variables z_1 and z_2), as polynomials in $\frac{1}{y_1}$ and $\frac{1}{y_2}$ they are of degree $\leq r$. Shimura's structure theorem on nearly holomorphic functions [30] says that all nearly holomorphic functions on $\mathbf{H}$ can be obtained from holomorphic functions by applying Maaß type operators

$$\delta_k := \frac{k}{2iy} + \frac{\partial}{\partial z}$$

and their iterates. This however is only true if the weight (i.e. $k+a+r$ or $k+r+b$) is bigger than $2r$, which is not necessarily true in our situation. We therefore use a weaker version of Shimura' theorem (see [31, Theorem 3.3]), valid under the assumption "$w > 1 + r$", where w is the weight at hand and r is the degree of the nearly holomorphic function: Every such function f on $\mathbf{H}$ of degree $\leq r$ has an expression

$$f = f_{hol} + L_w(\tilde{f})$$

where f_{hol} is holomorphic and $\tilde{f}$ is again nearly holomorphic of degree $\leq r$; in this expression

$$L_w := \delta_{w-2}\left(y^2 \frac{\partial}{\partial \bar{z}}\right) = \frac{w}{2i} y \frac{\partial}{\partial \bar{z}} + y^2 \frac{\partial^2}{\partial z \partial \bar{z}}$$

is a "Laplacian" of weight w commuting with the $|_w$-action of $SL(2,\mathbf{R})$. We also point out that f_{hol} is uniquely determined by f (in particular, $f = f_{hol}$, if f is holomorphic) and we have $(f \mid_w M)_{hol} = (f_{hol}) \mid_w M$ for all $M \in SL(2,\mathbf{R})$.

If we apply this statement to $\nabla^r_{k+l}(a,b)F$, considered as function of z_1 and z_2, we get an expression of type

$$\nabla^r_{k+l}(a,b)F = f + L^{z_1}_{k+a+r} g_1 + L^{z_2}_{k+r+b} g_2 + L^{z_1}_{k+a+r} L^{z_2}_{k+r+b} h$$

where f, g_1, g_2, h are nearly holomorphic functions on $\mathbf{H}\times\mathbf{H}$, f being holomorphic in both variables, g_1 holomorphic in z_2, g_2 holomorphic in z_1. Note that (due to our assumption $k \geq 2$) Shimura's theorem is applicable here. An inspection of Shimura's proof (which is quite elementary for our case) shows that f is indeed of the form $f = \mathbf{D}F$, where $\mathbf{D}$ is a polynomial p in $\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_{12}}, \frac{\partial}{\partial z_2}$, evaluated in $z_{12} = 0$. This polynomial does not depend on F at all and it has the required transformation properties.

It remains however to show that $\mathbf{D}$ is not zero:

For this purpose, we consider the special function

$$\mathbf{z}_{12}^r : \begin{cases} \mathbf{H}_2 & \longrightarrow & \mathbf{C}[X_1, X_2]_l \\ Z & \longmapsto & z_{12}^r X_1^a X_2^b \end{cases}$$

It is easy to see that $\nabla_{k+l}^r(a,b)(\mathbf{z}_{12}^r)$ is then equal to the constant function $r!$, therefore

$$\nabla_{k+l}^r(a,b)(\mathbf{z}_{12}^r) = \mathbf{D}(\mathbf{z}_{12}^r) = r!,$$

in particular, $\mathbf{D}$ is non-zero and we may put

$$\mathcal{D}_{k,a,b}^r = p(X_1^2 \frac{\partial}{\partial z_1}, X_1 X_2 \frac{\partial}{\partial z_{12}}, X_2^2 \frac{\partial}{\partial z_2}),$$

evaluated at $z_{12} = 0$

We can then prove in the same way as above:

Corollary 2.12. *The assertions of Lemma 2.3 and Theorem 2.4 remain true if f_1, f_2 have weights $k_i = \alpha_i' + 2 + \gamma (i = 1, 2)$ with some $\gamma > 0$, if one replaces*

$$Y^{(2)}(\varphi_1, \varphi_2)\left(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}\right)$$

by

$$\tilde{\mathcal{D}}_{2,\alpha_1,\alpha_2}^{\gamma} Y^{(2)}(\varphi_1, \varphi_2)\,(z_1, z_2)$$

