# Theta liftings and theta series[1]

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**Introduction.**

Theta liftings have been considered both from a classical and from a representation theoretic point of view. In the classical setting, one considers (mainly holomorphic) theta series attached to (mainly positive definite) integral quadratic forms as Siegel (or Hilbert-Siegel) modular forms. One is then interested in a description of the linear relations between the members of a given set of such theta series and in a characterization of the space of modular forms spanned by these theta series. The theta series in such a set are usually quite restricted in type, e.g., they belong to full lattices of some fixed level or they are theta series with characteristic (thetanullwerte) attached to a single lattice but with varying characteristic. The representation theoretic approach considers the more general theta correspondence between automorphic forms on adelic orthogonal and symplectic (or metaplectic) groups defined using the oscillator (or Weil-) representation of the metaplectic group. Here one discusses for an irreducible representation space of automorphic forms on one of the groups whether it is in the image under the theta correspondence of a representation space of automorphic forms on the other group respectively whether its image under the correspondence is zero or not.

Although both types of question appear to be extremely similar, they are not quite the same. It is the purpose of this note to discuss some cases in which a transfer of results between the two settings is made possible by recent results of Moeglin and to describe some of the difficulties that occur in other cases.

**1. The problems**

Let $(V,q)$ be a non-degenerate quadratic space of even dimension $m=2k$ over $\mathbf{Q}$, denote by $B(x,y)=q(x+y)-q(x)-q(y)$ the associated symmetric bilinear form, let $L\subseteq V$ be an integral $\mathbf{Z}$-lattice of full rank on $V$ (i.e., $q(L)\subseteq \mathbf{Z}$, $\mathbf{Q}\otimes L\cong V$) of level $N$ (i.e., $q(L^\#)\mathbf{Z}=N^{-1}\mathbf{Z}$, where $L^\#=\{y\in V \mid B(y,L)\subseteq \mathbf{Z}\}$ is the dual lattice of $L$).
We denote the orthogonal group of the space $(V,q)$ (viewed as an algebraic group over $\mathbf{Q}$) by $H$ or $H_V$. The adelic orthogonal group $H(\mathbf{A})$ operates on the set of lattices on $V$ by $L\mapsto hL$ for $h=(h_p)_p\in H(\mathbf{A})$, where $(hL)\otimes\mathbf{Z}_p=h_p(L\otimes\mathbf{Z}_p)$. The stabilizer of $L$ under this action is $O_{\mathbf{A}}(L)$, its finite part is a compact open subgroup of the finite adeles.
The genus $\mathrm{gen}(L)$ of $L$ is the $H(\mathbf{A})-$ orbit of $L$, it consists of all lattices $K$ on $V$ with $L\otimes\mathbf{Z}_p$ isometric with respect to $q$ to $K\otimes\mathbf{Z}_p$ for all primes $p$; the number of isometry classes of lattices in a genus is finite.
For a double coset decomposition

$$H(\mathbf{A})=\dot{\bigcup}_{i=1}^{t} H(\mathbf{Q})h_iO_{\mathbf{A}}(L).$$

the $L_i:=h_iL$ give a set of representatives of these isometry classes.
We restrict attention to positive definite $q$, let $\mathfrak{H}_n$ denote the Siegel upper half space of degree (or genus) $n$, put

$$q(\mathbf{x})=(\frac{1}{2}B(x_i,x_j))\in M_n^{\mathrm{sym}}(\mathbf{Q})\qquad \text{for } \mathbf{x}=(x_1,\dots,x_n)\in V^n$$

and consider the theta series

$$\vartheta_L^{(n)}(Z)=\sum_{\mathbf{x}\in L^n}\exp(2\pi i\ \mathrm{tr}(q(\mathbf{x})Z)).$$

[1]Expanded version of talks at IAS Princeton Nov. 23, 1999 and at RIMS Kyoto Jan. 1998

Inhomogeneous theta series $\vartheta^{(n)}_{L,\mathbf{y}}(Z)$ are defined in the same way, but with summation over a coset $\mathbf{y}+L^n$ with $\mathbf{y}\in V^n$. They are sometimes also called theta series with a characteristic or theta series with congruence conditions.
More generally we can consider a harmonic form $P: V^n\otimes\mathbf{R}\longrightarrow W_\rho$ with values in some irreducible representation space $W_\rho$ of $\mathrm{GL}_n(\mathbf{C})$ and put

$$\vartheta^{(n)}_{L,\mathbf{y}}(P,z)=\sum_{\mathbf{x}\in\mathbf{y}+L^n}P(\mathbf{x})\exp(2\pi i\ \mathrm{tr}(q(\mathbf{x})Z))$$

