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Theta Series of Modular, Extremal, and Hermitian Lattices

Rudolf Scharlau, Alexander Schiemann, and Rainer Schulze-Pillot

Abstract. We continue here our investigation of extremal lattices (in the sense of [**4**] and [**6**]) begun in [**8**]. In particular we are concerned with hermitian lattices and their theta series. We present some experimental results concerning modular and hermitian lattices and the spaces spanned by their theta series of degrees 1 and 2.

Introduction

The construction of lattices with integral positive definite quadratic form that have a high minimum is of interest as well from a purely number theoretic or geometric point of view as for the application to dense lattice sphere packings. In dimensions exceeding 8 it is extremely difficult (if not impossible) to find lattices attaining or approximating the known upper bounds for the minimum or even Minkowski's bound that bounds from below the minimum of an average lattice. One has therefore looked at special classes of lattices that allow to give a sharper bound for the minimum. In particular, for (even) unimodular lattices the notion of analytic extremality has been defined and studied in [**4**]. It gives an upper bound for the minimum of such a lattice by considering the space of modular forms in which the theta series of the lattices are and discusses in which dimensions lattices attaining this bound (called analytically extremal lattices) exist. Recently Quebbemann noticed that some well-known dense lattices like the Coxeter-Todd lattice in dimension 12 and the Barnes-Wall lattice in dimension 16 are analytically extremal in a more general sense. In [**6, 7**] he introduced the concepts of modular and of strongly modular lattices of some level $\ell > 1$ and extended the notion of analytic extremality to this setting for certain $\ell > 1$. A survey of the current state of the classification of these analytically extremal lattices has been given in [**8**].

The main purpose of the present article is to supplement [**8**] by considering more closely the role that is played by the lattices with an additional hermitian structure in genera of modular lattices. The underlying unimodular hermitian $\mathcal{O}_F$-lattices (with $F = \mathbb{Q}(\sqrt{-\ell})$) have theta series of degree n that are modular forms on the hermitian half space $\mathbf{H}_n$. The restriction of such a hermitian theta series to the embedded Siegel half space $\mathfrak{H}_n \subseteq \mathbf{H}_n$ is just the ordinary theta series of

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the transfer of the hermitian $\mathcal{O}_F$-lattice to a $\mathbb{Z}$-lattice. We study the relations of the various spaces of theta series that occur in this way and report on our experimental results. These concern the relative proportions of modular lattices, strongly modular lattices and lattices with hermitian structure in some modular genera as well as the dimensions of spaces of modular forms generated by the respective sets of lattices. The experimental results lead to speculations about the basis problem for hermitian modular forms and about spaces of Siegel modular forms that are generated by restrictions of hermitian modular forms.

1. Modular and hermitian lattices

Let L be an integral positive definite lattice over $\mathbb{Z}$, i.e., a lattice L of full rank in a rational vector space V of dimension $m = 2k$ over $\mathbb{Q}$ with a positive definite bilinear form $b : V \times V \longrightarrow \mathbb{Q}$ such that $b(L, L) \subseteq \mathbb{Z}$. As usual L is called even if $b(x, x) \subseteq 2\mathbb{Z}$ for all $x \in L$, unimodular if $L = L^{\#} := \{y \in V \mid b(y, L) \subseteq \mathbb{Z}\}$, and modular of level ℓ if there is an orthogonal similitude of V of norm ℓ such that $\sigma(L^{\#}) = L$. An interesting subset of the set of modular lattices is formed by those having a hermitian or quaternion-hermitian structure: Let M be a unimodular lattice over the ring $\mathcal{O}_F$ of integers in $\mathbb{Q}(\sqrt{-\ell}) =: F$ (ℓ square free) with respect to the positive definite hermitian form $h : M \times M \longrightarrow \mathcal{O}_F$ and assume M to be even (i.e., $h(x, x) \in 2\mathbb{Z}$ for all $x \in M$) if $\ell \not\equiv 3 \mod 4$. Let L be M considered as a $\mathbb{Z}$-lattice with bilinear form $b(x, y) = \mathrm{tr}\ h(x, y)$ if $\ell \equiv 3 \mod 4$ and $b(x, y) = \frac{1}{2}\ \mathrm{tr}\ h(x, y)$ if $\ell \not\equiv 3 \mod 4$. Then (L, b) is even and modular of level ℓ; it is called the transfer of (M, h) to $\mathbb{Z}$. Similarly, if B is a definite quaternion algebra over $\mathbb{Q}$ ramified at ∞ and the primes dividing ℓ (again ℓ square free), R a maximal order in B, M an R-lattice with positive definite hermitian form $h : M \times M \longrightarrow R$ (hermitian with respect to the standard involution of B), then the $\mathbb{Z}$-lattice L obtained by regarding M as $\mathbb{Z}$-lattice with bilinear form $b(x, y) = \mathrm{tr}\ h(x, y)$ is even and modular of level ℓ. For genera of class numbers up to a few thousand (e.g. for field discriminant $|d_F| \leq 19$ and $\mathrm{rk}_{\mathcal{O}_F} M \leq 7$), unimodular hermitian lattices can be generated and classified by computer quite rapidly using the generalized neighbouring lattice method of Kneser [**9**]; tables can be found at `http://www.math.uni-sb.de/~schulzep/Hermitian-lattices`. These constructions and variations of them (see [**8**]) are the only known ways of systematically constructing modular lattices; they do, however, not at all exhaust the set of modular lattices.