Remark 2.13. *Application of the differential operator to $Y^{(2)}(\varphi_1, \varphi_2)\,(Z))$ before restriction to the diagonal does not change the $Sp_2(\mathbf{R})$-representation space of that function, i.e., we have found a different function in the same representation space whose period integral assumes the value that is predicted by the conjecture of Gross and Prasad. More precisely, (using remark 2.9 and some additional considerations) one can show that the vanishing of this predicted value is already sufficient for the vanishing of the period integral for all triples F', f_1', f_2' of functions in the Harish-Chandra modules generated by the original functions F, f_1, f_2. To obtain a similar statement for the local representations at the finite places not dividing the level one would have to show that a nonvanishing invariant linear functional on the tensor product of the representations is not zero on the product of the* *spherical (or class 1) vectors invariant under the maximal compact subgroup. This is expected to be true as well; we plan to come back to these problems in future work.*

3. Restriction to an embedded Hilbert modular surface

To avoid technical difficulties we deal here only with the simplest case: The quaternion algebra D is ramified at all primes p dividing the level N and we have $\nu_1 = \nu_2 = 0$, i.e., the Yoshida lifting is a scalar valued Siegel modular form of weight 2 and the order R we are considering is a maximal order. We put $F = \mathbf{Q}(\sqrt{N})$ and assume that N is such that the class number of F is 1. We denote by Δ the discriminant of F, by $a \mapsto a^\sigma$ its nontrivial automorphism and consider the basis $1, w$ with $w = \frac{\Delta+\sqrt{\Delta}}{2}$ of the ring $\mathfrak{o}_F$ of F. Denoting by C the matrix

$$C := \begin{pmatrix} 1 & 1 \\ w & \bar{w} \end{pmatrix}$$

we have the usual modular embedding

$$\iota : (z_1, z_2) \mapsto C \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} {}^tC \tag{3.1}$$

of $\mathbf{H} \times \mathbf{H}$ into the Siegel upper half plane $\mathbf{H}_2$ and

$$\tilde{\iota} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} C & 0 \\ 0 & {}^tC^{-1} \end{pmatrix} \begin{pmatrix} a & 0 & b & 0 \\ 0 & a^\sigma & 0 & b^\sigma \\ c & 0 & d & 0 \\ 0 & c^\sigma & 0 & d^\sigma \end{pmatrix} \begin{pmatrix} C^{-1} & 0 \\ 0 & {}^tC \end{pmatrix} \tag{3.2}$$

from $SL_2(F)$ into $Sp_2(\mathbf{Q})$.
We have then for $\gamma \in SL_2(F)$:

$$\tilde{\iota}(\gamma)(\iota((z_1, z_2)) = \tilde{\iota}(\gamma((z_1, z_2))) \tag{3.3}$$

with the usual actions of the groups $SL_2(F)$ on $\mathbf{H} \times \mathbf{H}$ and of $Sp_2(\mathbf{Q})$ on the Siegel upper half plane.
We put now

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and consider for a Siegel modular form f of weight k for the group $\Gamma \subseteq Sp_2(\mathbf{Q})$ the function

$$\tilde{f}(\tau_1, \tau_2) := f|_k J(\iota(\tau_1, \tau_2)). \tag{3.4}$$

Writing $\iota_0 = J \circ \iota, \widetilde{\iota_0}(\gamma) := J\iota(\gamma)J^{-1}$ we see that $\tilde{f}$ is a Hilbert modular form for the group $\widetilde{\iota_0}^{-1}(\Gamma)$.
By calculating $\widetilde{\iota_0}(\gamma)$ explicitly for $\gamma \in SL_2(F)$ one checks that $\widetilde{\iota_0}(\gamma)$ is in $\Gamma_0^{(2)}(N)$ if and only if γ is in $SL_2(\mathfrak{o}_F \oplus \mathfrak{d})$, the group of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, d \in \mathfrak{o}_F, c \in \mathfrak{d}, b \in \mathfrak{d}^{-1}$, where $\mathfrak{d}$ is the different of F.
If L is a $\mathbf{Z}$-lattice of (even) rank $m = 2k$ with quadratic form q and associated bilinear form $B(x, y) := q(x + y) - q(x) - q(y)$ satisfying $q(L) \subseteq \mathbf{Z}$ and $Nq(L^\#)\mathbf{Z} = \mathbf{Z}$ (N is the level of (L, q)) then it is shown in [4] that

