Extremal lattices

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0 Introduction

This paper deals with discrete subgroups of euclidean vector spaces, equivalently finitely generated free abelian groups (isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$) together with a positive definite quadratic form. Such a structure will be called a *lattice* for short, typically denoted by $L, M, \dots$, with values $(v, w) \in \mathbb{R}$ of the bilinear form (where $v, w \in V \supset L$, the enveloping $\mathbb{R}$-vector space). The general background of this report is provided by the sphere packing problem (construction of lattices with large minimum), by the theory of modular forms, and by the theory of finite matrix groups. Almost all lattices of interest for one of the mentioned areas are "*algebraic*" or even "*rational*", by which we mean that the form takes rational values on them: $(v, w) \in \mathbb{Q}$ for $v, w \in L$. After rescaling, that is, multiplying the form with some positive integral constant α, a rational lattice becomes *integral*: $(v, w) \in \mathbb{Z}$ for all $v, w \in L$. The rescaled lattice will be denoted by ${}^{\alpha}L$. By definition, a lattice is integral if and only if it is contained in its *dual lattice*

$$L^{\#} := \{y \in V \mid (x, y) \in \mathbb{Z} \text{ for all } x \in L\}\,.$$

H.-G. Quebbemann has observed that quite a few individual lattices which are well known, or even famous, in one or several of the above mentioned areas (like the Leech lattice Λ_{24}, the Barnes-Wall lattice BW_{16}, or the Coxeter-Todd lattice K_{12}, the Quebbemann lattice(s) Q_{32}), share a common structure: after integral normalization, they are similar to their respective dual lattice. That is, there exists a bijective linear map $\sigma : V \to V$, a *similarity*, and a positive integer ℓ, the *similarity factor*, such that

$$\sigma(L^{\#}) = L \text{ and } (\sigma x, \sigma y) = \ell(x, y) \text{ for all } x, y \in V\,.$$

Furthermore, these lattices are *even*, that is $(x, x) \in 2\mathbb{Z}$ for all $x \in L$. An even lattice similar to its dual, with similarity factor ℓ, is called a *modular lattice* of level ℓ. Under the above circumstances, the integer ℓ is indeed equal to what is usually called the *level* of a lattice L: it is the smallest natural number ℓ such that ${}^{\ell}L^{\#}$, the rescaled dual lattice, is again even. (It is readily checked that the level of an even integral lattice ℓ is equal to the exponent, or twice the exponent of the discriminant group $T(L) := L^{\#}/L$.)

In his basic papers [Que95] and [Que97], Quebbemann investigates the relationship between his notion of modularity for lattices, and the theory of modular forms. This leads him, for certain levels ℓ, to the notion of an *extremal* (modular) lattice. Roughly speaking, a lattice is extremal if its *minimum* or *minimal norm*

$$\min L := \min\{(x, x) \mid x \in L \setminus \{0\}\}$$

is as large as it is possible from the point of view of modular forms. (For self-dual lattices, $\ell = 1$, this notion is classical [CoSl93].) Since the determinant of an n-dimensional ℓ-modular lattice necessarily equals $\ell^{n/2}$, independently of the particular lattice, the assumption of extremality also maximizes the (center) *density* $\delta(L) := (\min L)^{n/2}/\sqrt{\det L}$ of the associated sphere packing. It is therefore of direct geometrical significance. To avoid an ambiguous notion, the property of being extremal is more precisely called *analytic extremality*, whereas the property of being *extreme* in the classical sense of geometry of numbers is called *geometric extremality*. Recall that L is extreme if the density function attains a local maximum at the "point" L in an appropriate space of matrices. Quebbemann's definition of analytic extremality is restricted to special values of the level ℓ, namely the numbers ℓ whose sum $\sigma_1(\ell)$ of the positive divisors divides 24. We shall comment on this in more detail in Section 1 below, but observe that these numbers include almost all levels which have so far been of interest for applications. As was mentioned above, the starting point of Quebbemann's work were common properties of some important known lattices. Once the definition of modularity and analytic extremality was given, it immediately stimulated further investigations on these and related lattices, and also led to the discovery of some "new" extremal lattices. What makes the subject particularly intriguing is the fact that the concept of extremal lattices gives rise to a finite classification problem. For each of the finitely many levels ℓ, there is an upper bound on the dimension n up to which extremal lattices of level ℓ could possibly exist. This comes from the fact that for large values of n, the (hypothetical) theta series of an extremal lattice of dimension $n = 2k$ and level ℓ, which is a uniquely determined modular form $F_{k,\ell}$, has a negative coefficient. For the remaining pairs (n, ℓ), one is faced with the questions of existence, uniqueness, and possibly full classification of extremal lattices.

It is the purpose of this paper to report on what is presently known about these problems. The results are to a large extent due to Plesken & Nebe and Nebe who found many extremal (and also many other interesting) lattices in the course of investigating finite rational matrix groups, to Quebbemann, to Bachoc and Bachoc & Nebe, and to the present authors, A. Schiemann and B. Hemkemeier. The original results of this paper are following. We extend the classical finiteness result for extremal modular forms (see [MOS75]) to other levels; we extend the results of [SchHem94] on the complete classification of "modular genera" of lattices in "small" dimensions (up to between 8 and 16, depending on ℓ), to the composite levels $\ell = 6, 14, 15$; we discuss rather exhaustively the question of a unimodular structure over (real or imaginary) quadratic fields for those "small" dimensions; we discuss examples of extremality for some levels other than those considered by Quebbemann. Our classification results, and also parts of the results on the existence of certain lattices, are essentially based on computer programs developed in a joint DFG-project and written by B. Hemkemeier, A. Schiemann, M. Stausberg and F. Wichelhaus. These programs generate lattices with Kneser's method of neighboring lattices [Kne57] and can be viewed as extensions of the program used in the work [SchHem94]. For further developments in the hermitian case, see [Schi98].

Our treatment will be complete and self-contained by giving a construction (or at least a precise reference) for each occurring extremal lattice, including the well known ones. We shall use without further explanation a few basic notions and facts about lattices (some of them were already mentioned). The reader may consult [Que95] or [SchVen94] for these, and the books [O'Me71], [Kit93], [Kne73] [MiHu73] or [Ser70] for general background about integral quadratic forms. In [Ser70], the reader also finds an exposition of the basic theory of modular forms, including its application to the simplest case of lattices, namely those of level one. As a condensed introduction to modular forms including a basic stock of widely used explicit formulas we recommend [Sko92].

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1 Basic definitions and some constructions

1.1 Strongly modular lattices

We briefly recall a few standard definitions. A *lattice* (in the sense of number theory) is a pair (L, b), where L is a free $\mathbb{Z}$-module of finite rank, say $\operatorname{rank} L = n$, and $b : L \times L \to \mathbb{Q}$ is a positive definite symmetric bilinear form. An *isometry* between two lattices (L_1, b_1) and (L_2, b_2) is a group isomorphism $\varphi : L_1 \to L_2$ which is compatible with the forms: $b_2(\varphi x, \varphi y) = b_1(x, y)$ for all $x, y \in L_1$. Sometimes L is considered to be embedded into a rational vector space, and by defining $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$ we obtain an equivalent category if we regard lattices as triples $(V, b; L)$, where (V, b) is a "rational quadratic form", that is, a vector space with a positive definite symmetric bilinear form, and L a "lattice on V", that is a finitely generated subgroup which spans the vector space V. In this form, the definition immediately extends to the case of a totally real number field F instead of $\mathbb{Q}$, replacing $\mathbb{Z}$ by the ring of integers $\mathfrak{o}_F$, and requiring that the form is *totally positive definite*: $b(x,x)^\sigma > 0$ for all embeddings $\sigma : F \to \mathbb{R}$ and all $x \in V$, $x \neq 0$. At some places, we also deal with the case that F is a totally complex number field endowed with an involution $\alpha \mapsto \overline{\alpha}$ whose fixed field F_0 is totally real, and with hermitian forms $h : V \times V \to F$ with respect to the specified involution: $h(\alpha x, \beta y) = \alpha\overline{\beta}h(x,y)$. If necessary, we will speak of *quadratic lattices*, respectively *hermitian lattices*. Notice that in the number field case, lattices need not be free as modules; the relevant structure theory of such modules (finitely generated and torsion free) can be found for instance in [O'Me71]. Lattices over $\mathfrak{o}_F$ are always free if the class number of F is one. In the rational case, the form b will usually not occur in the notation, we just write $(x,y) := b(x,y)$. Observe that often the letter L refers to the whole structure $(V, b; L)$ or at least (L, b), and not just to the module L.

Isometry of lattices will be denoted by $L \cong M$.

If (L, b) is a lattice, and $\alpha \in F$ is totally positive, then ${}^{\alpha}L$ denotes the "scaled" lattice $(L, \alpha b)$; similarly in the hermitian case with $\alpha \in F_0$. From the introduction, we recall the definition of the dual lattice $L^{\#}$. A lattice is called *integral* if $L \subseteq L^{\#}$ and *even* if $b(x,x) \in 2\mathfrak{o}_F$ for all $x \in L$.

Two lattices are in the same *genus* if they become isometric over all completions $\mathfrak{o}_\mathfrak{p}$ of $\mathfrak{o}_F$

$$\operatorname{gen} L = \operatorname{gen} M \iff L \otimes \mathfrak{o}_\mathfrak{p} \cong M \otimes \mathfrak{o}_\mathfrak{p} \text{ for all places } \mathfrak{p} \text{ (including } \infty) \text{ of } F\,.$$

The *class number* of a lattice, or rather of its genus $\mathcal{G} = \operatorname{gen} L$ is the number of isometry classes contained in $\mathcal{G}$, often denoted by $h(\mathcal{G})$.

The local theory of lattices, that is, the theory of lattices over $\mathfrak{o}_\mathfrak{p}$, and the theory of genera are well understood. Details will not play a role in this paper. We only mention the fact that every lattice over $\mathfrak{o}_\mathfrak{p}$ posseses a *Jordan-decomposition*

$${}^{p^{-r}}L_{-r} \perp {}^{p^{-r+1}}L_{-r+1} \perp \ldots \perp L_0 \perp {}^{p}L_1 \perp \ldots \perp {}^{p^s}L_s$$

where $p \in \mathfrak{o}_\mathfrak{p}$ is a (local) prime element and all L_i are self-dual $\mathfrak{o}_\mathfrak{p}$-lattices: $L_i = L_i^{\#}$. If $\mathfrak{p} \not| 2$, a Jordan decomposition is unique up to isometry. If $\mathfrak{p}|2$, this only holds if all L_i are even. The lattice is called *totally even* in this case.

The *Gram matrix* of a free lattice with respect to an $\mathfrak{o}_F$-basis $(v_1, \ldots, v_n)$ is the matrix $(b(v_i, v_j))_{i,j} \in F^{n\times n}$. The *determinant* of a free lattice is the determinant of any of its Gram matrices. It is well defined for $F = \mathbb{Q}$, and for arbitrary fields well defined modulo squares of units in $\mathfrak{o}_F$. In the non-free case, the determinant may be defined as an adele modulo squares of local units; often the determinant ideal, generated by the determinants of all Gram matrices of n linearly independent vectors in L, is a sufficiently fine invariant.

For a self-dual lattice L over a non-dyadic discrete valuation ring $\mathfrak{o}_\mathfrak{p}$, $\mathfrak{p} \not| 2$, the only invariant in addition to the dimension is the (square class of the) determinant $\det L \in \mathfrak{o}_\mathfrak{p}^*/\mathfrak{o}_\mathfrak{p}^{*2} \cong \{\pm 1\}$. In view of the above mentioned essentially unique Jordan decomposition this gives a full classification of non-dyadic local lattices. In the dyadic case, the situation is in several respects essentially more complicated.