A modular lattice is called strongly modular if for all exact divisors m of ℓ (i.e., $\gcd(m, \ell/m) = 1$) the partial dual $D_m L = {}^m(\frac{1}{m} L \cap L^{\#})$ is isometric to L (here the superscript m on the left indicates scaling of the bilinear form by the factor m). The hermitian constructions mentioned above do in general not lead to strongly modular lattices.

The class numbers collected in Table 1 and 2 give a rough idea of the proportions of modular, strongly modular, and hermitian lattices in some genera containing modular lattices. Notice that non-isometric hermitian unimodular lattices may lead to isometric $\mathbb{Z}$-lattices under transfer.

TABLE 1. Class numbers

l	level of a (strongly) modular even genus $\mathcal{G}$, l is squarefree
n	rank of the $\mathbb{Z}$-lattices in $\mathcal{G}$. If $l = p_1 \cdots p_k$ (p_i prime, $p_i < p_{i+1}$) we add a string of $k-1$ signs $+, -$. The i-th sign is $+$ if the unimodular p_i-Jordan component has square discriminant, $-$ otherwise.
$h_{\mathbb{Z}}$	number of classes in $\mathcal{G}$.
$h_{\mathcal{O}}$	number of classes in the corresponding hermitian $\mathcal{O}$-genus where $\mathcal{O}$ is the ring of integers in $\mathbb{Q}(\sqrt{-l})$.
$\mathcal{G}_h$	collection of lattices in $\mathcal{G}$ with a hermitian structure over $\mathbb{Q}(\sqrt{-l})$ (i.e. isometric to a transferred hermitian lattice).
$h_{\mathbb{Z},h}$	number of classes in $\mathcal{G}_h$.
h_m	number of modular classes in $\mathcal{G}$.
h_M	number of strongly modular classes in $\mathcal{G}$ (if l is not a prime).
$h_{M,h}$	number of strongly modular classes in $\mathcal{G}_h$ (if l is not a prime).

l	n	$h_{\mathbb{Z}}$	$h_{\mathcal{O}}$	$h_{\mathbb{Z},h}$	h_m
1	8	1	1	1	1
	16	2	3	2	2
	24	24			24
2	4	1	1	1	1
	8	1	2	1	1
	12	3	5	3	3
	16		26	16	
3	4	1	1	1	1
	6	1	1	1	1
	8	2	1	1	2
	10	3	1	1	3
	12	10	2	2	10
	14	29	2	2	19
	16	163	3	3	85
	18		4	4	
	20		6	6	
	22		8	8	
	24		20	19	
5	4	1	2	1	1
	8	5	13	5	5
	12	48	158	38	38
7	4	1	1	1	1
	6	3	2	2	3
	8	8	3	3	8
	10	30	5	4	26
	12	395	11	9	185
	14		26	18	
	16		71	48	
	18		291	173	
	20		2225	1246	
11	4	3	2	2	3
	6	5	2	2	5
	8	31	6	5	31
	10	297	10	8	141
11	12		39	28	
	14		112	72	
	16		1027	601	
13	4	1	2	1	1
	8	38	53	22	22
17	4	3	8	3	3
	8	118	354	82	82
19	4	3	2	2	3
	6	12	3	3	8
	8	182	12	10	116
	10		32	23	
	12		290	186	
	14		5225	2751	
23	4	6	9	5	6
	6	29	30	14	29
	8	512	126	57	354
	10		768	335	
	12		8895	4132	
29	4	6	18	6	6
	8	1714	2958	730	732
31	4	6	9	5	6
	6	51	39	19	45
	8	2554	216	102	1016
	10		2064	949	
43	4	7	3	3	5
	6	76	8	7	28
	8		69	52	
	10		1212	656	
67	4	13	4	4	7
	6	328	17	13	68
	8		301	198	
	10		19580	9956	
163	4	57	8	8	15
	6	11869	111	66	523
	8		11438	6196	

TABLE 2. Class numbers: Composite l

l	n	$h_{\mathbb{Z}}$	$h_{\mathcal{O}}$	$h_{\mathbb{Z},h}$	h_m	h_M	$h_{M,h}$
6	4+	1	1	1	1	1	1
	4−	1	1	1	1	1	1
	8+	8	9	6	6	6	6
	8−	5	7	5	5	1	1
	12+		161	78			
	12−		154	76			
10	4+	1	1	1	1	1	1
	4−	1	1	1	1	1	1
	8+	24	19	12	12	8	8
	8−	19	14	11	11	3	3
	12−		2531	1125			
14	4+	3	4	3	3	3	3
	4−	2	4	2	2	0	0
	8+	80	134	44	44	26	26
	8−	50	104	34	34	0	0
15	4+	2	2	2	2	0	0
	4−	3	2	2	3	3	3
	6+	9	5	4	9	0	0
	6−	9	5	4	9	0	0
	8+	91	14	11	55	29	7
	8−	68	14	10	48	2	0
	10+	3221	48	31	779	0	0
	10−	3221	48	31	779	0	0
	12+		238	146			
	12−		240	142			
	14+		2120	1142			
	14−		2120	1142			
21	4+	3	4	3	3	3	3
	4−	2	4	2	2	0	0
	8+	305	323	113	113	45	45
	8−	245	268	95	95	1	1
22	4+	4	2	2	2	2	2
	4−	1	1	1	1	1	1
	8+	504	127	78	78	56	56
	8−	255	83	49	49	5	5
30	4++	5	3	3	3	1	1
	4+−	5	3	3	3	1	1
	4−+	2	2	2	2	0	0
	4−−	2	2	2	2	0	0
	8++		559	286			
	8+−		427	223			
	8−+		358	186			
	8−−		330	176			

2. Theta series and extremal lattices

Let L be an even positive definite integral lattice with Gram matrix $A = (b(e_i, e_j))$ with respect to some basis $\{e_i\}$. By $d(L) = \det\ A$ we denote the discriminant of L, by $N = N(L)$ the level of L (i.e., N is the gcd of all N' such that

$N'A^{-1}$ is integral with even diagonal). If L is modular of level ℓ we have $N(L) = \ell$, $d(L) = \ell^{m/2}$.