(3.5)
$$\vartheta^{(2)}(L, q, Z)|_k J = c_1 \sum_{x_1 \in L, x_2 \in L^\#} \exp\bigl(2\pi i \mathrm{tr}\bigl(\begin{pmatrix} q(x_1) & B(x_1, x_2)/2 \\ B(x_1, x_2)/2 & q(x_2) \end{pmatrix} Z\bigr)\bigr),$$

where c_1 is a nonzero constant depending only on the genus of (L, q) and where $\vartheta^{(2)}(L, q, Z)$ is the usual theta series of degree 2 of (L, q). One checks therefore that, writing $K = \{x_1 + x_2 w \in L \otimes F \mid x_1 \in L, x_2 \in L^\#\}$, we have

$$\vartheta^{(2)}(L, q, \widetilde{\iota_0}(z_1, z_2)) = \vartheta(K, q, (z_1, z_2)), \tag{3.6}$$

where we denote by $\vartheta(K, q, (z_1, z_2)) = \sum_{y \in K} \exp(2\pi i(q(y)z_1 + q(y)^\sigma z_2))$ the theta series of the $\mathfrak{o}_F$-lattice K with the extended form q on it.
It is again easily checked that $L^\# \subseteq N^{-1}L$ implies that K is an integral unimodular $\mathfrak{o}_F$-lattice, and it is well known that then the theta series $\vartheta(K, q, (z_1, z_2))$ is a modular form of weight k for the group $SL_2(\mathfrak{o}_F \oplus \mathfrak{d})$.

Lemma 3.1. *Let $\tilde{D}$ be a quaternion algebra over F ramified at both infinite primes and let $\tilde{R}$ be a maximal order in $\tilde{D}$. Let*

$$\mathcal{A}(\tilde{D}_{\mathbf{A}}^\times, \tilde{R}_{\mathbf{A}}^\times, 0) =: \mathcal{A}(\tilde{D}_{\mathbf{A}}^\times, \tilde{R}_{\mathbf{A}}^\times)$$

be defined in the same way as in Section 1 for D and let $\mathcal{A}(\tilde{D}_{\mathbf{A}}^\times, \tilde{R}_{\mathbf{A}}^\times)$ be equipped with the natural action of Hecke operators $T(\mathfrak{p})$ for the $\mathfrak{p}$ not dividing N described by Brandt matrices as explained in [11]*. Then by associating to a Hecke eigenform $\psi \in \mathcal{A}(\tilde{D}_{\mathbf{A}}^\times, \tilde{R}_{\mathbf{A}}^\times)$ the Hilbert modular form*

$$f(z_1, z_2) = \int_{(\tilde{D}^\times \backslash \tilde{D}_{\mathbf{A}}^\times) \times (\tilde{D}^\times \backslash \tilde{D}_{\mathbf{A}}^\times)} \psi(y)\psi(y')\vartheta(y'\tilde{R}y^{-1}, (z_1, z_2))dy\, dy'$$

one gets a bijective correspondence between the ψ as above and the Hecke eigenforms of weight 2 *and trivial character for the group $\Gamma_0(\mathfrak{n}, \mathfrak{d})$ of precise level $\mathfrak{n}$ giving an explicit realization of the correspondence of Shimizu und Jacquet/Langlands* [29, 24]*. Here $\mathfrak{n}$ denotes the product of*

the prime ideals ramified in $\tilde{D}$ *and* $\Gamma_0(\mathfrak{n}, \mathfrak{d})$ *is the subgroup of* $SL_2(\mathfrak{o}_F \oplus \mathfrak{d})$ *whose lower left entries are in* $\mathfrak{n}\mathfrak{d}$.
The function $\rho(y, y')$ *on* $\tilde{D}_{\mathbf{A}}^\times \times \tilde{D}_{\mathbf{A}}^\times$ *given by setting* $\rho(y, y')$ *equal to the Petersson product of* f *with* $\vartheta(y'\tilde{R}y^{-1}, (z_1, z_2))$ *is proportional to* $(y, y') \mapsto \psi(y)\psi(y')$.