In this paper, we shall have to describe genera of lattices only in the rational case $F = \mathbb{Q}$. We use the *genus symbol* as introduced in [CoSl93], Chapter 15. This symbol is a string of local symbols, one for each prime $p = 2, 3, 5, \ldots$ dividing $2 \cdot \det L$. The local symbol at the prime $p \neq 2$ of a lattice L with Jordan decomposition (1.1) as above is the formal product

$$\prod_{j=-s}^{t} (p^j)^{\varepsilon_{j,p} n_{j,p}} \text{ with } \varepsilon_{j,p} = \left(\frac{\det L_j}{p}\right) \text{ and } n_{j,p} = \dim L_j \,.$$

We do not describe a dyadic symbol in full generality here. Among other things, the parity "even/odd" of the Jordan component belonging to $q = 2^t$ is recorded by a subscribed $q_{I\!I}$ respectively q_I.

As an example, consider the binary lattice B given by the Gram matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 8 \end{smallmatrix}\right)$, of determinant 15. Over all completions $\mathbb{Z}_p$, $p \neq 2$, it can be diagonalized as $B \otimes \mathbb{Z}_p \cong \langle 2, 30 \rangle \otimes \mathbb{Z}_p = \langle 2, 2 \cdot 3 \cdot 5 \rangle \otimes \mathbb{Z}_p$. So the 3-adic symbol is $1^{-1}3^{+1}$, and the 5-adic symbol is $1^{-1}5^{+1}$. The 2-adic symbol is $1^{+2}_{I\!I}$, where the $+$-sign expresses the fact that $\det L = \det L_{0,2} \equiv \pm 1 \bmod 8$. We shall usually suppress the unimodular parts of the local symbols, since the dimensions $n_{0,p}$ and the signs $\varepsilon_{0,p}$ are determined by the total dimension and determinant and the other $n_{j,p}$, $\varepsilon_{j,p}$, $j \geq 1$. Furthermore, we shall indicate the parity of the unimodular component at $p = 2$, which equals the parity of the total lattice, and the total dimension n by writing the symbol as $I_n(\ldots)$ respectively $I\!I_n(\ldots)$ and omitting also the component $2^{\varepsilon_{0,2} n_{0,2}}_{I/I\!I}$. So our final symbol for the above example B will be $I\!I_2(3^{+1}5^{+1})$. Similarly, the lattice $\left(\begin{smallmatrix} 4 & 1 \\ 1 & 4 \end{smallmatrix}\right)$ can be diagonalized as $\langle 1, 3 \cdot 5 \rangle \otimes \mathbb{Z}_p$ for $p \neq 2$, and thus has the symbol $I\!I_2(3^{-1}5^{-1})$.

In the following, we use the notation $m \| \ell$ if m and ℓ are integers, m divides ℓ, and m and ℓ/m relatively prime. In this case we say that m is an *exact divisor* of ℓ.

Definition 1.1 Let L be an integral lattice, ℓ the exponent of its discriminant group $L^{\#}/L$, and q an exact divisor of ℓ. The *partial dual* $D_q L$ of L is defined as

$$D_q L := {}^{q}\!\left(\tfrac{1}{q} L \cap L^{\#}\right).$$

The lattice $D_q L$ is integral again, and $D_q(D_q L) \cong L$ (canonically). If q and r are as above and coprime, the operators D_q and D_r (on isometry classes of lattices of fixed exponent ℓ) commute: $D_q D_r L = D_r D_q L$. Clearly, D_q is also defined for lattices over $\mathbb{Z}_p$ (of appropriate exponent), and the operator D_q commutes with localization and thus maps genera onto genera. For a $\mathbb{Z}_p$-lattice L with Jordan decomposition as above, and $q = p^s$, we have

$$D_q L \cong L_s \perp {}^{p}L_{s-1} \perp {}^{p^2}L_{s-2} \perp \ldots \perp {}^{p^s}L_0 \,,$$

so D_q acts by reversing the sequence of p-Jordan-components. The operator D_q does not affect the other localizations $L \otimes \mathbb{Z}_{p'}$, $p' \neq p$ as modules, but because of rescaling by q, it in general does change the isometry class of the quadratic form.

In this paper, only genera with square free exponent $\ell = \ell(L)$ will play a role. Furthermore, $n = 2k$ is even, and $\det L = \ell^k$, so that for each p, the unimodular and the p-modular Jordan component have the same dimension $n_{0,p} = n_{1,p} = k$. Even in this special situation, D_p need not preserve every genus. Obviously, D_p, $p \neq 2$ does so if (and only if) k is even, or p is a square mod q for all odd primes $q \neq p$

dividing ℓ. As an example $D_3(\mathit{I\!I}_2(3^{+1}5^{+1})) = D_5(\mathit{I\!I}_2(3^{+1}5^{+1})) = \mathit{I\!I}_2(3^{-1}5^{-1})$, and indeed $D_3 B \cong D_5 B \cong B'$, where B, B' are the above binary lattices of determinant 15.

Notice that a lattice of square-free exponent ℓ is totally even if and only if both L and $D_2 L$ are even. In this case, ℓ equals what is usually called the *level* of L: the smallest natural number m such that the rescaled dual lattice ${}^{m}L^{\#}$ is again even.

We now come to the most important definition of this section, the significance of which has first been pointed out by H.-G. Quebbemann.

Definition 1.2 An integral lattice L is called *strongly modular* if $D_m L \cong L$ for all exact divisors m of the exponent ℓ of $L^{\#}/L$. It is called *modular* if $D_\ell L \cong L$.

In the definition of strong modularity, one could restrict m to prime powers since $D_m L \cong \prod_{q|m} D_q L$, where q runs over the prime powers exactly dividing m. Except for the self-dual case, $\ell = 1$, a modular lattice must have even dimension, $n = 2k$, say. If L has even determinant and is totally even and modular, then its dimension is divisible by 4.

We refer to [Que97] for examples.

If F is a totally real number field and (L, b) a quadratic $\mathfrak{o}_F$-lattice of rank n, we can consider the $\mathbb{Q}$-valued scalar product $(x, y) = \operatorname{tr} b(x, y)$ on FL, where $\operatorname{tr} : F \to \mathbb{Q}$ denotes the trace. The $\mathbb{Z}$-lattice of rank $n \cdot [F : \mathbb{Q}]$ thus obtained is denoted by $L_{\mathbb{Z}}$. It is said to be obtained by *transfer* form L. The $\mathfrak{o}_F$-dual

$$L^d := \{y \in V \mid b(x, y) \in \mathfrak{o}_F \text{ for all } x \in L\}$$

and the $\mathbb{Z}$-dual are related by the formula

$$(L_{\mathbb{Z}})^{\#} = \mathcal{D}_{F/\mathbb{Q}} L^d\,,$$

where $\mathcal{D}_{F/\mathbb{Q}}$ denotes the different of F over $\mathbb{Q}$. (This follows immediately from the definition of the different, or rather its inverse, which is a fractional ideal in F.) An immediate consequence is the formula

$$\det L_{\mathbb{Z}} = d_F^n \cdot \mathrm{N}(\det L)$$

where d_F denotes the field discriminant of F and N denotes the norm. We leave it to the reader to formulate a more precise statement for relation between the discriminant groups L^d/L and $L^{\#}/L$ (in the integral case). When L is self-dual, $L^d = L$, it amounts to $d_F \cdot L^{\#} \subseteq L$, and indeed $L^{\#}/L \cong (\mathfrak{o}_F/\mathcal{D}_{F/\mathbb{Q}})^n$. Everything remains true for hermitian lattices subject to the above conditions.

The most important special case for particular constructions is that of a quadratic field $F = \mathbb{Q}(\sqrt{D})$, for a square free integer D. One has $d_F = D$ if $D \equiv 1 \bmod 4$, and $d_F = 4D$ if $D \equiv 2, 3 \bmod 4$. We redefine $(x, y) = \frac{1}{2} \operatorname{tr} b(x, y)$ if $D \equiv 2, 3 \bmod 4$; then the above formula holds in both cases, with $(\sqrt{D})$ instead of $\mathcal{D}_{\mathcal{F}/\mathbb{Q}}$, and D instead of d_F. The lattices $L_{\mathbb{Z}}$ obtained from self-dual lattices L over $\mathbb{Q}(\sqrt{D})$ are not only D-elementary, $L_{\mathbb{Z}}^{\#}/L_{\mathbb{Z}} \cong (\mathbb{Z}/D\mathbb{Z})^n$, but even modular. The desired similarity from $L_{\mathbb{Z}}^{\#}$ to $L_{\mathbb{Z}}$ is simply given by multiplication with $\sqrt{D}$.

In the real quadratic case, we obtain a non-trivial extension of the transfer construction, which we call *twisted transfer*, as follows. The scalar product on the $\mathbb{Z}$-module L is defined as $(x, y) = \mathrm{tr}(\alpha \cdot b(x, y))$ for some fixed totally positive element $\alpha \in F$. It is convenient to write $\alpha = \lambda/\delta$, where δ is a generator of the different of F over $\mathbb{Q}$ (which may be totally positive or not!). Then the above equalities read $(L_{\mathbb{Z}})^{\#} = \lambda L^d$, $\det L_{\mathbb{Z}} = (\mathrm{N}\lambda)^n \cdot \mathrm{N}(\det L)$, and $L_{\mathbb{Z}}$ is $\mathrm{N}\lambda$-elementary for self-dual L. Modularity of $L_{\mathbb{Z}}$ need not hold, but we have the following easy lemma.

Lemma 1.3 *Consider a $\mathbb{Z}$-lattice $L_{\mathbb{Z}}$ of level ℓ obtained by twisted transfer $(x, y) := \mathrm{tr}\left(\frac{\lambda}{\delta} b(x, y)\right)$ from a self-dual lattice (L, b) over a real quadratic field F, where $\mathrm{N}\lambda = \ell$ and $(\delta) = \mathcal{D}_{F/\mathbb{Q}}$. If $\mathrm{N}\lambda > 0$ and (L, b) is isometric to its conjugate lattice, then $L_{\mathbb{Z}}$ is ℓ-modular.*

Of course, by "conjugate" lattice we understand the lattice L with the form $b(x, y)'$, where $\beta \mapsto \beta'$ is the non-trivial field automorphism of F, and $\beta.v = \beta' v$ for $\beta \in F$, $v \in FL$ is the twisted module-structure.

1.2 Theta series

We recall a few well known facts from the analytic theory of quadratic forms, or lattices. If L is an even lattice of even dimension $n = 2k$ and level ℓ, we denote by

$$\Theta_L(q) = \sum_{m \geq 0} r_L(m) q^m \,, \quad r_L(m) := |\{x \in L \mid (x, x) = 2m\}|$$

its theta series, where as usual $q = e^{2\pi i z}$ and z is a variable in the upper half plane. This is a modular form of weight k for the group $\Gamma_0(\ell)$ and a certain quadratic character $\varepsilon : \Gamma_0(\ell) \to \{\pm 1\}$. Using standard notation for the action of $PSL_2(\mathbb{R})$ on modular forms of weight k, this means that

$$\Theta_L|_k \gamma = \varepsilon(\gamma) \Theta_L \,.$$

The character ε only depends on the determinant of L, and is trivial if this is a square. So in that case we have modular forms of weight k and level ℓ in the strict sense, i.e. invariant under $\Gamma_0(\ell)$. We denote by $\mathcal{M}_k(\ell, \varepsilon)$ the finite-dimensional complex vector space of these modular forms, and by $\mathcal{S}_k(\ell, \varepsilon)$ the subspace of cusp forms. If L and M are lattices in the same genus, then the difference $\Theta_L - \Theta_M$ is a cusp form.