For a positive integer n the theta series $\vartheta^{(n)}(L, Z)$ with Z in the Siegel upper half space $\mathfrak{H}_n = \{Z = X + iY \in M_n^{\mathrm{sym}}(\mathbb{C}) \mid Y \text{ positive definite}\}$ is given as

$$\vartheta^{(n)}(L, Z) = \sum_{\mathbf{x}=(x_1,\dots,x_n)\in L^n} \exp(2\pi i\ \mathrm{tr}(b(\mathbf{x})Z))$$

with $b(\mathbf{x}) := (\frac{1}{2}b(x_i, x_j))$.

It is known that $\vartheta^{(n)}(L, Z)$ is in the space $M_n^k(N, \chi)$ of Siegel modular forms of character $\chi\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left(\frac{(-1)^{m/2} d(L)}{\det(D)}\right)$ and weight $k = m/2$ with respect to the group

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{Z}) \mid C \equiv 0 \mod N \right\}$$

[**1**].

The Fricke element

$$w_N = \begin{pmatrix} 0 & -1_n \\ N1_n & 0 \end{pmatrix} \in G\mathrm{Sp}_n(\mathbb{Q})$$

of the group of symplectic similitudes normalizes $\Gamma_0^{(n)}(N)$ and induces via

$$F(Z) \longmapsto (F|_k w_N)(Z) = (N)^{-nk/2} \det(-Z)^{-k} F(-(NZ)^{-1})$$

an automorphism of $M_n^k(N, \chi)$ whose square is $(-\mathrm{Id})^{kn}$; $M_n^k(N, \chi)$ is hence decomposed in the eigenspaces $M_n^k(N, \chi, +)$ and $M_n^k(N, \chi, -)$ with eigenvalues $+1,\ -1$ for even nk and with eigenvalues $+i$ and $-i$ for odd nk.

Similarly, we can choose for each exact divisor m of N (i.e, $\gcd(m, \frac{N}{m}) = 1$) elements ${w'_m}^{(n)} \in \mathrm{Sp}_n(\mathbb{Z})$ satisfying

$${w'_m}^{(n)} \equiv \begin{cases} \mathrm{Id}_n & \mod N/m \\ \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} & \mod m \end{cases}$$

and obtain Atkin-Lehner elements

$$w_m^{(n)} = {w'_m}^{(n)} \cdot \begin{pmatrix} m1_n & 0 \\ 0 & 1_n \end{pmatrix}$$

in the normalizer of $\Gamma_0^{(n)}(N)$ with $(w_m^{(n)})^2 \cdot m^{-1} \in \Gamma_0^{(n)}(N)$ and with $w_{m_1}^{(n)} w_{m_2}^{(n)} \in w_{m_1 m_2}^{(n)} \Gamma_0^{(n)}(N)$ for exact divisors $m_1,\ m_2$ of N.

If the character χ is trivial the $w_m^{(n)}$ form therefore a family of pairwise commuting involutions, and the space $M_n^k(N, \chi)$ splits into the direct sum of the eigenspaces $M_n^k(N, \chi)^\epsilon$ for each character ϵ of the subgroup of the normalizer of $\Gamma_0^{(n)}$ in $G\mathrm{Sp}_n(\mathbb{Q})$ modulo $(\Gamma_0^{(n)}(N) \cdot \mathbb{Q}^\times 1_{2n})$ that is generated by the $w_m^{(n)}$. If χ is arbitrary it is usual to view the eigenforms of w_N as modular forms on the Fricke group

$$\Gamma_*^{(n)}(N) = \Gamma_0^{(n)}(N) \cup \Gamma_0^{(n)}(N) \begin{pmatrix} 0 & -\sqrt{N}^{-1} 1_n \\ \sqrt{N} 1_n & 0 \end{pmatrix}$$

with respect to the character $\chi\epsilon$ that is equal to χ on $\Gamma_0^{(n)}(N)$ and sends $\left(\begin{smallmatrix} 0 & -\sqrt{N}^{-1}1_n \\ \sqrt{N}1_n & 0 \end{smallmatrix}\right)$ to $\epsilon(w_N)$.

It is known [**7, 2**] that $\vartheta^{(n)}(L)|_{w_m}$ is a multiple of the theta series $\vartheta^{(n)}(D_mL)$ of the partial dual D_mL of L. The factor of proportionality has been explicitly calculated for $n=1$ by Quebbemann [**7**] and less explicitly for general n in [**2**]. In particular, even modular lattices of level ℓ have theta series in a space of modular forms (with character) for the Fricke group $\Gamma_*^{(n)}(\ell)$, and even strongly modular lattices have theta series in one of the eigenspaces $M_n^k(\ell,\chi)^\epsilon$. If $\mathcal{M} = \mathcal{M}(k,\ell,\chi,\epsilon)$ is one of these spaces for $n=1$, Quebbemann [**7**] has shown that extremality is definable for $\mathcal{M}$ for certain values of $k,\ \ell,\ \chi,\ \epsilon$ in the sense that with $d = \dim_{\mathbb{C}} \mathcal{M}$ the map associating to

$$f(z) = \sum_{\nu=0}^{\infty} a_f(\nu)\exp(2\pi i \nu z)$$

the vector $(a_f(0),\dots,a_f(d-1)) \in \mathbb{C}^d$ is bijective. In particular, for a lattice L with $\vartheta^{(1)}(L) \in \mathcal{M}$ the minimum $\mu(L) = \min_{x \in L\setminus\{0\}} b(x,x)$ is at most $2(d-1)$, any lattice attaining this minimum is called analytically extremal. For details and examples see [**6, 7, 8**].