[17] (we view here a harmonic form that is initially given as a function on the space of matrices $M_{mn}(\mathbf{R})$ as a function on $V^n\otimes\mathbf{R}$ by identifying the latter space with $M_{mn}(\mathbf{R})$ using an arbitrary fixed basis of $V\otimes\mathbf{R}$ that is orthonormal with respect to $q$) .

For $\mathbf{y}=\mathbf{0}$, $\vartheta^{(n)}_L(P,Z)$ is a $W_\rho$-valued Siegel modular form for $\Gamma^{(n)}_0(N)$ with character $\chi$ depending on the discriminant of $L$, for the inhomogenous theta series the transformation behaviour is slightly more complicated, see [17].

We want to consider here the following classical questions:

A) Find the linear relations between the $\vartheta^{(n)}_{L_i}$ for $L_i$ running through a set of representatives of the isometry classes of lattices in the genus of the lattice $L$.

B) Characterize the space of modular forms generated by the $\vartheta^{(n)}_{L_i}$ for $L_i$ as above.

The question B) (the basis problem) has been considered in many variants, for example

B') Let $N,\chi$ be given. Is the space $M^{(n)}_k(\Gamma_0(N),\chi)$ of modular forms of degree $n$, weight $k$ and character $\chi$ (or its subspace of cusp forms) generated by

a) theta series (maybe inhomogeneous) attached to (arbitrary) lattices with quadratic form
b) by theta series attached to full lattices
c) by theta series attached to full lattices of level $N$ or even in a fixed genus of lattices
d) by theta series with spherical harmonics as in any of the above points a)-c)

In each of these subproblems, determine an explicit representation of a given $F\in M^{(n)}_k(\Gamma_0(N),\chi)$.

Before describing some results on these problems let us look at the representation theoretic point of view:

Let $(U^{(n)},A)$ denote a $2n$-dimensional space over $\mathbf{Q}$ with non-degenerate alternating form $A$, $\psi:\mathbf{Q}_\mathbf{A}/\mathbf{Q}\longrightarrow S^1$ a nontrivial additive character. The symplectic group $G=G_n=\mathrm{Sp}(U^{(n)},A)=\mathrm{Sp}(n)$ and the orthogonal group $H$ form a dual reductive pair in the sense of Howe. Let $\omega=\omega_\psi$ denote the Weil representation of the adelic group $G(\mathbf{A})\times H(\mathbf{A})$ on the space $S(V(\mathbf{A})^n)$ of Schwartz-Bruhat functions on $V(\mathbf{A})^n$. For $f\in S(V(\mathbf{A})^n)$, $g\in G(\mathbf{A})$, $h\in H(\mathbf{A})$ define the theta kernel

$$\theta^{(V,n)}(g,h;f)=\theta(g,h;f):=\sum_{\mathbf{x}\in V(\mathbf{Q})^n}\omega(g)f(h^{-1}\mathbf{x}).$$

For a space $Y$ of cuspidal automorphic forms on $G(\mathbf{A})$ we write $\Theta^V(Y)$ for the space of theta lifts

$$\Theta_f^V(\varphi')(h) := \int_{G(\mathbf{Q})\backslash G(\mathbf{A})} \varphi'(g)\theta^{(V,n)}(g,h;f)dg$$

with $\varphi' \in Y$, $f \in S(V(\mathbf{A})^n)$, similarly for a space $X$ of automorphic forms on $H(\mathbf{A})$ we write $\Theta^U(X) = \Theta^n(X)$ for the space of theta lifts

$$\Theta_f^{(n)}(\varphi)(g) := \int_{H(\mathbf{Q})\backslash H(\mathbf{A})} \varphi(h)\theta^{(V,n)}(g,h;f)dh$$

of functions $f \in X$ with respect to $f \in S(V(\mathbf{A})^n)$ to automorphic forms $\Theta_f(\varphi)$ on $G(\mathbf{A})$. These theta lifts are well known to be automorphic forms on the respective groups.
We have then the questions:

- $\tilde{\mathrm{A}}$) Given a (cuspidal) irreducible automorphic representation $\pi$ of $H(\mathbf{A})$, find the first $n$ for which $\Theta^{(n)}(\pi)$ is nonzero.
- $\tilde{\mathrm{B}}$) Given a cuspidal irreducible automorphic representation $\pi'$ of $G_n(\mathbf{A})$, decide for given $(V, q)$ whether $\pi'$ is a lift of some representation $\pi$ of $H_V(\mathbf{A})$ as above .