In the following we shall assume that ℓ is square free. This includes the assumption that L is totally even, and ℓ is equal to the exponent of $L^{\#}/L$. Denote by $\Gamma_0^*(\ell)$ the normalizer of $\Gamma_0(\ell)$ in $PSL_2(\mathbb{R})$. The factor group $\Gamma_0^*(\ell)/\Gamma_0(\ell)$ is 2-elementary abelian, generated by certain cosets $W_m \Gamma_0(\ell)$, $m|\ell$, which are independent $\bmod\, \Gamma_0(\ell)$; for $\ell = m$ one obtains the Fricke involution W_ℓ represented by $\left(\begin{smallmatrix} 0 & -1 \\ \ell & 0 \end{smallmatrix}\right)$. Since ε is quadratic, W_ℓ acts on $\mathcal{M}_k(\ell, \varepsilon)$, and if ε is trivial, all the W_m act as commuting involutions on $\mathcal{M}_k(\ell, \varepsilon)$ (the *Atkin-Lehner involutions*). Hence, if ℓ is prime or ε is trivial, $\mathcal{M}_k(\ell, \varepsilon)$ splits into eigenspaces $\mathcal{M}_k(\ell, \chi)$ with respect to the characters $\chi : \Gamma_0^*(\ell) \to \mu_4 = \{\pm 1, \pm i\}$ extending ε.

A basic result now is the so-called Atkin-Lehner-identity which says that for any lattice L of level ℓ, and any divisor m of ℓ, the theta series of the partial dual, $\Theta_{D_m L}$ is proportional to the Atkin-Lehner-transform $\Theta_L|_k W_m$ of the original theta series, with a certain numerical factor depending only on the genus. See [Que97] for a precise statement and formula. In particular, if L is strongly modular, then Θ_L is an Atkin-Lehner-eigenform:

$$\Theta_L \in \mathcal{M}_k(\ell, \chi)\,,\ \chi = \chi_{\mathrm{gen}\, L}$$

for a certain character χ depending only on the genus of L.

The following dimension formula for prime levels is taken from [Que95].

Proposition 1.4 *For a prime number $\ell \leq 23$, even weight k and $\chi(W_\ell) = (-1)^{k/2}$, χ trivial on $\Gamma_0(\ell)$, the dimension of $\mathcal{M}_k(\ell, \chi)$ is equal to*

$$\dim \mathcal{M}_k(\ell, \chi) = \frac{k}{2} g_0(\ell) + 1 + \frac{1}{2}\lfloor \frac{k}{4} \rfloor \left(1 + \left(\frac{-1}{\ell}\right)\right) + \frac{1}{2}\lfloor \frac{k}{3} \rfloor \left(1 + \left(\frac{-3}{\ell}\right)\right)$$

with

$$g_0(\ell) = \begin{cases} 0 & \text{for} \quad \ell = 1, 2, 3, 5, 7, 13 \\ 1 & \text{for} \quad \ell = 11, 17, 19 \\ 2 & \text{for} \quad \ell = 23 \end{cases}$$

The following dimension formulae were derived from [SkoZa88]; see [Que97] for the levels $6, 14, 15$.

Proposition 1.5 *For even weight k and χ trivial on $\Gamma_0(\ell)$ one has:*

$$\dim \mathcal{S}_k(10, \chi) = \frac{1}{4} \cdot \left[\frac{3(k-1) + (-1)^{k/2}}{2} - 2 - \eta\chi(W_2) + (-1)^{k/2} \left(\chi(W_5) + \chi(W_{10})\right) \right]$$

with

$$\eta = \begin{cases} +1 & \text{if } k \equiv 2, 4 \bmod 8 \\ -1 & \text{if } k \equiv 0, 6 \bmod 8. \end{cases}$$

$$\dim \mathcal{S}_k(21, \chi) = \frac{1}{4}(\mathrm{tr}(W_1) + \chi(W_3)\, \mathrm{tr}(W_3) + \chi(W_7)\, \mathrm{tr}(W_7) + \chi(W_{21})\, \mathrm{tr}(W_{21}))$$

with

$$\mathrm{tr}(W_1) = \frac{8k - 14}{3} + \frac{2}{3} \begin{cases} 0 & k \equiv 4 \bmod 6 \\ -1 & k \equiv 2 \bmod 6 \\ +1 & k \equiv 0 \bmod 6 \end{cases},$$

$$\mathrm{tr}(W_3) = (-1)^{k/2} \frac{4}{3} + (-1)^{k/2} \frac{2}{3} \begin{cases} -2 & k \equiv 4 \bmod 6 \\ +1 & k \equiv 2 \bmod 6 \\ +1 & k \equiv 0 \bmod 6 \end{cases},$$

$$\mathrm{tr}(W_7) = 0,$$

$$\mathrm{tr}(W_{21}) = 2(-1)^{k/2}.$$

1.3 Extremality

In this subsection, we specify the general setup in which extremality of a lattice can be defined in terms of its theta series.

Definition 1.6 a) Let $\mathcal{M}$ be a subspace of $\mathcal{M}_k(L)$. We say that *extremality is definable* with respect to $\mathcal{M}$ if the projection $\mathcal{M} \to \mathbb{C}^d$ to the first $d = \dim \mathcal{M}$ coefficients of the q-expansion

$$f = \sum_{m \geq 0} a_m q^m \mapsto (a_0, a_1, \ldots, a_{d-1})$$

is injective. If this holds, the unique element $F =: F_{\mathcal{M}} \in \mathcal{M}$ with q-expansion

$$F = 1 + \sum_{m \geq d} a_m q^m$$

is called the *extremal modular form* in $\mathcal{M}$.

b) Let L be an even lattice of dimension $2k$ and level ℓ and $\mathcal{M}$ be a subspace of $\mathcal{M}_k(\ell, \varepsilon)$ with $\Theta_L \in \mathcal{M}$ (where as above ε denotes the character defined by the determinant of L). We say that L is *extremal with respect to* $\mathcal{M}$ if extremality is definable with respect to $\mathcal{M}$ and $\Theta_L = F_{\mathcal{M}}$.

Thus if L is extremal, then $\min L \geq 2 \dim \mathcal{M}$ is as large as the specified space $\mathcal{M}$ of modular forms allows. Notice that according to the definition we have chosen, "extremality" is defined under rather general circumstances, but a strong necessary condition for existence of extremal lattices is that the extremal modular form $F_{\mathcal{M}}$ should have non-negative coefficients. In section 2 we shall prove that, for certain levels ℓ, this holds for only finitely many k.

The general definition of extremality is not of much use as long as no restrictions on the space $\mathcal{M}$ are imposed. We shall not treat this problem in general. In the modular and strongly modular case, the Atkin-Lehner identity suggests the following choice of $\mathcal{M}$. Here k must be even for prime levels $\ell \equiv 1(4)$; we shall also assume that k is even if ℓ is composite. Then the character on $\Gamma_0(\ell)$ describing the action on theta series is trivial, and $\{\pm 1\}$-valued on the involutions W_m.

Definition 1.7 Consider a genus $\mathcal{G}$ of level ℓ, determinant ℓ^k, and containing (strongly) modular lattices (e.g. ℓ prime). Let δ be the character on $\Gamma_0(\ell)$ and let χ be the character on the group of involutions W_m describing $\mathcal{G}$:

$$\chi(W_p) = g_p(L) \text{ for } L \in \mathcal{G}$$

with $g_p(L)$ the Gaussian sum from [Que97].

A (strongly) modular lattice in $\mathcal{G}$ is called (*strongly*) *modular extremal* if it is extremal with respect to the subspace

$$\{f \in \mathcal{M}_k(\ell, \delta) \mid f|_k W_\ell = \chi(W_\ell) f\}$$

respectively

$$\mathcal{M}_k(\ell, \chi) = \{f \in \mathcal{M}_k(\ell, \delta) \mid f|_k W_m = \chi(W_m) f \;\forall\, m \| \ell\}.$$

The investigation of genera of small level ℓ and "small" (relative to ℓ) dimension n shows that often the lattices with largest known minimum among all lattices of level ℓ and determinant $\ell^{n/2}$ are strongly modular extremal lattices. However, not all strongly modular extremal lattices do have the largest occurring minimum, not even within their genus, as the following "negative" example shows.

Example. Consider the genus $\mathcal{G} = I\!I_8(3^{-4}5^{-4})$ of dimension 8 and level 15, represented for instance by $B \perp B' \perp B' \perp B'$, with B and B' as above. This genus does not contain "obvious", i.e. decomposable strongly modular lattices. (The reason is that $I\!I_4(3^2 5^2)$ has only two classes $B \perp B$ and $B' \perp B'$ and thus no strongly modular lattice). However, complete enumeration of $\mathcal{G}$, with class number $h(\mathcal{G}) = 68$ shows the following:

(a) $\mathcal{G}$ contains exactly 2 (classes of) strongly modular lattices; they have minimum 4.

(b) $\mathcal{G}$ contains a unique lattice with minimum 6; this lattice is not strongly modular.

(c) The strongly modular lattices with minimum 4 are strongly modular extremal.

Part (c) is verified as follows: for the appropriate character χ given by $\chi(W_3) = \chi(W_5) = -1$, the space $\mathcal{M}_4(15, \chi)$ is two-dimensional, the space of cusp forms $\mathcal{S}_4(15, \chi)$ is one-dimensional, and a non-zero cusp form $\Sigma_{n\geq 1} a_n q^n$ has $a_1 \neq 0$. Thus extremality is definable, and strongly modular lattices with minimum 4 are extremal.

Notice that parts (a) and (b) of this example show that even without using the notion of extremality, we can state the fact that the largest minimum in a genus is not always attained by a strongly modular lattice

2 Extremal modular forms

The rather general notion of extremality introduced in 1.6 has its origin in the investigation of the special cases of the full modular group ($\ell = 1$) in [MOS75] and certain special ones of the $\Gamma_0^*(\ell)$ in [Que95, Que97]. In the case $\ell = 1$ it was proved in [MOS75] (extending results of Siegel [Sie69]) that extremality is definable for all even weights, that the unique extremal modular form has (except for the zeroth coefficient 1) even integral Fourier coefficients $a(n)$, that $a(d)$ is positive and that $a(d+1)$ is negative for large weights. These results do not carry over to the general situation in which extremality is definable. In particular it is not known in general whether the d-th Fourier coefficient of the extremal modular form is non zero (hence what the minimum of an extremal lattice is) or whether the extremal modular form has (even) integral Fourier coefficients and hence is at all eligible for being the theta series of a lattice. For the situations considered in [Que95, Que97] however, we have the following result (of which part i) is from [Que95, Que97]). As usual, $\sigma_0(\ell)$ denotes the number of divisors and $\sigma_1(\ell)$ the sum of the divisors of ℓ.

Theorem 2.1 i) *Let ℓ be one of the integers* 1, 2, 3, 5, 6, 7, 11, 14, 15, 23 *and let χ be the character on $\Gamma_0^*(\ell)$ defined by $(\frac{-\ell}{d})^k$ on $\Gamma_0(\ell)$, by $\chi(W_2) = 1$, $\chi(W_3) = (-1)^{k/2}$ (case 6a) or $\chi(W_2) = (-1)^{k/2}$, $\chi(W_3) = 1$ (case 6b) for $\ell = 6$,* *by $\chi(W_2) = 1$, $\chi(W_7) = (-1)^{k/2}$ for $\ell = 14$, by $\chi(W_3) = 1$, $\chi(W_5) = (-1)^{k/2}$ for $\ell = 15$, and $\chi(W_\ell) = i^k$ for the remaining values of ℓ.*

Then extremality is definable for $\mathcal{M}_k(\ell, \chi)$.