Theta series of degree n for hermitian forms are defined as functions on the space

$$\mathbf{H}_n = \{Z \in M_n(\mathbb{C}) \mid i({}^t\overline{Z} - Z) > 0\},$$

on which the group

$$U_1^{(n)} = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_{2n}(\mathbb{C}) \mid {}^t\overline{g} \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \right\}$$

operates by fractional linear transformation

$$Z \longmapsto (AZ+B)(CZ+D)^{-1}$$

[**3, 11**]: Let $F = \mathbb{Q}(\sqrt{-\delta})$ with square free $\delta > 0$. For an even positive definite hermitian $\mathcal{O}_F$-lattice M as above we have

$$\vartheta^{(n)}(M,h,Z) := \vartheta^{(n)}(M,Z) := \sum_{\mathbf{x} \in M^n} \exp(\pi i\ \mathrm{tr}(h(\mathbf{x})Z))$$

with $h(\mathbf{x}) := (h(x_i,x_j))$ for $\mathbf{x} = (x_1,\dots,x_n) \in M^n$. It is known that $\vartheta^{(n)}(M,Z)$ is a holomorphic modular form of weight $k = \frac{\mathrm{rank}(M)}{2} = \frac{m}{2}$ for a congruence subgroup of $U_1(F)$ [**11, 10**]. More explicitly we have

Lemma 2.1. *Let (M,h) be a positive definite hermitian free $\mathcal{O}_F$-lattice with $h(x,x) \in \mathcal{O}_F$ for all $x \in M$ and assume $h(x,x) \in 2\mathcal{O}_F$ for all $x \in M$ if $\delta \not\equiv 3 \mod 4$. Let the fractional $\mathcal{O}_F$-ideal $\mathcal{N}$ be given by*

$$\mathcal{N}^{-1} = \begin{cases} \{h(x,x) \mid x \in L^\#\}\mathcal{O}_F & \text{if } \delta \equiv 3 \mod 4 \\ 2\{h(x,x) \mid x \in L^\#\}\mathcal{O}_F & \text{if } \delta \not\equiv 3 \mod 4. \end{cases}$$

Then $\vartheta^{(n)}(M,Z)$ is a holomorphic modular form on $\mathbf{H}_n$ of weight $\frac{m}{2}$ with respect to the group of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in U_1(F)$ such that $A,B,C,D \in M_n(\mathcal{O}_F)$, $C \in M_n(\delta\mathcal{N})$.

If $h(L,L) \subseteq \mathcal{O}_F$, B may be in $M_n(2(\mathfrak{d}_F)^{-1})$ for $\delta \not\equiv 3 \mod 4$, in $M_n((\mathfrak{d}_F)^{-1})$ for $\delta \equiv 3 \mod 4$.
If $\mathcal{N}h(L^\#, L^\#) \subseteq \mathcal{O}_F$, C may be in $M_n(2(\mathfrak{d}_F)^{-1}\delta\mathcal{N})$ for $\delta \not\equiv 3 \mod 4$, in $M_n(\delta(\mathfrak{d}_F)^{-1}\mathcal{N})$ for $\delta \equiv 3 \mod 4$.
The restriction of $\vartheta^{(n)}(M,Z)$ to $Z \in \mathfrak{H}_n \subseteq \mathbf{H}_n$ (as the subset of Z with ${}^t\overline{Z} + Z$ and $i({}^t\overline{Z} - Z)$ real) is $\vartheta^{(n)}(L,b,Z)$, where L, b is the transfer of M to $\mathbb{Z}$ as described in Section 1.

PROOF. The proof is essentially contained in [**11, 10**].
We give some details explicitly here: Using Eichler's trick (or the theta correspondence in the dual pair (U_0, U_1) inside $\mathrm{Sp}_{2mn}(\mathbb{Q})$, where U_0 is the unitary group of the space supporting M with hermitian form h) one obtains embeddings of $U_1(F)$ into $\mathrm{Sp}_{2mn}(\mathbb{Q})$ and of $\mathbf{H}_n$ into $\mathfrak{H}_{2mn}$ that are compatible with the actions of the groups on the respective upper half spaces.
We write $x = x^{(r)} + x^{(i)}\sqrt{-\delta}$ with rational $x^{(r)}$, $x^{(i)}$ for elements x of F or matrices x with entries in F.
For $Z = X + iY \in \mathbf{H}_n$ with hermitian X, Y we decompose $X = X^{(r)} + X^{(i)}\sqrt{-\delta}$, $Y = Y^{(r)} + Y^{(i)}\sqrt{-\delta}$, $Z^{(r)} = X^{(r)} + iY^{(r)}$, $Z^{(i)} = X^{(i)} + iY^{(i)}$ with real $X^{(r)}$, $X^{(i)}$, $Y^{(r)}$, $Y^{(i)}$.
We let H be the Gram matrix of (M,h) with respect to some basis of the lattice and $g = H^{-1}$. Then the embedding for $\delta \not\equiv 3 \mod 4$ is given by