To connect both setups we consider a lattice $L$ on $V$ with adelic orthogonal group $O_{\mathbf{A}}(L) \subseteq H(\mathbf{A})$ and double coset decomposition

$$H(\mathbf{A}) = \dot{\bigcup}_{i=1}^{t} H(\mathbf{Q})h_i O_{\mathbf{A}}(L)$$

as above. We let $\mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$ be the space of functions $\varphi:\ H(\mathbf{A}) \longrightarrow \mathbf{C}$ with $\varphi(\gamma h u) = \varphi(h)$ for $\gamma \in H(\mathbf{Q})$, $u \in O_{\mathbf{A}}(L)$. These functions are of course determined by their values at the finitely many $h_i$. On the space $\mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$ one has a natural action of the local Hecke algebras with respect to the compact open subgroups $O(L \otimes \mathbf{Z}_p)$; this action is expressed explicitly by Eichler's Anzahlmatrices (generalized Brandt matrices) and allows us to find a basis of eigenforms for all local Hecke algebras for the primes $p$ not dividing the level $N$ of the lattice.
Let $f_L^{(n)} \in S(V(\mathbf{A})^n)$ be given as $f_L^{(n)} = \prod_p f_p^{(n)}$ with $f_p^{(n)} = \mathbf{1}_{L_p^n}$ for finite $p$ and

$$f_\infty^{(n)}(\mathbf{x}) = \exp(-2\pi \mathrm{trace}(q(\mathbf{x}))).$$

Then for $\varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$ and $g = (g_\infty, 1, \dots)$ we have

$$\Theta_{f_L^{(n)}}^{(n)}(\varphi)(g) = \sum_{j=1}^{t} \frac{\varphi(h_j)}{|O(h_j L)|} \sum_{\mathbf{x} \in (h_j L)^n} \omega(g_\infty) \exp(-2\pi \cdot \mathrm{tr}(q(\mathbf{x}))),$$

and under the usual correspondence of automorphic forms on $G(\mathbf{A})$ and on the Siegel half space $\mathfrak{H}_n$ of degree $n$ this function corresponds to

$$\Theta_L^{(n)}(\varphi)(Z) := \Theta_{f_L^{(n)}}^{(n)}(\varphi)(Z) = \sum_{j=1}^{h} \frac{\varphi(h_j)}{|O(h_j L)|} \vartheta^{(n)}(h_j L, Z),$$

where $L_j := h_j L$ runs through a set of representatives of the isometry classes in the genus of $L$ (we identify in the sequel the form $\Theta^{(n)}_{f_L^{(n)}}(\varphi)(g)$ on the adelic symplectic group and the corresponding Siegel modular form and denote both of them by $\Theta^{(n)}_L(\varphi)$). For $\varphi = 1$ this yields Siegel's weighted average.
We obtain in this way all linear combinations of the theta series of the lattices in the given genus. It is convenient to restrict attention to linear combinations that arise in this way from a Hecke eigenform $\varphi$ in $\mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$; this allows in particular to attach Satake parameters and an automorphic $L$-function $\mathcal{L}(\varphi, s)$ (the standard $L$-function of the orthogonal group) to $\varphi$.
Similar expressions involving theta series with harmonic forms arise for $\varphi$ from a space of vector valued functions on $O_{\mathbf{A}}(V)$ that are right invariant under $\prod_{p\neq\infty} O(L_p)$ and transform according to some fixed irreducible representation of $H(\mathbf{R})$ under right translation by elements of $H(\mathbf{R})$, see [17, 7, 8]. To obtain inhomogeneous theta series one has to replace the characteristic function of the local lattice by the characteristic function of the coset considered.

