REPRESENTATION BY INTEGRAL QUADRATIC FORMS - A SURVEY

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Introduction

An integral symmetric matrix $S = (s_{ij}) \in M_m^{\mathrm{sym}}(\mathbb{Z})$ with $s_{ii} \in 2\mathbb{Z}$ gives rise to an integral quadratic form $q(\mathbf{x}) = \frac{1}{2}\,{}^t\mathbf{x}S\mathbf{x}$ on $\mathbb{Z}^m$. If S is positive definite, the number $r(q,t)$ of solutions $\mathbf{x} \in \mathbb{Z}^m$ of the equation $q(\mathbf{x}) = t$ is finite, and it is one of the classical tasks of number theory to study the qualitative question which numbers t are represented by q or the quantitative problem to determine the number $r(q,t)$ of representations of t by q either exactly or asymptotically.

Starting with the work of Euler, Legendre–Gauß and Lagrange–Jacobi on the number of ways in which an integer can be represented as a sum of two, three and four integral squares, many deep and beautiful results have been obtained concerning these problems, as well in this classical setting as in generalized settings like the study of representations with congruence conditions, representation numbers of forms q' of rank $n \leq m$ by q, representation numbers or measures by definite or indefinite forms over the ring of integers of a number field.

In this article I want to give a survey of what is known (and what is not known) about these questions. In particular we will discuss and slightly extend some recent results about representation of numbers by totally definite forms of rank 3 over the integers of a totally real number field in Section 5. We will also discuss some recent progress concerning effectivity of results. Another recent survey is [20].

1. Basic notions and problems

To fix some notations, we consider an integral symmetric nonsingular matrix $S \in M_m^{\mathrm{sym}}(\mathbb{Z})$ with even diagonal and the nondegenerate quadratic form

$$q(\mathbf{x}) = \frac{1}{2}{}^t\mathbf{x}S\mathbf{x} \tag{1.1}$$

on $\mathbb{Z}^m$ given by S.

We may equivalently consider a lattice $L = \bigoplus_{j=1}^m \mathbb{Z}e_j$ of full rank m in the rational vector space V with quadratic form $q : V \longrightarrow \mathbb{Q}$ (satisfying $q(L) \subseteq \mathbb{Z}$) and attached symmetric bilinear form $B(x,y) = q(x+y) - q(x) - q(y)$ and obtain S as the Gram matrix $S = (B(e_i, e_j))$ attached to q and the basis $(e_1, \ldots, e_m)$ of L.

We will also consider the situation where $\mathbb{Z}$ is replaced by the ring of integers $\mathfrak{o}_F$ of a number field F, in which case an $\mathfrak{o}_F$-lattice of rank m is by definition an $\mathfrak{o}_F$-submodule of the m-dimensional F vector space V which generates V over F and is finitely generated over $\mathfrak{o}_F$. In particular, a lattice is not necessarily free as an $\mathfrak{o}_F$-module, and considering lattices with quadratic form is a little more general than just considering Gram matrices in the number field situation.

If L is free, the discriminant dL is the square class of the determinant of a Gram matrix, in the general case the discriminant ideal $\mathfrak{d}L$ is the ideal generated by the determinants of the Gram matrices of the free sublattices of L of full rank. The level $N = N(L)$ is the inverse of the ideal generated by $q(L^{\#})$ (where $L^{\#}$ is the dual lattice) or a generator of that ideal if it is principal. Since in the number field situation the case that $\mathfrak{n}L := q(L)\mathfrak{o}_F$ is different from $\mathfrak{o}_F$ can not (as over $\mathbb{Z}$) be avoided by scaling the quadratic form, we denote by $\tilde{N}(L)$ the reduced level $\tilde{N}(L) = (\mathfrak{n}L)^{-1} \cdot N(L)$ and by $\tilde{\mathfrak{d}}(L)$ the reduced discriminant

$$\tilde{\mathfrak{d}}(L) = \mathfrak{d}(L) \cdot (\mathfrak{n}L)^{-\mathrm{rk}(L)}. \tag{1.2}$$

If F is totally real and q totally definite we consider for $t \in \mathfrak{o}_F$ the number

$$r(L,t) = r(L,q,t) = \#\{x \in L \mid q(x) = t\} \tag{1.3}$$

of representations of t by the quadratic lattice $(L,q) = (L,q_L)$; more generally we consider for another quadratic lattice (K,q_K) the number $r(L,K)$ of isometries (with respect to q_K, q_L) of K into L; if F is $\mathbb{Q}$ and T is a Gram matrix of (K,q_K) we have

$$r(L,K) = r(S,T) = \#\{X \in M(m \times n, \mathbb{Z}) \mid {}^tXSX = T\} \tag{1.4}$$

(If $K = \mathbb{Z}x$ is of rank one with $q(x) = t$, then $r(L,K) = r(L,q,t)$ above).