With

$$k_1(\ell) := \frac{12\sigma_0(\ell)}{\sigma_1(\ell)}$$

one has

$$d_\ell := \dim \mathcal{M}_k(\ell, \chi) = 1 + \left[\frac{k}{k_1(\ell)}\right].$$

ii) *In the cases above the extremal modular form has integral Fourier coefficients $a_k(n)$ (that are even for $n > 0$). Moreover, one has $a_k(d_\ell) > 0$ for all k and $a_k(d_\ell + 1) < 0$ for k large enough (depending on ℓ, χ).*

Proof 2.2 i) has been proven in [Que95, Que97], where it is also shown that $\mathcal{M}_k(\ell, \chi)$ has a basis consisting of the functions $\Theta_N^i \Delta_\ell^j$, where $N = N(\ell, \chi)$ denotes a strongly modular lattice of minimal dimension $k_0 = k_0(\ell)$ with respect to ℓ and χ,

$$\Delta_\ell(z) = \prod_{m|\ell} \eta(mz)^{\frac{24}{\sigma_1(\ell)}}.$$

This implies immediately that the extremal modular form has (even) integral Fourier coefficients and allows to deduce the assertion about $a(d_\ell)$ and $a(d_\ell + 1)$ in the same way as in [MOS75]; some of the details have been carried out in [Sze95]. We sketch the main steps briefly: With

$$\varphi := \frac{\Delta_\ell}{\Theta_N^{(k_1(\ell)/k_0(\ell))}}$$

one has an expansion

$$\Theta_N^{-(k/k_0(\ell))}(q) = \sum_{s=0}^{\infty} \alpha(s)\varphi(q)^s$$

in a sufficiently small disk $|q| \le r < 1$, where $\Theta_N(q)$ and $\varphi(q)$ denote the expansions with respect to $q = \exp\ (2\pi i z)$ of these functions. For the coefficients $\alpha(s)$ one has

$$\alpha(d_\ell) < 0,$$

and

$$\frac{\alpha(d_\ell + 1)}{\alpha(d_\ell)}$$

is asymptotic to

$$\frac{\Theta_N^{(k_1(\ell)/k_0(\ell))}(e^{-2\pi y_0})}{\Delta_\ell(e^{-2\pi y_0})},$$

where y_0 is the (unique) positive zero of the derivative of $(y \longmapsto \Delta_\ell(e^{-2\pi y}))$ and Θ_N and Δ_ℓ are again viewed as functions of q. Moreover one has

$$\begin{aligned} a(d_\ell) &= -\alpha(d_\ell) \\ a(d_\ell + 1) &= -\alpha(d_\ell + 1) + \alpha(d_\ell) \cdot \\ &\left(\frac{2k_1(\ell)}{\sigma_0(\ell)} \left(\left[\frac{k}{k_1(\ell)}\right] + 1 \right) + \left(\frac{k_1(\ell)}{k_0(\ell)} - \frac{k - k_1(\ell)\left[\frac{k}{k_1(\ell)}\right]}{k_0(\ell)} \right) \right), \end{aligned}$$

where $c_1 = c_1(\ell)$ is the first Fourier coefficient of Θ_N. The boundedness of the quotient

$$\frac{\alpha(d_\ell+1)}{\alpha(d_\ell)}$$

then implies the assertion.

Remark 1. a) It is not difficult to calculate y_0 in the above argument explicitly to high accuracy. If one then replaces the quotient $\frac{\alpha(d_\ell+1)}{\alpha(d_\ell)}$ by its asymptotic value given above, one obtains a heuristic bound for the maximal weight $k_{\max}(\ell,\chi)$ of an extremal modular form with positive coefficient $a(d_\ell+1)$. For $\ell = 1$ a value of $k_{\max}(\ell) \leq 20500$ is given in [MOS75]. For the other values of ℓ, χ considered here we have the following table:

ℓ	2	3	5	$6a$	$6b$	7	11	14	15	23
$k_{\max}$	3118	473	82	134	146	37	17	22	22	6

Table 2.1

We calculated the extremal modular form in the weights below this bound and found that the coefficient $a(d_\ell+1)$ becomes negative a little earlier than this bound. The following tables give the values obtained for the jump weights (i.e., k with $k_1(\ell)|k$) and general weights.

ℓ	2	3	5	$6a$	$6b$	7	11	14	15	23
$k_{\max}$	2936	378	40	100	100	12	6	16	14	2

Table 2.2

ℓ	2	3	5	$6a$	$6b$	7	11
$k_{\max}$	3062	412	58	118	122	26	11

Table 2.3

Using the formulas above one can calculate from the extremal theta series the coefficients $\alpha(d_\ell)$, $\alpha(d_{\ell+1})$ and finds that the quotient $\frac{\alpha(d_\ell+1)}{\alpha(d_\ell)}$ seems to tend to its asymptotic value quite slowly, but from below, so that beyond the values of k given above one should not expect any further extremal modular forms with positive coefficient $a(d_\ell+1)$. In particular, there should be no extremal lattices of the types considered and dimension bigger than the values of $2k_{\max}(\ell,\chi)$ given above (we will see that our list of known extremal lattices terminates much earlier in all cases except $\ell = 23$).

b) As in [MOS75] one can also prove the existence of a bound $\tilde{k}_{\max}(\ell,\chi,\nu)$ for $\nu \in \mathbb{N}$ such that any modular form in $M_k(\ell,\chi)$ with constant term 1 and vanishing Fourier coefficients $a(1),\ldots,a(d_\ell-\nu)$ of weight $k > \tilde{k}_{\max}(\ell,\chi,\nu)$ has at least one negative Fourier coefficient (and hence cannot be a theta series).

c) If $k = 2$ and the character χ on $\Gamma_0^*(\ell)$ is trivial the condition under which extremality is definable just means that the modular curve $H/\Gamma_0^*(\ell)$ has no Weierstraß point at the cusp ∞. A more general notion of Weierstraß points has been introduced by Petersson in [Pet49] and investigated by Smart in [Sma66]. A detailed investigation of the Weierstraß points for the situations of interest here and of the consequences for theta series of lattices will be the subject of future work.

Remark 2. As in [Que97] our discussion of the extremal modular form in this section is limited to those cases in which a direct generalization of the methods from the case of level 1 is possible. In particular we do not discuss all strongly modular genera of levels $6, 14, 15$ but only those in dimension $4r$ obtained by taking r-fold orthogonal sums of the 4-dimensional genera $G_4(2^-3^+) = I\!I_4(2_{I\!I}^{-2}3^{-2})$ (case6b), $G_4(2^+3^-) = I\!I_4(2_{I\!I}^{2}3^{2})$ (case 6a), $G_4(2^+7^-) = I\!I_4(2_{I\!I}^{2}7^{2}), G_4(3^+5^-) = I\!I_4(3^{-2}5^{-2})$ with the $G_4()$-notation as in [Que97].

Remark 3. E. Rains informed us that he can prove the nonexistence of strongly modular lattices in the genera $r \cdot G_4(2^-7^+) = I\!I_{4r}(2_{I\!I}^{-2r}7^{-2r})$ (r odd) and $G_4(2^+7^-) \perp r \cdot G_4(2^-7^+) = I\!I_{4r+4}(2_{I\!I}^{-(2r+2)}7^{-(2r+2)})$ (r odd) of dimension $4r$ resp. $4r+4$ and in certain other genera of levels $2N, 4N$ with $N \equiv -1 \bmod 8$. We have received a preprint version ([RaSl98]) during the final revision of this article.

We conclude this subsection with a table of minima of extremal lattices for the levels treated in Theorem 2.1, part i) up to minimum 12. An entry in brackets means that the extremal modular form has a negative coefficient.

n \ ℓ	1	2	3	5	6	7	11	14	15	23
4	–	2	2	2	2	2	4	4	4	6
6	–	–	2	–	–	4	4	–	–	(8)
8	2	2	2	4	4	4	6	6	6	(10)
10	–	–	2	–	–	4	6	–	–	(12)
12	–	2	4	4	4	6	8	8	8	
14	–	–	4	–	–	6	8	–	–	
16	2	4	4	6	6	6	(10)	10	10	
18	–	–	4	–	–	8	10	–	–	
20	–	4	4	6	6	8	(12)	12	12	
22	–	–	4	–	–	8				
24	4	4	6	8	8	10				
26	–	–	6	–	–	10				
28	–	4	6	8	8	10				
30	–	–	6	–	–	12				
32	4	6	6	10	10	12				
34	–	–	6	–	–	12				
36	–	6	8	10	10					
40	4	6	8	12	12					
44	–	6	8	12	12					
48	6	8	10							
52	–	8	10							
56	6	8	10							
60	–	8	12							
64	6	10	12							
68	–	10	12							
72	8	10								
76	–	10								
80	8	12								

Table 2.4: Minima of (hypothetical) strongly modular extremal lattices

3 On the Existence and uniqueness of extremal lattices

In this section we follow the program of Quebbemann [Que95], [Que97]. That is, we consider strongly modular lattices of level ℓ with $\sigma_1(\ell)$ (sum of divisors of ℓ) dividing 24. We recall from the previous section the table of minima of such lattices (if they exist). We give a survey of what is presently known about these lattices. Some of the results are “classical” in the sense that they were known before [Que95]. Many “new” extremal lattices occurring were found by Nebe and Plesken & Nebe in the course of their investigation of maximal finite rational matrix groups. Other new lattices were constructed by Bachoc using number fields, quaternions, and codes. A complete investigation of all cases of small dimension n (up to $n = 8, \dots, 16$, depending on the level ℓ) was obtained in [SchHem94] for prime levels and is completed in this paper. We survey the results according to the minimum $2, 4, 6, 8$, and distinguish between “minimal dimensions” (the smallest dimension for which the minimum in question could occur), and other (usually less interesting) dimensions. For the levels considered in this section, we are aware of only one extremal strongly modular lattice with minimum 10. This is no. 19 of [NePl95] of level $\ell = 15$; for strong modularity, see [Neb97]. Gram matrices for the lattices stabilized by an integral maximal finite matrix group are contained in the program package GAP [Scho97]. A big collection of Gram matrices of modular lattices is also part of the ‘Catalogue of lattices’ of G. Nebe and N.J.A. Sloane which can be obtained from `http://www.research.att.com/~njas/lattices/`.

3.1 Extremal lattices with minimum 2

This case occurs for the levels $\ell \leq 11$ and is a bit exceptional and trivial. Indeed, there is no real condition on the minimum, any strongly modular lattice of the appropriate dimension is extremal.

Minimal dimensions. This is the respective smallest dimension n_0 for which a totally even lattice L of level ℓ with $\ell \leq 11$ and determinant $\ell^{n/2}$ exists. For $\ell = 1$, one has $n_0 = 8$, for $\ell = 2, 5, 6$, one has $n_0 = 4$, for $\ell = 3, 7, 11$, one has $n_0 = 2$. In all these cases, the class number of the respective genus (for $\ell = 6$, there are two of them) is one. Thus in each case, an extremal lattice exists and is unique. For $\ell = 1, 2, 3$, it is the root lattice E_8, D_4, A_2, for $\ell = 7, 11$ the binary lattice with Gram matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 6 \end{smallmatrix}\right)$, respectively. For $\ell = 6$, the genus $I\!I_4(2_{I\!I}^{-2}3^{-2})$ consists of the root lattice $A_2\,{}^2\!A_2$, and the genus $I\!I_4(2_{I\!I}^{2}3^{2})$ consists of a reflective lattice (see [SchBla96]) with root system $\mathsf{C}_2\,{}^3\mathsf{C}_2$. A Gram matrix of this lattice is $\left(\begin{smallmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 1 & 1 & 4 & 0 \\ 1 & -1 & 0 & 4 \end{smallmatrix}\right)$; the orthogonal group is (necessarily) equal to the Weyl group, of order 64.