On the other hand, given $\varphi'$ on $G(\mathbf{A})$ that corresponds to a Siegel modular cusp form $F$ of weight $k = \frac{m}{2}$ we have

$$\Theta^{V}_{f_L^{(n)}}(\varphi')(h_j) = C \cdot \langle F, \vartheta^{(n)}(L_j)\rangle$$

with some constant $C \neq 0$ depending on the normalization chosen and with $\langle\ ,\ \rangle$ denoting the Petersson inner product. In particular, this theta lifting of $\varphi'$ is zero if and only if the form $F$ is orthogonal to the space generated by the theta series of the relevant type.

At this point we summarize some of the known results concerning the problems discussed so far.

A)

- By a result of Kitaoka [24] the theta series of degree $n$ of the lattices in a genus of positive definite quadratic forms in $m \leq n+1$ variables are linearly independent (the case of $m \leq n$ is essentially trivial). This has been generalized to theta series with spherical harmonics in [8]. The general picture is that linear relations with $n \geq m/2$ are more or less rare and interesting.
- $N = 1$ ($L$ even unimodular):
  Böcherer (1990) [4] characterized for $n \in \mathbf{N}$ the Hecke eigenforms $\varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$ such that $\Theta^{(n)}(\varphi) \neq 0$ is cuspidal (equivalently, $\Theta^{(r)}(\varphi) = 0$ for $r \leq n$) in terms of zeros and poles of $L(\varphi, s)$, using earlier results [3] on the standard $L$- functions of Siegel cusp eigenforms in the space spanned by the theta series of even unimodular lattices.

  For example, for $n = m/2$ the (necessary and sufficient) condition is that $L(\varphi, s)$ has neither zeroes nor poles for $s \geq 1, s \in \mathbf{N}$. The fact that both zeroes and poles play a role here comes from the appearance of quotients of standard $L$-functions in the description of the intertwining operators in Langlands' theory of Eisenstein series.
- $N$ square free, $m = 4, n = 2$, discriminant of $(L, q)$ equal to $N^2$: In [5] (1991) the (new) eigenforms $\varphi$ with $\Theta^{(2)}(\varphi) = 0$ were put in one-to-one correspondence with the newforms $f \in S_1^2(\Gamma_0(N))$ with vanishing central value of the Hecke $L$-function. As an arithmetic corollary it could be shown that the theta series of degree 2 of these lattices characterize the isometry class.
- In the case $m = 3, n = 1$ (that does not fit into our general condition that $m$ should be even) similar results were obtained as classical versions of a theorem of Waldspurger in [19, 6] (1987,1992).

Ã) – From work of Rallis [31] and Li [28] it follows that for $n \geq m$ the theta lift of any given cuspidal $\pi$ is nonzero (the quadratic form considered is arbitrary non-degenerate in all of Ã) ).

– Moeglin [29](1998) calculated for given $\pi$ on the orthogonal side the first $n$ for which $\Theta^{(n)}(\pi \otimes \chi) \neq 0$ holds for some character $\chi$ on the adelic orthogonal group that is trivial on the elements of determinant 1 (she actually proves a more general result that allows to build in an additional character on the spinor norms). The characterization is in terms of poles of Eisenstein series associated to $\pi$ on groups $H_{V'}(\mathbf{A})$ where $V'$ is in the Witt tower of $V$, i. e., it is a direct sum of $V$ and a direct sum of hyperbolic planes. Her work builds on earlier work of Kudla and Rallis [26] and on the study of Eisenstein series in a sequence of papers by Moeglin and Moeglin/Waldspurger.

– Roberts (1999) [34] proved for $n \geq m/2$ and locally everywhere tempered $\pi$ a sufficient condition for the non-vanishing of $\Theta^{(n)}(\pi)$ (non-vanishing of $L^S(\varphi, s)$ at $s = n+1-(m/2)$ plus further local conditions, in $L^S(\varphi, s)$ the places in the set $S$ of bad places are omitted from the Euler product). If his condition is satisfied than so is Moeglin's condition, but not vice versa, due to the existence of the so called "zero parasites" of $L^S(\varphi, s)$, see [29].

B), B') – For $n = N = 1$ and weight divisible by 4 Hecke proved [20] that all modular forms can be expressed as linear combinations of theta series attached to even unimodular lattices (case c) from above).