$$M = \begin{pmatrix} A\, B \\ C\, D \end{pmatrix} \longmapsto \begin{pmatrix} A'\, B' \\ C'\, D' \end{pmatrix} =: M_H \in \mathrm{Sp}_{2mn}(\mathbb{Q})$$

with

$$\begin{aligned}
A' &= \begin{pmatrix} A^{(r)} \otimes 1_m & -A^{(i)} \otimes 1_m \\ \delta A^{(i)} \otimes 1_m & A^{(r)} \otimes 1_m \end{pmatrix}, \\
B' &= \begin{pmatrix} B^{(r)} \otimes H^{(r)} + \delta B^{(i)} \otimes H^{(i)} & -\delta B^{(i)} \otimes H^{(r)} + \delta B^{(r)} \otimes H^{(i)} \\ -\delta B^{(r)} \otimes H^{(i)} + \delta B^{(i)} \otimes H^{(r)} & \delta(B^{(r)} \otimes H^{(r)} + \delta B^{(i)} \otimes H^{(i)}) \end{pmatrix}, \\
C' &= \begin{pmatrix} C^{(r)} \otimes G^{(r)} + \delta C^{(i)} \otimes G^{(i)} & C^{(r)} \otimes G^{(i)} - C^{(i)} \otimes G^{(r)} \\ C^{(i)} \otimes G^{(r)} - C^{(r)} \otimes G^{(i)} & \delta^{-1}(C^{(r)} \otimes G^{(r)} + \delta C^{(i)} \otimes G^{(i)}) \end{pmatrix}, \\
D' &= \begin{pmatrix} D^{(r)} \otimes 1_m & -\delta D^{(i)} \otimes 1_m \\ D^{(i)} \otimes 1_m & D^{(r)} \otimes 1_m \end{pmatrix}, \\
Z &\longmapsto \begin{pmatrix} Z^{(r)} \otimes H^{(r)} + \delta Z^{(i)} \otimes H^{(i)} & \delta(Z^{(r)} \otimes H^{(i)} - Z^{(i)} \otimes H^{(r)}) \\ \delta(Z^{(i)} \otimes H^{(r)} - Z^{(r)} \otimes H^{(i)}) & \delta(Z^{(r)} \otimes H^{(r)} + \delta Z^{(i)} \otimes H^{(i)}) \end{pmatrix} \\
&=: \tilde{Z} \otimes \tilde{H}.
\end{aligned}$$

For $\delta \equiv 3 \mod 4$ we obtain

$$\begin{aligned}
A' &= \begin{pmatrix} (A^{(r)} - A^{(i)}) \otimes 1_m & -2A^{(i)} \otimes 1_m \\ \frac{\delta+1}{2} A^{(i)} \otimes 1_m & (A^{(r)} + A^{(i)}) \otimes 1_m \end{pmatrix} \\
B' &= \begin{pmatrix} 2(B \otimes \overline{H})^{(r)} & -\delta(B \otimes \overline{H})^{(i)} + (B \otimes \overline{H})^{(r)} \\ \delta(B \otimes \overline{H}^{(i)} + (B \otimes \overline{H})^{(r)} & \frac{\delta+1}{2}(B \otimes \overline{H})^{(r)} \end{pmatrix}, \\
C' &= \begin{pmatrix} \frac{\delta+1}{2\delta}(C \otimes \overline{G})^{(r)} & -(C \otimes \overline{G})^{(i)} - \frac{1}{\delta}(C \otimes \overline{G})^{(r)} \\ (C \otimes \overline{G})^{(i)} - \frac{1}{\delta}(C \otimes \overline{G})^{(r)} & \frac{2}{\delta}(C \otimes \overline{G})^{(r)} \end{pmatrix}, \\
D' &= \begin{pmatrix} (D^{(r)} - D^{(i)} \otimes 1_m & -\frac{\delta+1}{2} D^{(i)} \otimes 1_m \\ 2D^{(i)} \otimes 1_m & (D^{(r)} + D^{(i)}) \otimes 1_m \end{pmatrix}, \\
Z &\longmapsto \begin{pmatrix} 2(Z \otimes \overline{H})^{(r)} & -\delta(Z \otimes \overline{G})^{(i)} + (Z \otimes \overline{H})^{(r)} \\ \delta(Z \otimes \overline{G})^{(i)} + (Z \otimes \overline{H})^{(r)} & \frac{\delta+1}{2}(Z \otimes \overline{H})^{(r)} \end{pmatrix}.
\end{aligned}$$

The assertions can be read off from these formulae. □

COROLLARY 2.2. a) *If (M, h) is integral unimodular and for $\delta \not\equiv 3 \mod 4$ even, the hermitian theta series of degree one of (M, h) and the ordinary theta series of its transfer (L, b) to $\mathbb{Z}$ coincide.*

b) *If the $\mathbb{Z}$-lattice (L, b) is the transfer to $\mathbb{Z}$ of the hermitian integral unimodular (even) lattice (M, h) as above, its theta series of degree n is a modular form for the Fricke group.*

PROOF. a) is clear since $\mathbf{H}_1 = \mathfrak{H}_1$.
b) follows from the lemma since $\Gamma_0^{(n)}(\delta)$ as well as $\begin{pmatrix} 0 & -\sqrt{-\delta^{-1}}1_n \\ \sqrt{-\delta}1_n & 0 \end{pmatrix}$ are in the subgroup of $U_1^{(n)}(F)$ with respect to which the hermitian theta series is a modular form (assertion b) follows of course also from the fact that (L, b) is modular). □