Other dimensions. For the following pairs (n, ℓ) with higher n modular lattices L of minimum 2 in the respective genus are still extremal.

$$\begin{aligned} (n,\ell) \;=\; & (16,1), (8,2), (12,2); \\ & (4,3), (6,3), (8,3), (10,3); \\ & (4,7). \end{aligned}$$

The known (and rather trivial) classification of all these lattices shows that they are all modular, and thus extremal. (The classification for $\ell = 2$ and 3 can be readily obtained from the classification in the higher-dimensional cases $(16, 2)$, $(12, 3)$ in [SchVen94] by restricting to decomposable lattices.) The class numbers are shown in the following small table.

$n \setminus \ell$	1	2	3	7
4	–	1	1	1
6	–	–	1	
8	1	1	2	
10	–	–	3	
12	–	3		
16	2			

Table 3.1: Some class numbers

3.2 Extremal lattices with minimum 4

Minimal dimensions. These dimensions $n(\ell, 4)$, for the various levels ℓ, are shown in the following table. It is known that in each case there exists an extremal lattice, and is unique up to isomorphism. The complete classification of all genera of lattices in question is also known, the class number is given in the third row of the table. We recall from Section 2 that for the composite levels $\ell = 6, 14, 15$, the genus in question is one out of two totally even genera of determinant $\ell^{n/2}$, namely $I\!I_8(2_{I\!I}^4 3^4)$, $I\!I_4(2_{I\!I}^2 7^2)$, $I\!I_4(3^{-2}5^{-2})$, respectively. In the fourth row the number of classes of strongly modular lattices is indicated.

ℓ	1	2	3	5	6	7	11	14	15	23
$n(\ell, 4)$	24	16	12	8	8	6	4	4	4	2
h	24	24	10	5	8	3	3	3	3	2
h_{sm}	24	16	10	5	6	3	3	3	3	2

Table 3.2: Minimal dimensions for extremal lattices with minimum 4, and class numbers

Other dimensions. The following table contains, for each level ℓ, the dimensions above the minimal dimension for which extremal lattices have minimum 4. The third row contains the class number of the respective genus, if this is known. In the missing cases (for $\ell = 1, 2, 3$), the class number is very large, and a full classification at present is out of reach. The last row contains the number of known isomorphism classes of extremal lattices. We see that extremal lattices always exist. They are unique in the two cases $(n, \ell) = (11, 6)$, $(7, 8)$ of small class number, and, more surprisingly, also for $(3, 14)$. In all other cases they are most probably not unique.

ℓ	1		2			3					5	6+	6−	7		11
n	32	40	20	24	28	14	16	18	20	22	12	12	12	8	10	6
h						29	163				48	308	284	8	30	5
h_{ext}	≥ 22	≥ 3	≥ 3	≥ 1	≥ 1	1	6	≥ 1	≥ 1	≥ 1	4	6	4	1	4	1

Table 3.3: Extremal lattices of minimum 4 in non-minimal dimensions

Description of the lattices in minimal dimensions.

$\boxed{\ell = 1}$ This is the famous lattice Λ_{24} of Leech [Lee64].Its uniqueness has been proved by Conway [Con69], using modular forms. Perhaps the most basic among the various constructions for Λ_{24} is the following one: Take the binary Golay code $\mathcal{G}$; this is the unique linear binary self-dual code of length 24 and minimum weight $d(\mathcal{G}) = 8$. The lattice $L(\mathcal{G})$ has the root system $24A_1 = \{\pm a_1, \dots, \pm a_{24}\}$ (no new roots occur since $d(\mathcal{G}) = 8$). Λ_{24} is the neighbor of $L(\mathcal{G})$ with respect to the vector $u = -\frac{3}{2}a_1 + \frac{1}{2}\sum_{i=2}^{24} a_i$ (it is obvious that this lattice is even and has minimum 4).

$\boxed{\ell = 2}$ This is the lattice $\Lambda_{16} = BW_{16}$ of Barnes and Wall [BaWa59]. Quebbemann in [Que95] has given a proof of its uniqueness similar in spirit to Conway's proof for the Leech lattice, using modular forms. The uniqueness is also a consequence of the classification in [SchVen94] of all lattices of minimum 2 in the genus $I\!I_{16}(2^{8}_{I\!I})$, combined with the mass formula. A basic construction using codes, again analogous to the case of the Leech lattice, is the following. Take the first-order Reed Muller code $\mathcal{R} = \mathcal{R}(1,4)$, i.e the binary linear code of length 16, dimension 5, and minimal weight 8, generated by the codewords 1^{16}, $1^8 0^8$, $1^4 0^4 1^4 0^4$, $1^2 0^2 1^2 0^2 1^2 0^2 1^2 0^2$, $1010\dots10$. The lattice $L^{\mathcal{R}}$ of determinant 2^{10} has a unique totally even overlattice of index 2, obtained by adjoining $\frac{1}{2}\sum_{i=1}^{16} e_i$. This is the desired lattice BW_{16}.

$\boxed{\ell = 3}$ The extremal lattice in this genus is the Coxeter-Todd lattice [CoTo53]. It is obtained by ordinary transfer from a certain self-dual hermitian lattice K_6 over the Eisenstein integers $\mathbb{Z}[\omega]$, $\omega^3 = 1$. The latter lattice is the unique 6-dimensional self-dual $\mathbb{Z}[\omega]$-lattice containing no vectors of norm 1 [Feit78]. (This clearly implies that $\min K_{6,\mathbb{Z}} \geq 4$.) A convenient construction for K_6 is by lifting the so called hexacode $\mathcal{C}_6$ over $\mathbb{F}_4$. This code is by definition generated by the rows of

$$\begin{array}{cccccc} 1 & 0 & 0 & 1 & \omega & \omega \\ 0 & 1 & 0 & \omega & 1 & \omega \\ 0 & 0 & 1 & \omega & \omega & 1 \end{array},$$

(where now ω is understood as an element of $\mathbb{F}_4 \setminus \mathbb{F}_2$). It is clearly self-dual with respect to the standard hermitian form an $\mathbb{F}_4^6$. Consequently, the set of all $\frac{1}{2}\sum x_i a_i$, $x_i \in \mathbb{Z}[\omega]$, $(\overline{x}_1, \dots, \overline{x}_6) \in \mathcal{C}_6$ has the desired properties. Alternatively, K_6 can be obtained by twisted transfer $(x,y) \mapsto \mathrm{tr}(h(x,y)/(5+\sqrt{5})/2)$ from the self-dual $\mathbb{Z}[\omega, \varepsilon_5]$-lattice $J_3(5)$ used below in the case of minimum 8 and $\ell = 15$. A full classification of the genus without the use of a computer has been given in [SchVen95].

$\boxed{\ell = 5}$ There exists a unique even unimodular lattice in dimension 4 over $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$. Its orthogonal group is the reflection group of type H_4; a Gram matrix is directly read off from the Coxeter diagram

$$\circ\!\!-\!\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\!-\!\!\circ\!\!\overset{5}{-\!\!\!-\!\!\!-}\!\!\circ$$

as follows: the diagonal entries are equal to 2, the ij-entry, for $i \neq j$, is $-2\cos\frac{\pi}{m_{ij}}$, where $m_{ij} \in \{2,3,5\}$ is the order of the corresponding product of reflections, as indicated in the diagram. Ordinary transfer of H_4 gives a 5-modular $\mathbb{Z}$-lattice, which is clearly of minimum 4. This is our desired lattice Q.

$\boxed{\ell = 6}$ An 8-dimensional strongly modular lattice of level 6 and with minimum 4 can be obtained very easily as $D_4 \otimes A_2$. It follows from the general theorem7.1.1 in [Kit93], or by an elementary computation, that the minimum is indeed not smaller than 4. Alternatively, this is immediately clear from the following number field construction of the same lattice: consider the field $F = \mathbb{Q}(\zeta_{24}) = \mathbb{Q}(i, \sqrt{2}, \sqrt{-3}) = \mathbb{Q}(\zeta_8, \zeta_3)$ of 24^{th} roots of unity, its ring of integers $\mathfrak{o}_F \cong \mathbb{Z}[\zeta_8] \otimes \mathbb{Z}[\zeta_3]$ with the twisted transfer $(x, y) = \mathrm{tr}_{\mathbb{Q}}^{F} \frac{x\overline{y}}{2(2-\sqrt{2})}$ of the unit form over $\mathfrak{o}_F$. It is indeed readily checked that $\mathfrak{o}_{\mathbb{Q}(\zeta_8)} = \mathbb{Z}[\zeta_8] = \mathbb{Z}\left[\frac{\sqrt{2}+\sqrt{-2}}{2}, i\right]$ with the hermitian form $\mathrm{tr}_{\mathbb{Q}}^{\mathbb{Q}(\zeta_8)}\left(\frac{x\overline{y}}{2(2-\sqrt{2})}\right)$ gives the D_4-lattice. Namely, this lattice L satisfies $L^{\#} = 2(2-\sqrt{2})\mathcal{D}^{-1}L = 2\sqrt{2}\mathcal{D}^{-1}L = (1-\zeta_8)L$, where $\mathcal{D} = (1-\zeta_8)^7$ is the different of $\mathbb{Q}(\zeta_8)$. This lattice is even since $\mathrm{tr}\, \frac{x\overline{x}}{2(2-\sqrt{2})} = 2\, \mathrm{tr}_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{2})} \frac{x\overline{x}}{2(2-\sqrt{2})}$ and $2(2-\sqrt{2})\mathbb{Z}[\sqrt{2}]$ is the different of $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$. and thus $\mathrm{tr}_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{2})} \frac{x\overline{x}}{2(2-\sqrt{2})} \in \mathbb{Z}$.

$\boxed{\ell = 7}$ There is as well-known self-dual hermitian $\mathbb{Z}[\alpha]$-lattice, $\alpha = \frac{1+\sqrt{-7}}{2}$, of dimension 3 and minimum 2, which we denote be J_3 (following [Coh76]). Its unitary group is $U(J_3) = 2 \times G_{168}$, where $G_{168} \cong L_3(2) \cong L_2(7)$ is a famous simple group of order 168. The group $U(J_3)$ is a primitive irreducible complex reflection group, occurring as no. 24 in the basic list of such groups given by Shephard and Todd [ShTo54]. A Gram matrix for J_3 showing up in this context is

$$J_3 \cong \begin{pmatrix} 2 & \overline{\alpha} & 1 \\ \alpha & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

It also occurs in the work of Mimura [Mim82], where hermitian lattices over imaginary quadratic fields, generated by norm 2 vectors, are classified. (The significance of the field $\mathbb{Q}(\sqrt{-7})$ in this "direct" approach by the way is that α is the unique imaginary quadratic number with trace 1 and norm 2.) The desired extremal 7-modular lattice is obtained from J_3 by ordinary transfer. It is isometric to the Barnes lattice P_6 and to the Craig lattice $A_6^{(3)}$.

$\boxed{\ell = 11}$ Due to the fact that $\mathrm{N}\pi = 3$, where $\pi = \frac{1+\sqrt{-11}}{2}$, there is an obvious binary unimodular $\mathbb{Z}[\pi]$-lattice with minimum 2, given by the Gram matrix $\begin{pmatrix} 2 & \overline{\pi} \\ \pi & 2 \end{pmatrix}$. From this a 4-dimensional extremal 11-modular lattice is obtained by ordinary transfer.