If q is not totally definite, one studies instead the representation measure (Darstellungsmaß) $\mu(L,K)$ or $\mu(S,T)$ [61, 45], which is defined in terms of the measure of a fundamental domain of a (real or adelic) orthogonal group modulo the action of a discrete subgroup. We will usually write $r(L,K)$ for $\mu(L,K)$ if this cannot cause confusion.
In the discussion of measures we will always exclude the cases that $(FK)^\perp$ or FL is a hyperbolic plane, in which the measures become infinite.
The starting point of all considerations is the following

Local-global principle (Hasse–Minkowski):

If (L,q_L) and (K,q_K) are nondegenerate integral quadratic $\mathfrak{o}_F$-lattices as above, then (K,q_K) is represented by some lattice in the genus of $(L,q) = (L,q_L)$ if and only if $K_v := K \otimes \mathfrak{o}_{F_v}$ is represented by $L_v = L \otimes \mathfrak{o}_{F_v}$ for all places v of F (i.e., if K is represented by L locally everywhere).

Here the genus of (L,q) is the set of all quadratic $\mathfrak{o}_F$-lattices that are isometric to some lattice on V that is in the orbit of L under the action of the adelic orthogonal group $O_V(\mathbb{A})$ of V.

It is part of this local-global principle that this characterization of the genus is equivalent to each of the following:

a) (L,q_L) and (M,q_M) are in the same genus if L and M represent each other locally everywhere.
b) (L,q_L) and (M,q_M) are in the same genus if and only if $(L \otimes F, q_L)$ and $(M \otimes F, q_M)$ have the same signature at all archimedean places of F and if for each ideal $\mathfrak{a}$ of $\mathfrak{o}_F$ there is a linear isomorphism from L to M that is an isometry modulo $\mathfrak{a}$.

It is well-known that only this weak version of local-global principle is valid, classical examples are the binary forms $x^2 + 55y^2$ and $11x^2 + 5y^2$ belonging to the same genus, where the second form represents e.g. 5 and 11 but not the first, and the Ramanujan form $x^2 + y^2 + 10z^2$, for which the numbers not of the form $4^k(16n+6)$ are represented locally everywhere and for which there are 18 numbers known that are represented locally everywhere but not by the form itself globally.

The quantitative version of the local-global principle is

Siegel's main theorem:[60, 61, 62, 40]

For definite L and K as above let $O(L)$ be the (finite) group of isometries of (L, q_L) onto itself and put

$$m(\text{gen } L) = \sum_{i=1}^{h} \frac{1}{|O(L_i)|}, \tag{1.5}$$

where $L_1, \dots, L_h$ is a full set of representatives of the isometry classes $\text{cls}(L_i)$ of lattices in the genus $\text{gen}(L)$ of L (which is a finite set by reduction theory).
Then the average number of representations

$$r(\text{gen } L, K) := \frac{1}{m(\text{gen } L)} \cdot \sum_{i=1}^{h} \frac{r(L_i, K)}{|O(L_i)|}$$

is equal to

$$r(\text{gen } L, K) = c_{n,m,F} \prod_{v} \alpha_v(L, K), \tag{1.6}$$

where the product is over all places v of F, $c_{n,m,F}$ is a constant depending on F, $m = \text{rk}(L)$, $n = \text{rk}(K)$, the local representation densities $\alpha_v(L, K)$ are determined for nonarchimedean v by counting the number of representations modulo $\mathfrak{p}_v^j$ of K by L for large enough j and the product of the $\alpha_v(L, K)$ over the archimedean v is (up to a constant depending on the signatures of q_L, q_K) equal to

$$N_{\mathbb{Q}}^{F}((\tilde{\mathfrak{d}}L)^{-\text{rk}(L)/2}) \cdot N_{\mathbb{Q}}^{F}((\mathfrak{d}K \cdot (\mathfrak{n}L)^{-\text{rk}(K)})^{\frac{\text{rk}(L)-\text{rk}(K)-1}{2}}. \tag{1.7}$$

If (L, q_L) is indefinite, an analogous formula is true, with $\frac{r(L_i,K)}{|O(L_i)|}$ replaced by the representation measure $\mu(L_i, K)$ and $m(\text{gen } L)$ replaced by the measure $\mu(\text{gen } L)$ of the genus.