Proof. The first part of this Lemma is due to Shimizu [29] (taking into account that by [11] the group $\Gamma_0(\mathfrak{n}, \mathfrak{d})$ is the correct transformation group for the theta series in question). Let $\hat{\rho}$ be the function on the adelic orthogonal group of $\tilde{D}$ induced by ρ and let $\hat{\psi}$ be the function on the adelic orthogonal group of $\tilde{D}$ induced by $(y, y') \mapsto \psi(y)\psi(y')$. The function $\hat{\psi}$ generates an irreducible representation space of $\tilde{D}_{\mathbf{A}}^\times$ whose theta lifting to $SL_2(F_{\mathbf{A}})$ is generated by f, and $\hat{\rho}$ is a vector in the theta lifting of this latter representation of $SL_2(F_{\mathbf{A}})$, which by [25] coincides with the original representation space generated by $\hat{\psi}$. Since both $\hat{\rho}, \hat{\psi}$ are invariant under the same maximal compact subgroup of $\tilde{D}_{\mathbf{A}}^\times$, the uniqueness of such a vector implies that they must coincide up to proportionality. That $\hat{\rho}$ is not zero follows from the obvious fact that f by its construction can not be orthogonal to all the theta series.

Lemma 3.2. *With the above notations let* φ_1, φ_2 *in* $\mathcal{A}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times, 0)$ *be Hecke eigenforms with the same eigenvalue under the involutions* $\widetilde{w_p}$ *for the* $p \mid N$ *with associated newforms* h_1, h_2 *of weight* 2 *and level* N. *Let* f *be a Hilbert modular form of weight* 2 *for the group* $SL_2(\mathfrak{o}_F \oplus \mathfrak{d})$ *that corresponds in the way described in Lemma 3.1 to the function* $\psi \in \mathcal{A}(\tilde{D}_{\mathbf{A}}^\times, \tilde{R}_{\mathbf{A}}^\times)$ *for* $\tilde{D} = D \otimes F$ *and* $\tilde{R}$ *being the maximal order in* $\tilde{R}$ *containing* R. *Then the value of the period integral*

$$\int_{SL_2(\mathfrak{o}_F \oplus \mathfrak{d})\backslash H \times H} (Y^{(2)}(\varphi_1, \varphi_2)|_2 J(\iota((z_1, z_2)))f((z_1, z_2))dz_1 dz_2 \tag{3.7}$$

is equal to

$$c_2\langle f, f\rangle(\sum_i \varphi_1(y_i)\psi(y_i))(\sum_i \varphi_2(y_i)\psi(y_i)), \tag{3.8}$$

where we identify y_i *with* $y_i \otimes 1 \in \tilde{D}$ *and where* c_2 *is some constant depending only on* N.

In order to interpret the value obtained in (3.8) in the same way as in Section 2 as the central critical value of an L-function, we review briefly the integral representation of the L-function that one obtains when one replaces in a triple (h, f_1, f_2) of elliptic cusp forms the pair (f_1, f_2) by one Hilbert cusp form f for a real quadratic field.

For the moment, both the Hilbert cusp form f and the elliptic cusp form h can be of arbitrary even weight k. Now we consider the Siegel type Eisenstein series of weight k, defined on $\mathbf{H}_3$ by

$$E_3^k(W,s) = \sum_{\gamma=\begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma_0^3(N)_\infty \backslash \Gamma_0^3(N)} det(CW+D)^{-k} det(\Im(\gamma < W >)^s$$