$\boxed{\ell = 14}$ An extremal 4-dimensional strongly modular lattice of level 14 is obtained by ordinary transfer from the binary $\mathbb{Z}[\alpha]$-lattice (of determinant 2) given by $\begin{pmatrix} 2 & \overline{\alpha} \\ \alpha & 2 \end{pmatrix}$, where $\alpha = \frac{1+\sqrt{-7}}{2}$ as usual. The modularity with respect to the prime 2 is not directly given by α, but comes from the fact that $\begin{pmatrix} 2 & \overline{\alpha} \\ \alpha & 2 \end{pmatrix}$ is isometric to its complex conjugate (cf. Proposition 1.3).

$\boxed{\ell = 15}$ The trace form $\mathrm{tr}_{\mathbb{Q}}^{F}(x\overline{y})$ on $\mathfrak{o}_F$, where $F = \mathbb{Q}(\sqrt{5}, \sqrt{-3})$, clearly gives a strongly modular lattice of level 15 and minimum 4.

Description of the lattices in other dimensions.

$\boxed{\ell = 1}$ One might suspect that many among, perhaps most of the 32-dimensional unimodular lattices have minimum 4. However, despite the efforts of several authors (notably Koch and Venkov, Koch and Nebe, and Blaschke-Requate), only relatively few such lattice have been obtained by a structural algebraic construction, and it is difficult to show that the lattices constructed are indeed non-isomorphic. A few lattices have been constructed and distinguished using particular automorphisms [Que92], [BQS95]. A attempt to the systematic construction of all extremal lattices regardless of their automorphisms is the notion of neighbor defect $\delta(L)$ of [KoVe89, KoVe91]. This number is defined as 32 minus the largest rank r of the root system of a neighboring lattice M (the root system $\mathsf{R}(M)$ is necessarily of the form $r\mathsf{A}_1$). So the lattices of neighbor defect 0 are by definition the 2-neighbors of the ordinary 'lifts' $L(C)$, where C is a self-dual binary code of length $n = 32$ and minimal weight 8. The smallest values for δ are $0, 8, 12, 14, 15$, and in these cases the "defect lattice" is unique. The lattices with neighbor defect 0 and 8 are classified [KoVe89, KoVe91], [KoNe93], there are $5 + 10$ of them. 7 pairwise non-isomorphic lattices of neighbor defect 12 are known, by [Bla96].

It is an easy consequence of the Minkowski-Siegel mass formula that there are at least $8.45 \cdot 10^{51}$ classes of extremal even unimodular lattices in dimension 40; see Peters [Pet83]. The first explicit example of such a lattice has been given by McKay; see [CoSl93], Chapter 8.5. Later extremal unimodular lattices emerged in a standard way from the investigation of self-dual binary codes C of length 40 and minimal weight 8; see above. Ozeki investigated in [Oze88] these lattices for three particular codes and showed that they have distinct Siegel theta series of degree two. Using the above class number estimate, Peters showed in [Pet90] that the Siegel theta series of degree 2 are not a classifying invariant for the set of all extremal even unimodular lattices in dimension 40. See also [CaSl97] and [CoSl93], Chapter 7.7, page 194, for further references.

$\boxed{\ell = 2}$ In dimension 20, two extremal lattices are known from [PlNe95], p. 49; see also [BQS95]. A third one, with automorphism group of order $2^{14}3^2 5$ has been found in the course of a projected complete enumeration of this genus, using a suitable refinement of the computer program of [SchHem94]. C. Bachoc and B. Venkov have announced a proof of the fact that there are no further extremal lattices in this genus.

In [Bac95, Bac97], Theorem 6.7, it is shown, by a uniform construction using codes, that, for all three dimensions in question, 20, 24 and 28, an extremal lattice can be obtained from a suitable self-dual lattice over a maximal order of the quaternion algebra $\mathbb{Q}_{2,\infty}$.

$\boxed{\ell = 3}$ For dimension 14, see [SchHem94]. For dimensions 16 to 22, such lattices come from self-dual hermitian lattices over the Eisenstein integers, by [Feit78]; see also [Bac97], Theorem 6.7.

$\boxed{\ell = 5}$ Complete enumeration of the 12-dimensional genus in [SchHem94] has shown that there are 4 extremal lattices. A. Schiemann in [Schi98] has shown that there are no less than 29 classes of self-dual hermitian lattices over $\mathbb{Z}(\sqrt{-5})$ giving rise to such $\mathbb{Z}$-lattices.

$\boxed{\ell = 6}$ For both genera in question, of dimension 12, we have obtained by computer the full classification of all lattices, and the determination of the strongly-modular ones among those with minimum 4. The 'plus-genus' $I\!I_{12}(2^6 3^6)$ is the one coming from even self-dual hermitian lattices over $\mathbb{Z}(\sqrt{-6})$. This genus has been classified completely in [Schi98]. There are 161 classes, 41 of them have minimum 4. It turns out that all 6 extremal lattices in $I\!I_{12}(2^6 3^6)$ have one or several self-dual hermitian structures. Conversely, each hermitian lattice with minimum 4 and the correct number 84 of minimal vectors (there are 14 such lattices) actually gives rise to a strongly modular $\mathbb{Z}$-lattice.

$\boxed{\ell = 7}$ The classification of all lattices in dimensions 8 and 10 is again taken from [SchHem94]. The existence of a self-dual hermitian structure on such lattices is mentioned in [Bac97] and completely worked out in [Schi98].

$\boxed{\ell = 11}$ See [SchHem94]. No appropriate hermitian lattice exists, by [Schi98].

3.3 Extremal lattices with minimum 6

Minimal dimensions. The minimal dimension $n(\ell, 6)$ is given in the following table. Extremal lattices do not exist for $\ell = 7$, they exist in all other cases. The class number is very large and not known if $\ell \leq 6$; for $\ell \geq 7$ it is shown in the third row of the table. The fourth row collects the known information about the number of extremal lattices. For $\ell = 3, 5$ uniqueness of extremal lattices is an open question.

ℓ	1	2	3	5	6	7	11	14	15	23
$n(\ell,6)$	48	32	24	16	16	12	8	8	8	4
h						395	31	80	91	6
h_{ext}	≥ 3	≥ 3	≥ 1	≥ 1	≥ 5	0	1	1	2	1

Table 3.4: Extremal lattices with minimum 6 in minimal dimensions

Other dimensions. The following table collects the known information about the existence of extremal lattices of minimum 6 above the minimal dimension. The class numbers are very large and not known except for $(n, \ell) = (10, 11)$.

ℓ	1		2			3					5	6	7		11
n	56	64	36	40	44	26	28	30	32	34	20	20	14	16	10
h_{ext}	≥ 1	≥ 1	≥ 3	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1	≥ 1		≥ 1		≥ 1	2

Table 3.5: Extremal lattices of minimum 6 in non-minimal dimensions

Description of the lattices in minimal dimensions.

$\boxed{\ell = 1}$ If an extremal self-dual ternary code C of length 48 is extremal, that is, has minimal distance 15, then the ordinary lift $L(C)$ is an odd unimodular lattice, whose (pairs of) minimal vectors are precisely the canonical basis vectors of ${}^3I_{48} \subset L(C)$. The 2-neighbor $\Lambda(C) := L(C)(u)$ for the vector $u = \frac{1}{3}(-5, 1, \ldots, 1)$ is even and has minimum 6. There are two such codes known, a Pless doubly circulant code and a quadratic residue code, giving rise to lattices called P_{48p}, respectively P_{48q} in [CoSl93], pp. 148 f. and 195. In view of well known automorphism groups of the respective code, their automorphism group contains $2L_2(23) \times S_3$ and $2L_2(47)$, respectively. The absence of a common integral overgroup in $GL_{48}(\mathbb{Q})$ (see [CoSl93], refering to J. Thompson) implies that P_{48p} and P_{48q} are non-isometric (and their groups in fact indentical to the above groups). Recently, G. Nebe has reconstructed these two lattices and their groups, correcting an error about the second group, and has constructed a third extremal unimodular lattice; see [Neb96].

$\boxed{\ell = 2}$ Extremal 2-modular lattice in dimension 32 have been considered by H.-G. Quebbemann already in 1984. They are famous because of the fact that they give rise to the densest known sphere-packing in this dimension, surpassing the laminated lattices Λ_{32}. In [Que84b], such a lattice is constructed using the quaternion group of order 8 acting on 8 copies of the D_4-root lattice. A different construction is given in [Que87], namely as a self-dual lattice over $\mathbb{Z}[\zeta_8]$. In [Bac95], Théorème 5.2, C. Bachoc has constructed an 8-dimensional non-integral quaternionic lattice over the Hurwitz integers such that the underlying $\mathbb{Z}$-lattice is 2-modular extremal. It is shown in [BaNe97] that no integral (and thus self-dual) such quaternionic lattice exists. Another 2-modular extremal lattice has been constructed in [Neb96], Theorem 5.1. It is known from (highly non-trivial!) computer calculations [PlSo97] that the four constructions mentioned actually give rise to non-isometric $\mathbb{Z}$-lattices. In fact, the orthogonal groups are known, and are pairwise distinct.

$\boxed{\ell = 3}$ An extremal 3-modular lattice in dimension 24 has been constructed by Nebe, using a certain 24-dimensional rational representation of the group $SL_2(3) \times SL_2(13)$; see [Neb95], p. 39, Satz (VI. 8). This matrix group by the way is not maximal finite and thus does not directly belong to the subject of [Neb95]. It however occurs systematically in the course of the investigation of invariant lattices for the cyclic groups C_{52} and C_{78} (see the Introduction of [Neb95]).

$\boxed{\ell = 5}$ According to Nebe and Plesken [NePl95], the group $2.\mathrm{Alt}_{10}$ occurs as an irreducible maximal finite matrix group in dimension 16, leaving invariant a 5-elementary lattice of determinant 5^8 and minimum 6. A numerical verification using the Gram matrix of loc. cit. shows that this lattice is modular. In [Neb97] a more conceptual proof is given, based on the observation that a similarily between $L^\#$ and L must normalize the orthogonal group of L.

$\boxed{\ell = 6}$ Everything is completely analogous to the case $\ell = 5$. This time, the group is a quotient of $Sp_4(3) \times C_3 \times SL_2(3)$,of order $2^{10}3^6 5$. Four other extremal lattices, with groups of orders $2^4 3^6$, $2^8 3^4$, $2^6 3^4 5, 2^{10} 3^3$ have been found by computer.

$\boxed{\ell = 7}$ An extremal 7-modular lattice with minimum 6 does not exist in the minimal dimension $n = 12$ by [SchHem94].

$\boxed{\ell = 11}$ The unique 4-dimensional even unimodular lattice H_4 over $\mathbb{Q}(\sqrt{5})$ gives rise to an 8-dimensional 11-elementary lattice of minimum 6 by twisted transfer with $\frac{1+3\sqrt{5}}{2\sqrt{5}}$ of norm $\frac{11}{5}$. It follows from Lemma 1.3 that this lattice is 11-modular. It is the unique lattice with minimum 6 in its genus, and does not posses a self-dual hermitian structure over $\mathbb{Z}\left[\frac{1+\sqrt{-11}}{2}\right]$.