– For $n = 1$ and prime level Eichler proved in 1956 [13] that for trivial character and weight 2 all cusp forms can be expressed as linear combinations of theta series attached to the ideals of maximal orders in a quaternion division algebra. This was generalized by Eichler and others to include higher weights using spherical harmonics and to include other levels and nontrivial characters, see [14, 15, 35, 21] and the references given there. The final result is in [21], where it is shown that one gets everything if one uses primitive nebentype forms and their twists in addition to the theta series attached to the ideals of certain orders in definite quaternion division algebras. In all cases the level of the lattices used is the level of the modular form represented or a divisor thereof (cases c), d) from above).

– For $n = 1$ Waldspurger [40] proved 1977 for arbitrary level $N$ and even weight $k \geq m$ with either $8 \mid m$ or $m \geq 12$ and $4 \mid m$ that all cusp forms of weight $k$ and trivial character can be represented by theta series with spherical harmonics attached to lattices of rank $m$ and of level dividing $N$. For primitive quadratic nebentype he proved that the Hecke closure of the appropriate space of theta series generates the space of cusp forms; it is not known in this case whether the space of theta series in question is closed under the action of the Hecke operators (cases b), d) from above).

– For $n > m$ (singular weights) Freitag [16] proved that all modular forms for $\Gamma_0^{(n)}(N)$ are linear combinations of theta series of lattices of level $N$, for $n = m$ see [1] .

– For $N = 1$, $n$ arbitrary Böcherer proved [3] that a Hecke eigenform $F$ of weight $k$ and degree $n$ is a linear combination of theta series of even unimodular lattices of rank $2k$ if and only if its standard $L$-function has no zero at $(m/2) - n$ if $m/2 \geq n$ resp. if it has a simple pole at $n + 1 - (m/2)$ if $n > m/2$. This condition is satisfied for all $F$ if $n \leq (m/4)$. He proved a similar result for theta series with harmonic forms, see also work of Weissauer [41].

– For $N$ square-free, $m/2 \leq n \leq m$ it was proved in [5] that a Hecke eigenform $F$ of level $N$ and trivial character can be expressed by theta series of lattices of level $N$ if the standard $L$-

function (as a product over the primes not dividing $N$) has a simple pole at $s = n + 1 - (m/2)$ and some additional technical conditions are satisfied.

– Igusa and Salvati Manni [22, 36, 37, 38] studied spaces of modular forms spanned by products of "Thetanullwerte", i.e., special inhomogenous theta series (case a) from above). An interesting new result in the case $n = 1$ is in [10].

$\tilde{\mathrm{B}}$) We will seee below that results of Moeglin imply that problem $\tilde{\mathrm{B}}$ is equivalent to $\tilde{\mathrm{A}}$ with the roles of the symplectic and orthogonal group reversed, i.e., it is equivalent to asking whether the given cuspidal representation $\pi'$ of $G_n(\mathbf{A})$ has a nonzero cuspidal lifting to the given orthogonal group. We have here again results by Kudla and Rallis [26] giving a sufficient condition in terms of the behaviour of the standard $L$-function for $\pi'$ to have a nonzero lifting to some $V$ of prescribed discriminant (but not necessarily a given special $V$); these have been sharpened by Moeglin [29] to a (necessary and sufficient) characterization by poles of Eisenstein series attached to $\pi'$ on symplectic groups of higher rank.

Our questions concerning the relation between results in the classical and in the representation theoretic setting are then

?A) Let $\pi$ be an irreducible cuspidal representation of $H(\mathbf{A})$. Assume that we can prove for a special lattice $L$ on $V$ and for a nonzero form $\varphi$ in $\mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L)) \cap \pi$ that $\Theta_L^{(n)}(\varphi) = 0$ holds, i. e., we have a classical vanishing result. We want to know whether this already implies the representation theoretic vanishing $\Theta^{(n)}(\pi) = \{0\}$.