Here and in the sequel we denote by G_∞ the subgroup of G defined by "$C = 0$", where G is any group of symplectic matrices.
We restrict this Eisenstein series to $W = \begin{pmatrix} \tau & 0 \\ 0 & Z \end{pmatrix}$ with $\tau \in \mathbf{H}$, $Z \in \mathbf{H_2}$ and furthermore we consider then the modular embedding with respect to Z.
In this way we get a function $E(\tau, z_1, z_2, s)$, which behaves like a modular form for τ and like a Hilbert modular form for (z_1, z_2) of weight k. We want to compute the twofold integral

$$I(f,h,s) := \int_{SL_2(\mathfrak{o}_F \oplus \mathfrak{d}) \backslash \mathbf{H}^2} \int_{\Gamma_0(N) \backslash \mathbf{H}} \overline{h(\tau) f(z_1, z_2)} E(\tau, z_1, z_2, s) d\tau^* dz_1^* dz_2^*$$

where $dz^* = y^{k-2} dxdy$ for $z = x + iy \in \mathbf{H}$.
This can be done in several ways: One can relate this integral to similar ones in [26] or in [14] (both these works are in an adelic setting) or one can try to do it along classical lines as in [13, 28, 9]. We sketch the latter approach here (for class number one, h being a normalized newform of level N.)
The inner integration over τ (which can be done with $Z \in \mathbf{H}_2$ instead of the embedded (z_1, z_2)) is the same as in the papers mentioned above, producing an L-factor $L_2(h, 2s + 2k - 2)$ (with $L_2(,)$ denoting the symmetric square L-function) times a Klingen type Eisenstein series $E_{2,1}(h, s)$, which is defined as follows: We denote by $C_{2,1}$ the maximal parabolic subgroup of $Sp(2)$ for which the last line is of the form $(0, 0, 0, *)$ and we put $C_{2,1}(N) = C_{2,1}(\mathbf{Q}) \cap \Gamma_0^2(N)$. Furthermore we define a function $h_s(Z)$ on $\mathbf{H}_2$ by

$$h_s(Z) = h(z_1) \left(\frac{det(Y)}{y_1} \right)^s,$$

where $z_1 = x_1 + iy_1$ denotes the entry in the upper left corner of $Z = X + iY \in \mathbf{H}_2$. Then we put

$$E_{2,1}(h,s)(Z) = \sum_{\gamma \in C_{2,1} \backslash \Gamma_0^2(N)} h_s(Z) \mid_k \gamma$$

To do the second integration, one needs information on certain cosets: This is the only new ingredient entering the picture:

Lemma 3.3. *A complete set of representatives for* $C_{2,1}(N)\backslash\Gamma_0^2(N)$ *is given by*

$$\{d(M)J^{-1}\tilde{\iota}_0(\gamma)\}$$

with γ *running over* $SL_2(\mathfrak{o}_F\oplus\mathfrak{d})_\infty\backslash SL_2(\mathfrak{o}_F\oplus\mathfrak{d})$*, and* $M = \begin{pmatrix} * & * \\ u & v \end{pmatrix}$ *running over those elements of* $SL(2,\mathbf{Z})_\infty\backslash SL(2,\mathbf{Z})$ *with* $v\equiv 0(N)$*, where* d *denotes the standard embedding of* $GL(2)$ *in* $Sp(2)$ *given by* $d(M) = \begin{pmatrix} (M^{-1})^t & 0 \\ 0 & M \end{pmatrix}$.

This lemma is related to the double coset decomposition

$$C_{2,1}(N)\backslash\Gamma_0^2(N)/\tilde{\iota}_0(SL_2(\mathfrak{o}_F\oplus\mathfrak{d}))$$

and somewhat analogous to the coset decomposition in [28, p.692]; we omit the proof.

We may now do the usual unfolding to get

$$\begin{aligned}(3.9)\qquad &\int_{SL_2(\mathfrak{o}_F\oplus\mathfrak{d})\backslash\mathbf{H}^2} \overline{h(z_1,z_2)}\widetilde{E_{2,1}(h,*,s)}(z_1,z_2)dz_1^*dz_2^* \\ &= \int_{SL_2(\mathfrak{o}_F\oplus\mathfrak{d})_\infty\backslash\mathbf{H}^2} \sum_{M=\begin{pmatrix} * & * \\ v & u\end{pmatrix}} h((v,u)C\begin{pmatrix} z_1 & 0 \\ 0 & z_2\end{pmatrix}C^t\begin{pmatrix} v \\ u\end{pmatrix}) \\ &\qquad\times\overline{f(z_1,z_2)}\left(\frac{Dy_1y_4}{(v+u\omega)^2y_1+(v+u\bar{\omega}^2y_2}\right)^s dz_1^*dz_2^*\end{aligned}$$