$\boxed{\ell = 14}$ There is a unique 4-dimensional even unimodular lattice over $\mathbb{Q}(\sqrt{2})$, called Δ_4' in the literature. Let $\pi = 3 - \sqrt{2}$ of norm 7, and consider $L = \Delta'_{4,\mathbb{Z}}$ with twisted transfer $\mathrm{tr}\left(\frac{\pi}{2} \cdot -\right)$. Since $\mathrm{tr}\left(\frac{\pi}{2}(u + v\sqrt{2})\right) = 3u - 2v$ and $|v| \le u/\sqrt{2}$ for totally positive $u + v\sqrt{2}$, it readily follows that $\min L = 6$. Furthermore, L is strongly modular: the similarily with multiplier 2 is given by multiplication with $\sqrt{2}$, the desired similarily of norm 7 is $x \mapsto \pi'\sigma(x)$, where $\beta \mapsto \beta'$ is the non-trivial automorphism of $\mathbb{Q}(\sqrt{2})$, and σ an isomorphism from Δ_4' onto its conjugate lattice (i.e. σ is antilinear and $(\sigma x, \sigma y) = (x, y)' \in \mathbb{Z}[\sqrt{2}]$ for any two $x, y \in \Delta_4'$; see Lemma 1.3 above). The orthogonal group of L is identical to $\mathrm{O}(\Delta_4')$, which is the semi-direct product of C_2 with the Weyl group $W(F_4)$, of order $2^8 3^2$. A complete enumeration of the genus $I\!I_8(2_{I\!I}^{+4}7^{+4})$ (with class number 80) shows that L is the unique lattice with minimum 6 in this genus.

$\boxed{\ell = 15}$ Even unimodular lattices Δ over $\mathbb{Z}[\sqrt{6}]$ give 15-elementary $\mathbb{Z}$-lattices by twisted transfer with $\beta = \frac{-3+2\sqrt{6}}{2\sqrt{6}}$. Since $\mathrm{tr}\left(\beta(u + v\sqrt{6})\right) = 2u - 3v$ and $u \geq \sqrt{6}|v|$ for totally positive $u + v\sqrt{6}$, it is readily checked that $\min \Delta_{\mathbb{Z}} \geq 6$ if Δ does no represent 2. By [Sch94], there exists one such $\mathbb{Z}[\sqrt{6}]$-lattice in dimension 4. Arguing as in the case $\ell = 14$, one shows strong modularity of $\Delta_{\mathbb{Z}}$. A complete enumeration of the genus $I\!I_8(3^{+4}5^{+4})$ (with class number 91) shows that it contains 4 lattices with minimum 6. Two of these are strongly modular, with orthogonal groups of order $2^4 3^2$, respectively $2^5 3^2$; the other two are modular, but not strongly modular. (The latter lattices have larger groups, of orders $2^4 3^3$ and $2^{13} 3^2$).

Description of the lattices in other dimensions.

$\boxed{\ell = 1}$ An extremal unimodular lattice in dimension 56 has been obtained by combining a construction of Ozeki with the existence of a ternary $[56, 28, 15]$-code; see [Oze89], Example 5. The "Quebbemann-Craig-lattice" $B^{(4)}_{56,1}$, self-dual over $\mathbb{Z}(\sqrt{-29})$, is another (not necessarily non-isometric) example; see [BQS95].

In dimension 64, one has the well known lattice Q_{64} of Quebbemann; see [Que84b] or [CoSl93], Chapter 8.3.

$\boxed{\ell = 2}$ Already for $n = 36$, one may suspect that a large portion of this huge genus of lattices consists of extremal ones. However, no attempt to prove a result like this is known. A systematic construction of three such lattices $B^{(m)}_{36,2}, m = 3, 4, 5$ is contained in [BQS95] ("Quebbemann-Craig-lattices"). They are self-dual over $\mathbb{Z}[\sqrt{-2}]$ and admit an automorphism of order 19. Since the number 82080 of minimal vectors in these lattices is sufficiently small and since one automorphism of large order 19 is already known, the computer program of [PlSo97] allows to calculate the orthogonal groups. The three groups are of different orders $2^4 \cdot 3^2 \cdot 5 \cdot 19$, $2^2 \cdot 3^2 \cdot 19$, $2^6 \cdot 3^3 \cdot 5 \cdot 19$ in particular, the lattices are pairwise non-isometric. Another construction [Bac97] which is partly systematic (a quaternionic code construction) and partly ad-hoc (a 2-step neighboring) also gives a lattice with the desired properties; it is very unlikely, but not completely impossible that this lattice should be isometric to one of the $B^{(m)}_{36,2}$.

In dimensions 40 and 44, an example again based on quaternions and codes is given in [Bac97]; nothing else is known.

$\boxed{\ell = 3}$ In dimensions 26, 28 and 32, such lattices have been obtained first in [Bac97], using codes. This method failed in dimensions 30 and 34. In these dimensions, A. Schiemann has constructed extremal 3-modular lattices as a self-dual hermitian lattices, using a sophisticated computer search. (The same method was succesfully applied also for the dimensions 26, 28 and 32.)

$\boxed{\ell = 6}$ Using the same tools as in the case $\ell = 3$, A. Schiemann has found a 10-dimensional self-dual hermitian lattice over $\mathbb{Z}(\sqrt{-6})$. Its transfer to $\mathbb{Z}$ turned out to be strongly modular.

$\boxed{\ell = 7}$ For $n = 14$, the existence is open. In [Schi98], A. Schiemann shows that such a lattice cannot have a self-dual hermitian structure over $\mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$. For $n = 16$, it is readily checked that the well-known unique 8-dimensional self-dual $\mathbb{Z}[\sqrt{2}]$-lattice with minimum 4 (see [HsHu89]) gives rise to a 7-elementary lattice of minimum 6 by twisted transfer $\operatorname{tr}\left(\frac{1+2\sqrt{2}}{2\sqrt{2}}\cdot -\right)$. (Notice that $\operatorname{tr}\left(\frac{1+2\sqrt{2}}{2\sqrt{2}}(u+v\sqrt{2})\right) = 2u+v$ and use some easy estimates.) A numerical verification using Gram matrices proves strong modularity. Schiemann's work reveals the surprising fact that also in this dimension, no appropriate $\mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$-lattice exists.

$\boxed{\ell = 11}$ These 10-dimensional lattices have been obtained by computer classification of the whole genus in [SchHem94].

3.4 Extremal lattices with minimum 8

The genera (possibly) containing extremal lattices of minimum 8 all have so large class numbers that a full classification is out of reach (and would probably be not very interesting). The only exception to this statement is the case $(n, \ell) = (6, 23)$, where however extremal lattices do not exist. Also, a full classification or even just a good estimation of the number of extremal lattices (if they exist) seems to be intractable. Therefore, we limit ourselves in the following table to indicate what is known about the existence of extremal lattices. "yes" means that there exists at least one extremal lattice; in all these cases, one might suspect that there exist in fact many, but so far nobody seems to have worked on the problem of producing as many as possible non-isomorphic ones. "no" means that it is proved that extremal lattices do not exist; at present, there are only two such cases. The cases $\ell = 1, 3, 7$ in dimensions $72, 36, 18$, respectively, are the most fascinating ones, since here extremal lattices would be more dense than all known lattices in these dimensions. We suspect that they do not exist. C. Bachoc and B. Venkov have announced a proof of this fact for $\ell = 7$. Based on numerical experience, we also conjecture that an extremal 11-modular extremal lattice in dimension 14 does not exist.

ℓ	1			2				3					
n	72	80	88	48	52	56	60	36	38	40	42	44	46
	?	yes		yes				?		yes			

ℓ	5		6		7			11		14	15	23
n	24	28	24	28	18	20	22	12	14	12	12	6
	yes		yes		?	yes		no		yes	yes	no

Table 3.6: Existence of extremal lattices with minimum 8

The non-existence for $\ell = 23$ is trivial, since the corresponding extremal modular form (of weight 3 and non-trivial character) reads

$$1 + 66q^4 + 24q^5 - 22q^6 + \ldots$$

and thus cannot be a theta series.

The non-existence result for $(n, \ell) = (12, 11)$ has been proved by Nebe and Venkov and is of a more sophisticated nature. In this case, the extremal modular form has non-negative coefficients "as far as one can see". It is however possible to derive from the assumed knowledge of this theta series sufficiently many restrictions on the second degree theta series of the lattice in question to succeed in showing that the corresponding space of Siegel modular forms does not contain such a series with non-negative coefficients.

$\boxed{\ell = 1}$ In dimension 80, it is not possible any more to verify by computer the minimum of an arbitrary lattice (given by a gram matrix, say). Therefore, we have to distinguish between the construction of an appropriate lattice and the verification of its minimum. A plausible candidate of an extremal unimodular lattice in dimension 80 has been given in [SPi93], essentially going back to [Que81], using an automorphism of order 79. Two candidates with automorphisms of order 41, and a unimodular hermitian structure over $\mathbb{Z}(\sqrt{-41})$ have been constructed in [BQS95]. The first construction, using codes, where the minimum could actually be verified was given by C. Bachoc; see [BaNe98]. The method is explained in [Bac97]. In this case, it gives rise to two non-isometric lattices. A different construction relates one of these to the well-known 20-dimensional extremal 7-modular lattice; see below.

$\boxed{\ell = 2}$ A 48-dimensional 2-modular lattice with minimum 8 has been constructed by Bachoc in [Bac97], Theorem 6.7. Quebbemann has observed that such a lattice can be easily obtained by the so-called η-construction from the Leech lattice with its structure over the Hurwitz order. See the end of [Bac97] for details.

$\boxed{\ell = 3}$ In [BaNe98] an extremal 40-dimensional 3-modular lattice is constructed as the tensor product over $\mathbb{Z}[\alpha]$, $\alpha = (1 + \sqrt{-7})/2$ of the 10-dimensional $\mathbb{Z}[\alpha]$-lattice mentioned below in the case $\ell = 7$, with a maximal order in the quaternion algebra $\mathbb{Q}_{3,\infty}$ ramified at the places 3 and ∞. A code construction for the same lattice is also given.

$\boxed{\ell = 5}$ There is a well known, in fact unique, 12-dimensional even unimodular lattice with minimum 4 over $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$; see [CoHs87]. By ordinary transfer, this gives a 24-dimensional 5-modular lattice with minimum 8.

$\boxed{\ell = 6}$ Two lattices with minimum 8 in the appropriate 24-dimensional genus are exhibited in [Neb95] (numbers 16 and 17). In [Neb97] it is shown that they are strongly modular.

$\boxed{\ell = 7}$ A 10-dimensional self-dual hermitian $\mathbb{Z}[\alpha]$-lattice, $\alpha = (1 + \sqrt{-7})/2$, giving rise to a 20-dimensional 7-modular $\mathbb{Z}$-lattice with minimum 8 and automorphism group $2.M_{22}.2$ has already been described in the Atlas [Con85].

$\boxed{\ell = 14}$ A 12-dimensional lattice of determinant $2^6 7^6$ and minimum 8, invariant under the group $L_2(7) \times D_8$ is exhibited in section VII, p. 36 of [PlNe95]. Strong modularity can be verified using the Gram matrix given on p. 64 of [PlNe95].

$\boxed{\ell = 15}$ From the theory of complex reflection groups [Coh76], p. 408, the following unimodular hermitian lattice over $\mathbb{Z}[\omega, \varepsilon_5]$, $\omega = \frac{-1+\sqrt{-3}}{2}$, $\varepsilon_5 = \frac{1+\sqrt{5}}{2}$ with minimum 2 is known:

$$J_3(5) \cong \begin{pmatrix} 2 & -\overline{\omega}\varepsilon_5 & -1 \\ -\omega\varepsilon_5 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Ordinary transfer gives a 12-dimensional 15-elementary $\mathbb{Z}$-lattice which is clearly strongly modular (use $\sqrt{-3}$ and $\sqrt{5}$) and has minimum 8. This lattice is isometric to the one presented in section VII, p. 36 of [PlNe95]. One further lattice of minimum 8, which also turned out to be strongly modular, was found by computer search.

4 Further genera containing extremal lattices

In this section we give numerical results on those strongly modular genera of lattices of square free level $\ell \leq 21$ which are not covered by Section 3. These are the levels 10, 13, 17 and 21 and part of the genera of levels 6, 14 and 15. For small enough dimensions it is shown by explicit consideration of the Atkin-Lehner action on modular forms that extremality is definable. Various extremal modular forms are calculated, and results on the existence of extremal lattices similar in spirit to those of Section 3 are given. It turns out that a certain lattice of dimension 16, level 21 and minimum 12 occurring in the work of Nebe and Plesken [NePl95] is extremal.