A version of this theorem for representations with congruence conditions is valid as well [2]. In fact, in the arithmetic proof the formula given above is obtained by summing up a similar formula that is valid for each genus of representations (i. e., a set of representations that can locally everywhere be transformed into each other by an isometry of the lattice) over the finitely many genera of representations of K by L, and if one sums only over those genera satisfying a specific congruence condition, one obtains the main theorem for representations with congruence conditions. Similarly, in the representation theoretic proof using Weil's ideas [66, 47], a formula for representations with congruence conditions is simply obtained by using a different test function.

In view of these two basic local-global results, work on the representation properties of integral quadratic forms usually deals with the following problems

> **Problem A:** Describe as precisely as possible the set of lattices of some fixed rank n that are represented locally everywhere by L but are not represented by L, in particular, try to prove that this set is finite or at least very small in some meaningful sense.
>
> **Problem B:** Try to either compute the difference $r(L, K) - r(\text{gen } L, K)$ as explicitly as possible or to prove that it is small compared to $r(\text{gen } L, K)$ for K that are sufficiently large in some sense, e.g. have large minimum.

> **Problem C:** Compute $r(\text{gen } L, K)$ explicitly, i.e., compute the local representation densities $\alpha_v(L,K)$ either by giving formulae or by giving efficient algorithmic procedures, and bound $r(\text{gen } L, K)$ from below by giving bounds for the local densities.

It is both arithmetically interesting and helpful for the solution of the above problems to consider at the same time primitive representations:

A representation $\sigma : K \longrightarrow L$ is called primitive, if $L \cap (F \cdot \sigma(K)) = \sigma(K)$, where $F \cdot \sigma(K)$ is the F-subspace of V generated by $\sigma(K)$ (in particular, if $F = \mathbb{Q}$ and $t \in \mathbb{Z}$, a vector $x \in L$ representing t is a primitive representation if $\frac{x}{a} \notin L$ for $a \in \mathbb{Z} \setminus \{\pm 1\}$). The Hasse-Minkowski local-global principle and Siegel's main theorem are valid for primitive representations too.

We will denote by $r^*(L,K)$ the number of primitive representations of K by L, by $q_n^*(L)$ the set of lattices of rank n primitively represented by L and by $q_n(\text{gen } L)$ (resp. $q_n^*(\text{gen } L)$) the set of lattices of rank n represented (primitively) locally everywhere by L (and hence by some lattice in the genus of L).

For $c > 0$ we will say that K is represented by L (or by the genus of L) with imprimitivity bounded by c, if there is a representation $\sigma : K \longrightarrow L$ (respectively $\sigma : K \longrightarrow L'$ for some L' in the genus of L) such that

$$(L \cap (F \cdot \sigma(K)) : \sigma(K)) \leq c \tag{1.8}$$

(resp. the same inequality with some $L' \in \text{gen } L$ in place of L).

The Hasse-Minkowski local-global principle implies that K (of the right signatures at the archimedean places) is represented by $\text{gen}(L)$ with imprimitivity bounded by c if and only if for all nonarchimedean v there are local representations $\sigma_v : K_v \longrightarrow L_v$ such that the product of the corresponding local indices is bounded by c (since K_v is unimodular and hence maximal at almost all v this product is always a finite product).

We write $q_{n,c}(L)$ resp. $q_{n,c}(\text{gen } L)$ for the set of lattices of rank n that are represented by L with imprimitivity bounded by c. In the case $n = 1$ (to which we will restrict attention in most of this article) we will omit the subscript n in the above notations and simply write $q(L)$, $q_c(L)$ etc.