Using the Fourier expansions of f and F,

$$h(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}, \qquad f(z_1,z_2) = \sum_{\nu\in\mathbf{0}_F,\nu\gg 0} A(\nu)e^{2\pi itr(\nu\cdot z)}$$

one can (after some standard calculations) write the integral above as

$$\gamma(s)\sum_n\sum_{\sim\backslash(u,v)} a(n)\overline{A(n(v+u\omega)^2)}n^{-s-2k+2}N(v+u\omega)^{-2s-2k+2}$$

where we use the following equivalence relation: two pairs (u,v) and (u',v') are called equivalent iff $v+u\omega$ and $v'+u'\omega$ are equal up to a unit from $\mathfrak{o}_F$ as a factor.

Assume now in addition that h is a normalized eigenfunction of all

Hecke operators; then we define the L-function $L(h\otimes f,s)$ as an Euler product over all primes p with Euler factors (at least for p coprime to N)

$$L_p(h\otimes f,s):=L_p^{Asai}(f,\alpha_p p^{-s})L_p^{Asai}(f,\beta_p p^{-s})$$

where we use the Euler factors $L_p^{Asai}(f,s)$ of the Asai-L-function attached to f (see [1]) and α_p and β_p are the Satake-p-parameters attached to the eigenform h (normalized to have absolute values $p^{\frac{k-1}{2}}$). We will write later $L(h,f;s)$ to denote the shift of this L-function that is normalized to have functional equation under $s\mapsto 1-s$.
By standard calculation, we see that the integral above is, after multiplication by $L_2(h,2s+2k-2)$, equal to the L-function $L(h\otimes f,s+2k-2)$ (up to elementary factors; the condition $v\equiv 0(N)$ also creates some extra contribution for p-Euler factors with $p\mid N$). This calculation of course requires some formal calculations similar to those given e.g. in [13].

Remark 3.4. *If the class number H of F is different from one, then the orbit structure is more complicated. One gets H different sets of representatives of the type described in the lemma above (each one twisted by a matrix in $SL_2(F)$ mapping a cusp into ∞). After unfolding, one gets then a Dirichlet series also involving Fourier coefficients of f at all the H different cusps. If we assume that f is the first component (i.e. the one corresponding to the principal ideal class) in a tuple of H Hilbert modular forms such that the corresponding adelic modular form is an eigenform of all Hecke operators, then it is possible (but quite unpleasant) to transfer that Dirichlet series into the Euler product in question.*

Now we return to the case of weight 2. We can compute the integral $I(f,h,s)$ at $s=0$ not only by unfolding as above but also by using the Siegel-Weil formula for the Eisenstein series in the integrand. Then one gets in the same way as in [7] that the square of the right hand side of (3.8) is (up to an explicit constant) the product of the central critical values of the L-functions attached to the pairs h_1,f and h_2,f as above:

Theorem 3.5. *Let $\varphi_1,\varphi_2,h_1,h_2,f,\psi$ be as in Lemma 3.2. Then the square of the period integral*

$$\int_{SL_2(\mathfrak{o}_F\oplus\mathfrak{d})\backslash\mathbf{H}\times\mathbf{H}}(Y^{(2)}(\varphi_1,\varphi_2)|_2J(\iota((z_1,z_2)))f((z_1,z_2))dz_1dz_2 \tag{3.10}$$

is equal to

$$\frac{c_3}{\langle h_1, h_1\rangle\langle h_2, h_2\rangle} L(h_1, f; 1/2) L(h_2, f; 1/2), \tag{3.11}$$

where c_3 *is an explicitly computable nonzero number depending only on* N *and the product* L*-function* $L(h, f; s)$ *is normalized to have its functional equation under* $s \mapsto 1 - s$.
In particular the period integral is nonzero if and only if the central critical value of $L(h_1, f; s) L(h_2, f, s)$ *is nonzero.*

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