$\boxed{\ell = 6}$ After Section 3, only the genera $I\!I_{2k}(2_{I\!I}^{-k}3^{-k})$ with $k \equiv 0(4)$ remain to be considered. So the character values are $\chi = (--)$, and the dimension formula 1.4 gives

$$\dim \mathcal{S}_k(6, --) = \frac{k}{4} - 1\,.$$

Dimension 8. Since $\dim \mathcal{S}_4(6, --) = 0$, extremality is definable and strongly modular lattices of minimum 2 in $I\!I_8(2_{I\!I}^{-4}3^{-4})$ are already extremal. There is exactly one such lattice (the class number of the whole genus is 5).

Dimension 16. The essentially unique non-zero cusp form in the one-dimensional space $\mathcal{S}_8(6, --)$ has non-zero first coefficient. Therefore extremality is definable, the extremal modular form is

$$F_{8,6,--} = [1, 0, 42, 832, 5754, \ldots]\,,$$

(writing the coefficients of $1, q, q^2, \ldots$ as a row vector). No lattice with this theta series could be found.

$\boxed{\ell = 10}$ In each dimension $2k$ with even k, there exist two genera $I\!I_{2k}(2_{I\!I}^{\pm k}5^{\pm k})$ of totally even 10-elementary lattices of determinant 10^k. The signs of the genus symbol coincide with the character values. Recall the dimension formula Proposition 1.5 for $\mathcal{S}_k(10, \chi)$.

Dimension 4. Both genera have class number 1, there are no non-zero cusp forms. The respective unique lattice is necessarily strongly-modular and extremal.

Dimension 8. The genus $I\!I_8(2_{I\!I}^{4}5^{4})$ has class number 24, the genus $I\!I_8(2_{I\!I}^{-4}5^{-4})$ has class number 19, and 8, respectively 3 of the lattices are strongly modular. In both cases, the space of cusp forms is one-dimensional, so strongly modular lattices of minimum 4 are extremal. There are 3, respectively 1 such lattices. Their theta series are $[1, 0, 24, 48, 216, 288, 672, \ldots]$, respectively $[1, 0, 10, 120, 110, 320, 520, \ldots]$.

Dimension 12. Both genera are too large to be classified completely.

In the case of $I\!I_{12}(2_{I\!I}^{6}5^{-6})$, the space of cusp forms is two dimensional, extremality is definable, and therefore strongly modular lattices of minimum 6 would be extremal. No such lattice is known. Extensive computer search has produced only two lattices with minimum 6. They are 10-modular and transformed into each other by D_2 (or D_5). For $I\!I_{12}(2_{I\!I}^{-6}5^{6})$, the space of cusp forms is one-dimensional, extremality is definable, and extremal lattices (of minimum 4) exist. In fact, there is an abundance of minimum 4 lattices in this genus, with various different theta series, therefore it was helpful for the search to calculate first the extremal modular form

$$F_{6,10,-+} = [1, 0, 12, 136, 444, 1872, \ldots] .$$

Dimension 16. For the genus $I\!I_{16}(2_{I\!I}^{8} 5^{8})$, the space of cusp forms is 3-dimensional. This means that the minimum 8 lattice in this genus given in [NePl95] (see [Neb97] for strong modularity) is actually extremal. Two further such lattices (with smaller groups of orders $2^{10} \cdot 3$ and $2^9 \cdot 3^2$) have been found by computer search.

In the case of $I\!I_{16}(2_{I\!I}^{-8} 5^{-8})$, the space of cusp forms is two dimensional. The extremal modular form is

$$F_{8,10,--} = [1, 0, 0, 100, 770, 3968, \ldots] ,$$

no lattice with this theta series has been found.

$\boxed{\ell = 13}$ Even 13-elementary lattices of dimension $n = 2k$ and determinant 13^k exist only in dimensions divisible by 4, and the relevant modular forms belong to the trivial character. The dimension formula Proposition 1.4 applies.

Dimension 4. We consider the genus $I\!I_4(13^{-2})$, with character value $\delta = -$. The dimension formula Proposition 1.4 gives $\dim \mathcal{M}_2(13, -) = 0$, i.e. there are no non-zero cusp forms. Thus extremality is definable, and extremal lattices have minimum 2. The class number of the genus is 1, the unique lattice is of course modular and has minimum 2.

Dimension 8. We have to consider modular forms of weight 4 and trivial character $\chi = \chi_0$. The dimension of this space is known from Proposition 1.4: $\dim \mathcal{M}_4(13, +) = 3$. The genus theta series (see [Kri95]) reads

$$\Theta_{4,13,+} = \frac{1}{170}(E_4(q) + 169 E_4(q^{13})) = 1 + \frac{24}{17}(q + 9q^2 + 28q^3 + 73q^4 + \ldots) .$$

A basis of the space of cusp forms is given by the following two newforms:

$$\begin{array}{rcl} C_1 & = & [0, 1, 0, 4, -4, \ldots] \\ C_2 & = & [0, 0, 1, -3, 1, \ldots] \end{array}$$

Clearly, extremality is definable, and extremal lattices are the modular lattices with minimum 6. The extremal modular form is

$$F_{4,13,+} = \Theta_{4,13,+} - \frac{24}{17} C_1 - \frac{24 \cdot 9}{17} C_2 = [1, 0, 0, 72, 96, \ldots].$$

Complete enumeration of the genus $I\!I_8(13^{+4})$, with class number 37, shows that there is exactly one lattice with minimum 6. This lattice is modular, and thus modular extremal.

$\boxed{\ell = 14}$ The genera $I\!I_{2k}(2_{I\!I}^{k} 7^{k})$ with even k have already been treated in Section 3 (existence of strongly modular extremal lattices is known for dimensions 4, 8 and 12). In the cases $I\!I_{2k}(2_{I\!I}^{-k} 7^{-k})$, nothing is to be done since strongly modular lattices do not exist, according to the result of Rains and Sloane mentioned in Remark 3 of Section 2.

$\boxed{\ell = 15}$ The genera $I\!I_{2k}(3^{\varepsilon k}5^{\varepsilon k})$ with even k and $\varepsilon = (-1)^{k/2}$ have already been treated in Section 3. For the genera with $\varepsilon = -(-1)^{k/2}$, we have the following partial, less interesting results.

Dimension 4. For $I\!I_4(3^2 5^2)$, with character values $\chi = (-+)$, the space of cusp forms is trivial, the class number is 2, no strongly modular lattice exists.

Dimension 8. This case has been treated in the example after Definition 1.7.

Dimension 12. For $I\!I_{12}(3^6 5^6)$, with character values $\chi = (-+)$, the space of cusp forms is 2-dimensional, extremality is definable. The extremal modular form is

$$F_{6,15,++} = [1, 0, 132, 156, 552, 1056, \ldots] .$$

Eight lattices with this theta series have been found, six of them are modular, none is strongly modular.

$\boxed{\ell = 17}$ Like in the case $\ell = 13$, lattices of the type considered exist only in dimensions divisible by 4, and the relevant modular forms belong to the trivial character. The dimension formula Proposition 1.4 applies.

Dimension 4. The corresponding space of modular forms has dimension $\dim \mathcal{M}_2(17, +) = 2$. A cusp form has a non-zero first coefficient. Thus extremality is definable, and extremal lattices have minimum 4. The class number of $I\!I_4(17^{+2})$ is 3, there is a unique lattice with minimum 4 which is modular and thus modular extremal.

Dimension 8. The dimension formula gives $\dim \mathcal{M}_4(17, +) = 4$. The genus theta series reads

$$\Theta_{4,17,+} = 1 + \frac{24}{29}[0, 1, 9, 8, 73, 126, 252, \ldots]$$

As a basis of the cusp forms, one can take

$$\begin{aligned} C_1 &= [0, 1, -10, 12, 38, -2, 84, \ldots] \\ C_2 &= [0, 0, -2, 1, 10, -2, -20, \ldots] \\ C_3 &= [0, 0, 0, -1, -3, 0, 6, \ldots]. \end{aligned}$$

Because of the triangular form of this "matrix", extremality is definable. The coefficient of q^6 of the extremal modular form $F_{4,17,+}$ is calculated as -48, thus extremal lattices cannot exist.

$\boxed{\ell = 19}$ The genus $I\!I_{2k}(19^{(-1)^k k})$ exists and is modular for all even dimensions $2k$ (this of course holds for any prime $\ell \equiv 3 \bmod 4$). We have checked that extremality is definable for $k = 1, \ldots, 5$, and calculated the dimension $\dim \mathcal{S}_k(19, (-1)^k) = 0, 1, 2, 3, 3$ respectively (for even k, see Proposition 1.4). Complete enumeration of the genus is possible for $k \leq 4$, with class numbers $1, 3, 11, 181$. It turns out that in each case, there is a unique lattice with minimum $2, 4, 6, 8$ respectively. This lattice is modular and thus extremal. In dimension 10, we were unable to find a lattice of minimum 8.

$\boxed{\ell = 21}$ For each dimension $2k$ with even k there are two appropriate genera of lattices $I\!I_{2k}(3^{\varepsilon k}7^{\delta k})$. For $k \equiv 2 \bmod 4$, the character values are opposite to the signs of the genus symbol: $\chi(W_3) = -\varepsilon$, $\chi(W_7) = -\delta$. The dimension formula Proposition

1.5 applies; for the values $k = 2, 4, 6, 8$, we have checked that extremality is definable. In the following, we write d_χ instead of $\dim \mathcal{S}_k(21,\chi)$ for short.

Dimension 4. For the genus $I\!I_4(3^{-2}7^2)$, we have $d_\chi = 0$, so extremality is trivially definable. However, no strongly modular lattice exists. (By the way, the class number is 2, and one of the two lattices has minimum 4.) In the case of $I\!I_4(3^2 7^{-2})$, we have $d_\chi = 1$, the class number is 3, all lattices are strongly modular, one of them has minimum 4 and is thus extremal.

Dimension 8. The class number of $I\!I_8(3^4 7^4)$ is 305, and 45 lattices are strongly modular. The dimension $d_\chi = 2$, so the strongly modular lattices of minimum 6 are extremal; there are 6 such lattices. This genus by the way contains one lattice of minimum 8, which is necessarily not strongly modular. The class number of $I\!I_8(3^{-4}7^{-4})$ is 245, exactly one of these lattices is strongly modular. This lattice has minimum 4. On the other hand, the dimension formula gives $d_\chi = 2$ again, so extremal lattices would have minimum 6, and no extremal lattice exists. (The genus contains 14 lattices of minimum 6 and no lattice with larger minimum.)

Dimension 12. Evaluation of the dimension formula shows that extremality would require minimum 8 for the genus $I\!I_{12}(3^{-6}7^6)$ and minimum 10 for $I\!I_{12}(3^6 7^{-6})$. The extremal modular forms are

$$F_{6,21,-+} = [1, 0, 0, 0, 64, 194, 474, 864, \ldots] .$$

$$F_{6,21,+-} = [1, 0, 0, 0, 0, 378, 504, 756, \ldots] .$$

In the first case many lattices with minimum 8 are known, but no one with the desired theta series. In the second case, the largest known minimum in the genus is 8, and not 10.

Dimension 16. For the plus-genus, one has $d_\chi = 5$, and there exists indeed a strongly modular lattice of minimum 12 in this genus, by [NePl95]. No other lattice of minimum 12 has been found in this genus. For the minus-genus, the dimension is $d_\chi = 4$, but no lattice of minimum 10 could be found.

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