2. Local densities

The problem of computing the local densities at a place v has in principle been completely solved by Hironaka and Sato for non-dyadic v ($2 \in \mathfrak{o}_{F_v}^*$) in [21]. The formulae given there are explicit but rather complicated. Yang [71] gives explicit formulae for $\text{rk}(K) \leq 2$ with v non-dyadic and $\text{rk}(K) = 1$ with $\mathfrak{o}_{F_v} = \mathbb{Z}_2$, which are much easier to handle. Katsurada gives an explicit formula for the case that L is a sum of hyperbolic planes in [35] and recursion formulas in [33] and (jointly with Hisasue) in [34]; these results lead to an explicit formula for the Fourier coefficients of Siegel-Eisenstein series in [35] and to rationality results for certain power series attached to local densities.

The most complete textbook reference on local densities is [40]. For our present purpose we note here that in Theorem 5.6.5 of [40] it is shown that the local density $\alpha_v(L,K)$ can be bounded from below as

$$\alpha_v(L,K) \geq c_1^{\text{rk}(K)+1-\text{rk}(L)} \cdot c_2(L,v) \tag{2.1}$$

for K that are locally represented by L with imprimitivity bounded by c_1, where $c_2(L,v)$ depends only on L (and $\text{rk}(K)$).

An analogous estimate

$$\alpha_v(L,K) \geq c_3(L,v) \tag{2.2}$$

is valid for all K with $\alpha_v(L,K) \neq 0$ if the Witt index $\mathrm{ind}(F_vL)$ of F_vL is $\geq \mathrm{rk}(K)$ by Theorem 5.6.5 of [40]; this can also be concluded from the estimate (2.1) since in that case it is easy to show that K_v is represented by L_v with imprimitivity bounded by some constant depending only on L_v (more precisely, on the index of L_v in a maximal lattice on F_vL). We notice that $\mathrm{ind}(F_vL) \geq \mathrm{rk}(K)$ is always true for $\mathrm{rk}(L) \geq 2 \cdot \mathrm{rk}(K) + 3$.

To give explicit values for these constants $c_2(L,v)$, $c_3(L,v)$ for the v dividing $\mathfrak{d}L$ is a tedious matter.
A rough estimate for the value of $c_2(L,v)$ in terms of the level and the discriminant of L_v has been given in [58] for the case $\mathrm{rk}(K) = 1$, in the same case Hanke describes in [18] a recursive procedure giving a good value for $c_2(L,v)$ depending on the local isometry class of L_v but no explicit formula in terms of level and discriminant of L_v. Since this procedure is easy to implement in a computer program, it is probably the best method for use in experimental investigations (an implementation for PARI/GP using modular forms data from William Stein's program HECKE) has been announced in [18]). With the help of Katsurada's recursion formulas it should not be too difficult to extend this procedure to higher $\mathrm{rk}(K)$, this is the subject of diploma thesis work in the author's group in Saarbrücken.

The remaining factors $\alpha_v(L,K)$ for the v with L_v unimodular can be estimated using Proposition 5.6.2, Lemma 5.6.10, Corollary 5.6.2 and Exercise 1 of Section 5 of [40] in such a way that one controls the infinite product $\prod_v \alpha_v(L,K)$. When one does this, one may obtain for $\mathrm{rk}(L) < 2\mathrm{rk}(K) + 3$ (apart from factors whose product over all v converges to some fixed number independent of K) factors of the type $(1 \pm N_{\mathfrak{p}_v}^{-1})$ for the $\mathfrak{p}$ dividing $\det(K_v)$; their product can be bounded from below by $(dK)^{-\epsilon}$ for any ϵ.

If $\mathrm{rk}(L) = \mathrm{rk}(K) + 2$ one may obtain factors $(1 - \chi_v(\mathfrak{p}_v)N_{\mathfrak{p}_v}^{-1})^{-1}$ for almost all v, where $\chi = \prod_v \chi_v$ is the quadratic character attached to the discriminant of the orthogonal complement $(FK)^\perp$ of FK in FL. Their product may be (ineffectively) bounded by $(dK)^{-\epsilon}$ as well by Siegel's bound on $L(s,\chi)$ at $s = 1$.