Equivalently we may assume that we have $\pi \cap \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L)) \neq \{0\}$ and $\Theta^{(n)}(\pi) \neq \{0\}$ and study the question under which conditions on $\varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L)) \cap \pi$ and on $\pi$ we can conclude that

$$\Theta_L^{(n)}(\varphi) \neq 0.$$

?B) Let $\pi'$ be an irreducible cuspidal automorphic representation of $G_n(\mathbf{A})$ such that $\pi' = \Theta^{(n)}(\pi)$ for some $\pi$ of $H(\mathbf{A})$ and let $F$ be a Siegel modular form of level $N$ and weight $k = \frac{m}{2}$ corresponding to some $\varphi' \in \pi'$. Then there exist $\varphi \in \pi$ and $f \in S(V_{\mathbf{A}}^n)$ with

$$\varphi' = \Theta_f^{(n)}(\varphi).$$

In classical terms this amounts to saying that $F$ can be written as a linear combination of some possibly inhomogenous theta series attached to some possibly quite horrible lattices on the space in question, that is, we have a solution to B' a) in its vaguest form from our list above. We want, however, to find such a linear combination in a more restricted class of theta series (for example in order to compute the Fourier coefficients explicitly), see B' c) above.

We ask therefore under which conditions on $F$ and on $\pi'$ we can conclude that $F$ can more specifically be written as $F = \Theta_L^{(n)}(\varphi)(Z)$ for some lattice $L$ on $V$ of level $N$ and $\varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$.

## 2. Results

Our main tool will be the following theorem of Moeglin [30] (which is valid for arbitrary non-degenerate $q$ on $V$):

**Theorem (Moeglin).** *Let $\pi$ be an irreducible subrepresentation of the space of cuspidal automorphic* *forms on the adelic orthogonal group $H(\mathbf{A})$ and suppose that $\Theta^{(n)}(\pi)$ contains a nonzero cusp form. Then $\Theta^{(n)}(\pi)$ is an irreducible cuspidal representation, and one has $\Theta^V(\Theta^{(n)}(\pi)) = \pi$. Conversely for $\pi'$ a cuspidal irreducible automorphic representation of $G(\mathbf{A})$, $\Theta^{(V)}(\pi')$ is irreducible cuspidal if it contains a nonzero cusp form and*

$$\Theta^{(n)}\Theta^V(\pi') = \pi'.$$

*In particular, $\pi'$ is the image under the theta correspondence of a cuspidal automorphic representation $\pi$ of $H_V(\mathbf{A})$ (problem $\tilde{B}$) if and only if $\Theta_V(\pi')$ contains a nonzero cusp form.*

**Remark.** For our present situation of positive definite $V$ the cuspidality assumption is vacuous for forms on $H(\mathbf{A})$ and can hence be omitted.

**Proposition 1.** *Let $\pi$ be an irreducible subrepresentation of the space of automorphic forms on $H(\mathbf{A})$ such that $\Theta^{(n)}(\pi)$ contains a nonzero cusp form. Let $L$ be a lattice on $V$ satisfying*

> $(*)$ *For all $r \in \mathbf{N}$ and for all $\varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$ the Siegel modular form $\Theta_L^{(r)}(\varphi)$ is cuspidal if and only if its image under Siegel's $\Phi$-operator is zero (equivalently: For all $r \in \mathbf{N}$ and for all $\varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$ with $\Theta_L^{(r)}(\varphi) = 0$ the lift $\Theta_L^{(r+1)}(\varphi)$ is cuspidal).*

*Then $\Theta_L^{(n)}(\varphi) \neq 0$ for all $\varphi \neq 0$ in $\mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L)) \cap \pi$.*

*Proof.* It is known [31] that there is $r$ with $\Theta^{(r)}(\pi) \neq \{\mathbf{0}\}$ and that for the smallest such $r$ the lifting $\Theta^{(r)}(\pi)$ is cuspidal, $\Theta^{(r')}(\pi)$ non-cuspidal for $r' > r$. Also, the linear independence of theta series of degree bigger than the dimension implies that $\Theta_L^{(s)}(\varphi) \neq 0$ for some $s$. If we chose $s$ minimal with this property, then by $(*)$ the form $\Theta_L^{(s)}(\varphi)$ is cuspidal, hence $\Theta^{(s)}(\pi)$ is cuspidal by Moeglin's theorem. It follows that $s = r = n$ holds.