We are interested in finding and classifying modular extremal lattices or in proving the nonexistence for certain levels and dimensions and in the same questions concerning lattices with hermitian structure. This leads to some questions concerning spaces of modular forms generated by theta series of $\mathbb{Z}$-lattices and by theta series of lattices with hermitian structure:

a) Which part of a space of modular forms for the Fricke group containing theta series of a modular genus of $\mathbb{Z}$-lattices is generated by the theta series of modular lattices in that space?
b) Which part of a space of modular forms as in a) is generated by theta series of lattices with hermitian structure?
c) Which part of some space of hermitian modular forms is generated by the hermitian theta series of hermitian lattices that are in that space? (Basis problem for hermitian modular forms)

If in a) or b) the space generated by the relevant theta series in degree 1 were a strict subspace this would imply that the notion of analytic extremality should be refined by restricting attention to the space of modular forms generated by the theta series in question. The same result in degree $n \geq 2$ could lead to proofs for the nonexistence of extremal modular lattices along the lines of [**5**]. We notice that for $n \geq 2$ the space of modular forms on the Siegel space $\mathfrak{H}_n$ that is generated by theta series of $\mathbb{Z}$-lattices with hermitian structure is the space of restrictions to $\mathfrak{H}_n$ of a space of hermitian modular forms on $\mathbf{H}_n$. In particular, this space will in general not be invariant under the action of Hecke operators even if the underlying space of hermitian theta series is Hecke invariant (since the Hecke action and the restriction process do not commute). In these cases it is therefore also of interest to study the Hecke-closure of the subspace of Siegel modular forms under investigation.

3. Experimental results and speculations

We conducted quite extensive computer calculations concerning the questions raised in Section 2. We give some of the results in the following tables.

In these tables, the dimension of the vector space spanned by a fixed set of modular forms is computed as follows: For each of the modular forms in the spanning set we compute the vector of all Fourier coefficients with index below a certain bound (in the case of degree one modular forms) or with the trace of the indexing reduced symmetric 2×2-matrix below a certain bound (in the case of Siegel modular forms of degree 2). In the case of degree one the bound is (using a well known result of Hecke) chosen such that a modular form is zero if this vector is zero. In the case of degree 2 such a bound is not known and we fix it as high as the running time of our program permits and such that the computed value of the dimension has apparently stabilized upon increasing the bound up to this value. The dimension given is then the dimension of the space spanned by these vectors of Fourier coefficients and is therefore only a lower bound for the dimension of the space in the case of degree 2. However, in cases in which our table indicates that for spaces $W_1 \subseteq W_2$ of modular forms we have $\dim W_1 < \dim W_2$, this means that the spaces of beginning parts of Fourier expansions of the modular forms in the respective spaces are not equal and hence implies that indeed $W_1 \neq W_2$ is true (even though either one or both of the dimensions given might be smaller than the respective correct value).

The action of the Hecke operator $T(p)$ on the modular forms given is calculated from this vector of Fourier coefficients. For this we use in the case of degree 2 the explicit formulas given in [**12**], with the obvious modifications for the case of modular forms with nontrivial character (nebentype) and odd weight.

Calculating the action of the Hecke operator $T(p)$ in this way on a modular form replaces the bound c on the trace of the indexing matrix by $[c/p]$. While the dimensions of the spaces spanned by binary Siegel theta series are likely to be the exact values, the given bounds for dimensions of Hecke closures come in some cases from such short vectors that they are quite likely to miss the correct value.

TABLE 3. Dimensions of spans of theta series

Notation (l, n, $\mathcal{G}$, $\mathcal{G}_h$ as above (Tables 1,2)).:

t — $\dim \langle \vartheta^{(1)}(L) \mid L \in \mathcal{G} \rangle$.

t_F — $\dim \langle \vartheta^{(1)}(L) + \vartheta^{(1)}(D_l L) \mid L \in \mathcal{G} \rangle$.

t_h — $\dim \langle \vartheta^{(1)}(L) \mid L \in \mathcal{G}_h \rangle$.

b — lower bound for $\dim \langle \vartheta^{(2)}(L) \mid L \in \mathcal{G} \rangle$.

$b_{T(p)}$ — lower bound for the dimension of the closure of $\langle \vartheta^{(2)}(L) \mid L \in \mathcal{G} \rangle$ under Hecke operators $T(p)$.

b_F — lower bound for $\dim \langle \vartheta^{(2)}(L) + \vartheta^{(2)}(D_l L) \mid L \in \mathcal{G} \rangle$.

b_{FT+} — lower bound for the dimension of the closure of $\langle \vartheta^{(2)}(L) + \vartheta^{(2)}(D_l L) \mid L \in \mathcal{G} \rangle$ under operations $\prod T_{p_i}$ with $\prod \chi(p_i) = 1$.

b_{F-T+} — lower bound for the dimension of the closure of $\langle \vartheta^{(2)}(L) - \vartheta^{(2)}(D_l L) \mid L \in \mathcal{G} \rangle$ under operations $\prod T_{p_i}$ with $\prod \chi(p_i) = 1$.

b_h — lower bound for $\dim \langle \vartheta^{(2)}(L) \mid L \in \mathcal{G}_h \rangle$ where $\mathcal{G}_h$ as above.

b_{hT+} — lower bound for the dimension of the closure of $\langle \vartheta^{(2)}(L) \mid L \in \mathcal{G}_h \rangle$ under operations $\prod T_{p_i}$ with $\prod \chi(p_i) = 1$.