The result is summarized in the following lemma:

Lemma 2.1. *If* $\mathrm{rk}(L) \geq 2 \cdot \mathrm{rk}(K) + 3$, *there is a constant* $C > 0$ *(depending on* L *and on* $\mathrm{rk}(K)$*) such that*

$$r(\mathrm{gen}\ L, K) \geq C \cdot (\mathfrak{d}K \cdot (\mathfrak{n}L)^{-\mathrm{rk}(K)})^{(\mathrm{rk}(L)-\mathrm{rk}(K)-1)/2} \tag{2.3}$$

holds for all non-degenerate K *of fixed rank for which* $r(\mathrm{gen}\ L, K) \neq 0$.
If $\mathrm{rk}(L) \geq \mathrm{rk}(K) + 2$ *and if one restricts attention to* K *that are represented by* $\mathrm{gen}(L)$ *with imprimitivity bounded by some fixed* t, *one has an estimate*

$$r(\mathrm{gen}\ L, K) \geq C_{\epsilon,t} \cdot N_{\mathbb{Q}}^F(\mathfrak{d}K \cdot (\mathfrak{n}L)^{-\mathrm{rk}(K)})^{\frac{\mathrm{rk}(L)-\mathrm{rk}(K)-1}{2}-\epsilon} \tag{2.4}$$

for all $\epsilon > 0$, *where the constant* $C_{\epsilon,t} > 0$ *is not effective if* $\mathrm{rk}(L) = \mathrm{rk}(K) + 2$ *(due to the possibility of a Landau-Siegel zero of a Dirichlet L-series).*
If $2 \cdot \mathrm{rk}(K) + 1 \leq \mathrm{rk}(L) < 2 \cdot \mathrm{rk}(K) + 3$, *the condition on* K *of representability with bounded imprimitivity is satisfied if one has either one of the following conditions:*

- K *is represented by* L_v *with imprimitivity bounded by some fixed* t_v *for the finitely many places* v *for which* $\mathrm{ind}(F_vL) < \mathrm{rk}(K)$
- *For some fixed integer* s *one has* $\mathfrak{p}_v^s \nmid \det(K_v)$ *for all* v *with* $\mathrm{ind}(F_vL) < \mathrm{rk}(K)$.

In particular, if $\mathrm{rk}(K) = 1$*, this last condition restricts divisibility by those* $\mathfrak{p}_v$ *for which* L_v *is anisotropic, and it becomes equivalent to representability of* K *by* $\mathrm{gen}(L)$ *with imprimitivity bounded by some fixed* t*.*

3. Indefinite lattices

We will not say much about indefinite lattices here and refer to [29] and the references given there instead. The main tool in the arithmetic approach is here the strong approximation theorem for the spin group [15, 14, 44] which implies that for $\mathrm{rk}(L) \geq 3$ the isometry class of L is equal to its spinor genus for indefinite L, a lattice (M, q_M) being in the spinor genus of L if it is isometric to a lattice in the orbit of the image in $O_V(\mathbb{A})$ of the adelic spin group $\mathrm{Spin}_V(\mathbb{A})$ under the covering map

$$\mathrm{Spin}_V(\mathbb{A}) \longrightarrow SO_V(\mathbb{A}).$$

Since computations for spinor genera can be reduced to a large extent to local computations, the representation behaviour of indefinite lattices (L, q) is (thanks to [45, 65] and subsequent work quoted in [25, 29], see also [69, 70, 68, 59]) completely understood for $\mathrm{rk}(K) \leq \mathrm{rk}(L) - 3$ and reasonably well understood for $\mathrm{rk}(L) = 3$, $\mathrm{rk}(K) = 1$.

In the general case of $\mathrm{rk}(K) = \mathrm{rk}(L) - 2$ (and K nondegenerate) the results of [45, 65, 24] imply that $r(\mathrm{gen}\ L, K) = \mu(L, K)$ except for K for which the binary space generated by the orthogonal complement of K belongs to a certain finite (and usually rather small) set of spaces depending on L, but the knowledge about the behaviour in these exceptional classes is still not satisfactory. There are interesting results for the case of codimension 1 in [69, 70, 68], but again our knowledge in this case is far from perfect.

4. Results for $\mathrm{rk}(K) = 1$, $F = \mathbb{Q}$

Restricting attention from now on to totally definite lattices (L, q), we summarize first what is known about the most classical case, the case of representation of numbers by a form (or lattice) over $\mathbb{Z}$.