**Corollary 1:** *Let $L$ be a lattice of square free level $N$, $\pi$ an irreducible subrepresentation of the space of automorphic forms on $H(\mathbf{A})$ and $0 \neq \varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L)) \cap \pi$ with $\Theta_L^{(n)}(\varphi) = 0$.*
*Then $\Theta_L^{(n)}(\pi) = 0$.*
*In particular we have*

a) *Let $L$ be an even unimodular lattice, let $\varphi$ be an eigenform in $\mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$ and denote by $\pi$ an irreducible component of the representation of the adelic orthogonal group generated by $\varphi$. Then the theta lifting $\Theta^{(n)}(\pi)$ is zero if and only if the conditions given in [4] for the vanishing of $\Theta_L^{(n)}(\varphi)$ are satisfied.*

b) *Let $V$ be a definite quaternion algebra over $\mathbf{Q}$, equipped with the (reduced) norm form as quadratic form, $L$ an Eichler order in $V$ of square free level $N$. Let $\varphi$ be an eigenform in $\mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$ and denote by $\pi$ the representation generated by $\varphi$. Then the theta lifting $\Theta^{(2)}(\pi)$ is zero if and only if the conditions given in [5] for the vanishing of $\Theta_L^{(2)}(\varphi)$ are satisfied.*

*Proof.* The condition $*$ is satisfied for $L$ of square free level [5] (and trivial for level 1).

**Remark.**

a) For general $N$ little seems to be known about the validity of condition $*$.

b) If there is $r \in \mathbf{N}$ with $\Theta_L^{(r)}(\varphi) = 0$, $\theta_L^{(r+1)}(\varphi)$ noncuspidal, then $\theta^{(r+1)}(\pi)$ is not cuspidal and hence by Rallis' result $\Theta^{(r)}(\pi) \neq 0$. Condition $(*)$ is therefore necessary for the validity of the conclusion of Proposition 1.

c) Statement and proof of the proposition and the corollary can be transferred to theta series with spherical harmonics (liftings of $\varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L), \tau)$ for an irreducible representation $(Z_\tau, \tau)$ of $H(\mathbf{R})$).

d) For locally everywhere tempered $\pi$ (i.e., all Satake parameters have absolute value 1) the same condition for the non-vanishing of $\Theta^{(n)}(\pi)$ as in part a) of the corollary can also be deduced from the non-vanishing result of Roberts [34]

**Proposition 2.** *Let $\pi'$ be an irreducible subrepresentation of the space of cuspidal automorphic forms on $G_n(\mathbf{A})$ containing a function $\varphi'$ corresponding to the Siegel modular form $F$ of weight $k$ for $\Gamma_0^{(n)}(N)$. Assume that $\pi := \Theta^V(\pi') \neq \{\mathbf{0}\}$ and that*

> *$(**)$ there is a lattice $L$ on $V$ of level $N'|N$ such that $\mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L)) \cap \pi \neq \{\mathbf{0}\}$ and such that $(*)$ from Proposition 1 holds.*

*Then $\pi'$ contains a form corresponding to a Siegel cusp form $\tilde{F} \neq 0$ of weight $k$ for $\Gamma_0^{(n)}(N')$ such that $\tilde{F}$ is a linear combination of theta series attached to lattices in the genus of $L$.*
*In particular we have*

a) *If $n = 1$ and $N$ is the level of $\pi'$ then $F$ itself is a linear combination of theta series attached to lattices in the genus of $L$.*

b) *If $N = 1$ then $(**)$ is satisfied and $F$ is a linear combination of theta series attached to even unimodular lattices.*

c) *If $N$ is square free and odd then $(**)$ is satisfied and $F$ is in the space generated by the $\tilde{F}|T$ where $T$ runs through the the $p$-parts of the Hecke algebra of $\Gamma_0^{(n)}(N)$ for the $p$ dividing $N$.*