b_{hT-} — lower bound for the dimension of the closure of $\langle \vartheta^{(2)}(L) \mid L \in \mathcal{G}_h \rangle$ under operations $\prod T_{p_i}$ with $\prod \chi(p_i) = -1$.

l	n	t	t_F	t_h	b	$b_{T(p)}$	b_F	b_{FT+}	b_{F-T+}	b_h	b_{hT+}	b_{hT-}
1	8	1	1	1	1	1	1	1	0	1	1	0
	16	1	1	1	1	1	1	1	0	1	1	0
2	4	1	1	1	1	1	1	1	0	1	1	0
	8	1	1	1	1	1	1	1	0	1	1	0
	12	1	1	1	2	2	2	2	0	2	2	0
3	4	1	1	1	1	1	1	1	0	1	1	0
	6	1	1	1	1	2	1	1	0	1	1	1
	8	1	1	1	2	2	2	2	0	1	2	0
	10	1	1	1	2	3	2	2	0	1	2	1
	12	2	2	2	4	4	4	4	0	2	4	0
	14	2	2	2	6	7	5	5	1	2	2	2
	16	2	2	2	7	7	6	6	1	3	5	0
5	4	1	1	1	1	1	1	1	0	1	1	0
	8	2	2	2	3	3	3	3	0	3	3	0
	12	2	2	2	7	7	6	6	1	6	6	0
6	4+	1	1	1	1	1	1	1	0	1	1	0
	4−	1	1	1	1	1	1	1	0	1	1	0
	8+	2	2	2	6	6	5	5	1	5	5	0
	8−	2	2	2	4	4	4	4	0	4	4	0
7	4	1	1	1	1	1	1	1	0	1	1	0
	6	2	2	2	3	4	3	3	0	2	3	1
	8	2	2	2	5	5	5	5	0	3	5	0
	10	2	2	2	7	8	6	6	1	3	6	1
	12	4	3	3	15	15	11	11	4	5	11	0
10	4+	1	1	1	1	1	1	1	0	1	1	0
	4−	1	1	1	1	1	1	1	0	1	1	0
	8+	4	3	3	14	14	10	10	4	9	10	0
	8−	4	3	3	12	12	9	9	3	8	9	0

TABLE 3 (continued):

l	n	t	t_F	t_h	b	$b_{T(p)}$	b_F	b_{FT+}	b_{F-T+}	b_h	b_{hT+}	b_{hT-}
11	4	2	2	2	3	3	3	3	0	2	3	0
	6	2	2	2	4	5	4	4	0	2	3	2
	8	3	3	3	10	10	10	10	0	5	10	0
	10	4	3	3	21	24	15	15	6	7	9	7
13	4	1	1	1	1	1	1	1	0	1	1	0
	8	4	3	3	15	15	11	11	4	10	11	0
14	4+	2	2	2	3	3	3	3	0	3	3	0
	4−	2	2	2	2	2	2	2	0	2	2	0
	8+	5	4	4	28	28	20	20	8	19	20	0
	8−	5	4	4	22	22	17	17	5	16	17	0
15	4+	2	2	2	2	2	2	2	0	2	2	0
	4−	2	2	2	3	3	3	3	0	2	3	0
	6+	3	3	3	8	14	8	12	0	4	12	2
	6−	3	3	3	8	14	8	12	0	4	12	2
	8+	5	4	4	32	32	23	23	9	10	21	0
	8−	5	4	4	28	28	21	21	7	8	18	0
	10+	7	5	5	71	72	44	45	37	18	36	0
	10−	7	5	5	71	72	44	45	32	18	36	0
17	4	2	2	2	3	3	3	3	0	3	3	0
	8	5	4	4	25	25	19	19	6	18	19	0
19	4	2	2	2	3	3	3	3	0	2	3	0
	6	4	3	3	10	13	8	8	2	3	4	6
	8	5	4	4	32	32	24	24	8	10	24	0
21	4+	2	2	2	3	3	3	3	0	3	3	0
	4−	2	2	2	2	2	2	2	0	2	2	0
	8+	7	5	5	67	67	42	42	25	39	42	0
	8−	7	5	5	59	59	38	38	21	35	38	0
22	4+	3	2	2	4	4	3	3	1	2	3	0
	4−	1	1	1	1	1	1	1	0	1	1	0
	8+	8	5	5	79	79	45	45	47	39	45	0
	8−	8	5	5	65	65	38	38	27	32	38	0
23	4	3	3	3	6	6	6	6	0	5	6	0
	6	4	4	4	16	17	16	16	0	11	16	1
	8	6	5	5	47	47	36	36	11	23	36	0
29	4	3	3	3	6	6	6	6	0	6	6	0
	8	8	6	6	83	83	56	56	27	53	56	0
30	4++	4	3	3	5	5	4	4	1	3	4	0
	4+−	4	3	3	5	5	4	4	1	3	4	0
	4−+	2	2	2	2	2	2	2	0	2	2	0
	4−−	2	2	2	2	2	2	2	0	2	2	0
31	4	3	3	3	6	6	6	6	0	5	6	0
	6	6	5	5	27	30	24	24	3	14	23	3
	8	8	6	6	98	98	66	66	32	37	66	0
43	4	4	3	3	7	7	6	6	1	3	6	0
	6	8	5	5	54	61	35	35	26	7	18	21
67	4	6	4	4	13	13	10	10	3	4	10	0
	6	12	7	7	164	175	95	95	70	13	26	39
163	4	14	8	8	57	57	36	36	34	8	36	0

Our computations seem to suggest the following conclusions:

a) In all examples, in degree 1 the space of modular forms for the Fricke group containing theta series of $\mathbb{Z}$-lattices with hermitian structure is spanned by these theta series. This suggests that the basis problem for theta series of degree 1 of (even) unimodular hermitian $\mathcal{O}_F$-lattices has a positive answer, i.e., the space of modular forms with the correct multiplier for the congruence subgroup of $U_1(F)$ described in Lemma 2.1 appears to be generated by theta series of (even) unimodular hermitian $\mathcal{O}_F$-lattices. One of the present authors (Schulze-Pillot) is working on a proof of this jointly with S. Böcherer.
In particular, it seems not to be possible to improve the notion of analytic extremality for $\mathbb{Z}$-lattices with hermitian structure.

b) Concerning the question whether in a genus of $\mathbb{Z}$-lattices containing modular extremal lattices some or all of the extremal lattices have a hermitian structure, no pattern appears to be recognizable, any possible behavior seems to occur.

c) In degree 2, the space generated by the theta series of $\mathbb{Z}$-lattices with hermitian structure (= restrictions of hermitian theta series) is in general strictly smaller than the subspace of modular forms for the Fricke group that is generated by all theta series of $\mathbb{Z}$-lattices in that genus. This might give the possibility to prove non-existence of extremal modular $\mathbb{Z}$-lattices with hermitian structures analytically in special cases with similar methods as in [**5**]. It indicates also that the space $\mathcal{R}_h$ of restrictions to $\mathfrak{H}_n$ of hermitian modular forms coming from theta series of hermitian lattices on $\mathfrak{H}_n$ is in general a proper subspace of the space $\mathcal{M}_*$ of Siegel modular forms for the relevant Fricke group that is generated by $\mathbb{Z}$-lattices in the genus of the transfer to $\mathbb{Z}$ of the hermitian lattices considered (and the corresponding weight and multiplier).

As indicated at the end of the previous section it makes sense to modify the problem and ask the following question:

Given a congruence subgroup Δ of $U_1(F)$ of type described in Lemma 2.1. Is the Hecke closure of the space of Siegel modular forms that arise as restrictions to $\mathfrak{H}_n$ of the hermitian modular forms for Δ of given weight and multiplier equal to the space of all Siegel modular forms for $\Delta \cap \mathrm{Sp}_n(\mathbb{R})$ of corresponding weight and multiplier?

Our tables provide some partial answers:

First of all, the closure $\mathcal{T}$ under the action of the Hecke operators $T(p)$ (for some small primes p) of the space of Siegel modular forms generated by the theta series of degree 2 of all lattices in a genus of prime level is in some cases of rank $\equiv 2 \bmod 4$ (and hence of modular forms of odd weight and nebentype $\chi \neq 1$) strictly larger than this space. In other words, in these cases (which of course occur only for $\chi(p) = -1$)) the space generated by degree 2 theta series is not Hecke invariant (for $\chi(p) = 1$ it is well known [**1**] how the action of $T(p)$ on a theta series is expressed by theta series). These seem to be the first examples of this type (notice that our genera are modular and hence the well known process of adding the theta series from the genera of rescaled (partial) duals does not change the set of theta series considered). In all these cases we computed the action of the Hecke operator on Siegel's weighted average of the theta series of the lattices in the genus (the genus theta series) and found that its image is not contained in the space of theta series at hand. Lynne Walling informs us that she is close to proving that the genus theta series is always an eigenfunction of $(T(p))^2$, this is confirmed by our computations in all cases checked. In the case of level 15 and dimensions 6 and 10 the action of $T(2)$ (with $\chi(2) = +1$) sends the theta series of one of the two genera into a multiple of the theta series of the other genus. It should be possible to prove this fact more generally using Andrianov's formulae. In fact it is tempting to speculate whether for arbitrary p the Fourier coefficients of the effect of $T(p)$ on the genus theta series can be calculated in terms of local densities of lattices over $\mathbb{Z}_\ell$ for all primes ℓ where in the case of $\chi(p) = -1$ these local lattices don't fit together as completions of a global lattice. We hope to clarify this point in future work.

The closure $\mathcal{T}_h$ under the action of the Hecke operators of the space $\mathcal{R}_h$ of restrictions to $\mathfrak{H}_n$ of hermitian modular forms coming from theta series of hermitian lattices on $\mathfrak{H}_n$ is again usually strictly larger than $\mathcal{R}_h$. In most cases of trivial character (or rank $\equiv 0 \bmod 4$) it coincides with $\mathcal{T}$. However, in some cases (level 3, rank 16, and level 15, rank 8) there seem to be small differences in dimension; these may still be due to an insufficient range of computation as indicated in the beginning of this section (notice that here the Hecke operator at $p = 3$ is not available and that the computational problem becomes worse for larger p).

In the case of nebentype (or of rank $\equiv 2 \bmod 4$) the situation seems to be more difficult and needs further investigation.

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Fachbereich Mathematik,Universität Dortmund,Vogelpothsweg 87,D-44227 Dortmund,

E-mail address: `Rudolf.Scharlau@math.uni-dortmund.de`

Fachbereich 9 Mathematik, Universität des Saarlandes, Postfach 15 11 50, D-66041 Saarbrücken,

E-mail address: `aschiem@math.uni-sb.de`

Fachbereich 9 Mathematik, Universität des Saarlandes, Postfach 15 11 50, D-66041 Saarbrücken

E-mail address: `schulzep@math.uni-sb.de`