*Proof.* By Moeglin's theorem $\pi$ is irreducible. For $0 \neq \varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L)) \cap \pi$ we have by Proposition 1 $\tilde{F} := \theta_L^{(n)}(\varphi) \neq 0$. Again by Moeglin's theorem $\tilde{F} \in \pi'$ holds. In the case that $n = 1$ and $N$ is the level of $\pi'$ then by [11] $\tilde{F}$ and $F$ are both proportional to the unique newform in $\pi'$. to $F$. In the case $N = 1$ the cusp form $F$ of level 1 in $\pi'$ is unique since for each $p$, the $\mathrm{Sp}_n(\mathbf{Z}_p)$-fixed vector in an irreducible local representation is unique up to scalars. The assertion follows. In the case that $N$ is square free and odd, by the results of Aubert and Shu-Yen Pan [2, 39] the local representation $\pi_p$ contains for each $p|N$ a vector invariant under the orthogonal group of some lattice of level $p$ or 1 on $V_p$. The resulting function $\varphi \in \mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L))$ satisfies $(*)$ as in the corollary to Proposition 1, hence $(**)$ is satisfied. For the $p \nmid N$ the $\mathrm{Sp}_n(\mathbf{Z}_p)$-fixed vector in $\pi'_p$ is unique up to scalars as above and for the $p$ dividing $N$ the vectors fixed by the local version $K_p^N$ of $\Gamma_0^{(n)}(N)$ form an irreducible module under the Hecke algebra of $K_p^N$ by Proposition 2.2.4 of [12].

**Remark.**

a) Notice that Moeglin's theorem implies that the form corresponding to $\tilde{F}$ is indeed in $\pi'$ and not only in a representation space isomorphic to $\pi'$.

b) Let $n = 1$ and let $V$ be a definite quaternion algebra $D$ equipped with the reduced norm as quadratic form, assume that the irreducible cuspidal automorphic representation $\pi'$ of $GL_2(\mathbf{A})$ is in the image of the set of automorphic representations of the adelic multiplicative group of this quaternion algebra under the Jacquet/Langlands correspondence (i.e., for each prime at which $D$ is ramified the $p$-adic component $\pi'_p$ is either special or supercuspidal). It is known that this implies that $\pi'$ is in the image of the theta correspondence from the adelic orthogonal group of $V$. Let $F$ be the unique newform of level $N$ in $pi'$.. Then by Moeglin's theorem $\pi := \Theta^{(1)}(\pi')$ is nonzero and cuspidal irreducible. The description of the local Jacquet/Langlands correspondence gives that there is a lattice $L$ of level $N$ in $V$ and a $\varphi$ in $\mathcal{A}(H(\mathbf{A}), O_{\mathbf{A}}(L)) \cap \pi$. With a little extra effort it should be possible to show that $L$ can be chosen to be a special order of this level as in [21]; it would then follow in an elementary way that $\Theta_L^{(2)}(\phi) \in \pi'$ is nonzero and of level $N$, hence proportional to $F$. This will then imply that $F$ is an explicit linear combination of theta series attached to the ideals of the special orders of the same level as proven in [21]. For completeness we recall that the condition on the type of $\pi'_p$ for the primes ramified in $D$ excludes for example $F$ of nebentype with prime level $p$: supercuspidal representations do not have prime level, and special representations have the wrong central character at $p$.

A different way to derive these implications in some cases has been sketched in Section 9 of [21].

**Proposition 3.** *Let $F$ be a cuspidal Siegel modular form of weight $k$ divisible by 4 with respect to $\mathrm{Sp}_n(\mathbf{Z})$ and assume that $F$ is a linear combination of arbitrary theta series (possibly inhomogenous) attached to lattices on the space $V$ of dimension $2k$ carrying even unimodular positive definite lattices. In addition assume that the form on $Sp_n(\mathbf{A})$ corresponding to $F$ generates an irreducible subrepresentation of the space of automorphic forms on the adelic symplectic group. Then $F$ is a linear combination of theta series attached to even unimodular lattices on $V$.*

*Proof.* Denote by $\pi'$ the space of automorphic forms generated by the automorphic form on $Sp_n(\mathbf{A})$ corresponding to$F$. By Moeglin's theorem $\Theta^V(\pi') =: \pi$ is an irreducible representation of $H(\mathbf{A})$ with $\Theta_{(n)}(\pi) = \pi'$. By the local theta correspondence [25, 32] $\pi$ has to contain a vector $\tilde{\varphi}$ invariant under the group $O_{\mathbf{A}}(L)$ for a lattice $L$ of level 1. The assertion follows now from Proposition 2.

**Remark.** If $F$ is already given as a linear combination of theta series attached to full lattices, the assertion can also be deduced using the computations of traces of theta series in [9, 27]. These computations give similar results for general levels without requiring the irreducibility of the space of automorphic forms generated by $F$.

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