Theorem 4.1. (Kloosterman, Tartakovski)
Let (L, q) *be integral positive definite with* $\mathrm{rk}(L) = m \geq 5$*. Then* L *represents all sufficiently large numbers* a *in* $q(\mathrm{gen}\ L)$*. The same is true for* $m = 4$ *if one restricts attention to* a *in* $q_c(\mathrm{gen}\ L)$ *for some fixed* c *or (what is the same) for some fixed* s *to the set*

$$\{a \in q(\mathrm{gen}\ L) \mid p^s \nmid a \text{ if } L_p \text{ is anisotropic}\}. \tag{4.1}$$

In both cases one has an asymptotic formula

$$r(L, a) = r(\mathrm{gen}\ L, a) + O(a^{\frac{m}{2}-1-\delta}) \tag{4.2}$$

for some $\delta > 0$*, with*

$$r(\mathrm{gen}\ L, a) \geq C \cdot a^{\frac{m}{2}-1-\eta} \tag{4.3}$$

for all $\eta > 0$*, for some constant* $C(\eta)$ *for all* a *in the respective subset of* $q(\mathrm{gen}\ L)$*, where the additional* η *appears only for* $m = 4$*.*

This theorem has been proved for diagonal forms by Kloosterman in his dissertation (Groningen 1924) for $m \geq 5$ and in [42] for $m = 4$ with an error term

$$O(a^{\frac{m}{4}+\epsilon}) + O(a^{\frac{m}{2}-1-\frac{1}{4}+\epsilon}) \quad \text{for } m \geq 5 \tag{4.4}$$

and with an error term

$$O(m^{\frac{17}{18}+\epsilon}) \quad \text{for } m = 4 \tag{4.5}$$

using the Hardy-Littlewood-Ramanujan circle method (where of course the main term is not yet identified as Siegel's average, but arises as some multiple of $a^{\frac{m}{2}-1}$ times the singular series). Tartakovski generalized it to arbitrary quadratic forms in [63], for a modern version of this proof see [31].

It is nowadays usually proved using modular forms theory: With

$$\vartheta(L,z) := \vartheta(L,q,z) = \sum_{a=0}^{\infty} r(L,q,a)\exp(2\pi i a z) \tag{4.6}$$

and $\vartheta(\text{gen } L, z) := \vartheta(\text{gen } L, q, z)$ the corresponding weighted average over the genus, it is known that $\vartheta(\text{gen } L, z)$ and $\vartheta(L,z)$ are modular forms of weight $k = \frac{m}{2}$ and quadratic character for the group $\Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$, where N is the level of (L,q).

Since the difference $\vartheta(\text{gen } L, z) - \vartheta(L,z)$ is a cusp form, one can apply estimates for Fourier coefficients of cusp forms to bound the error term. This gives an error term

$$O(\sigma_0(a) \cdot a^{\frac{m}{4}-\frac{1}{2}}), \tag{4.7}$$

if m is even (by Deligne's proof of the Ramanujan-Petersson conjecture) and

$$O(a^{\frac{m}{4}-\frac{2}{7}+\epsilon}) \quad \text{for odd } m \geq 5 \tag{4.8}$$

by Iwaniec's estimate [30] for Fourier coefficients of cusp forms of half integral weight (extended from square free integers to all integers by using the Shimura correspondence).

That $\vartheta(\text{gen } L, z) - \vartheta(L,z)$ is a cusp form, is usally mentioned without proof; it can easily be proved by using the fact that the value of the theta series of L at a cusp is expressed (using the transformation formula for theta series) by a Gauss sum over a quotient L/cL for some c and hence depends only on the congruence properties of (L,q), i.e. on the genus.

The implied constants in these estimates did not receive much attention; only quite recently the increased interest in explicit and effective results lead to some work concerning their values. In particular, continuing work of Fomenko [16] the case $m = 4$ was studied in [58], where the local densities were estimated from below and the coefficients of the cuspidal part

$$\vartheta(L,z) - \vartheta(\text{gen } L, z)$$

of the theta series were estimated by computing a bound for the Petersson norm of this cusp form. The result obtained was:

Proposition 4.2. *Let a be primitively represented by all $L_p, rk(L) = 4$. Then a is represented by L if*

$$\frac{\varphi(a)}{a^{1/2}\sigma_0(a)} \geq 6.418 \prod_{p|N} C_p \tag{4.9}$$

holds, where

$$C_p = N_p^5 \frac{p^2-1}{2} \cdot \sqrt{(1+\frac{1}{p})} \tag{4.10}$$

in general (with N_p denoting the p-part of the level N of L),

$$C_p = N_p^{9/2} \frac{(1+\frac{1}{p})^{3/2}}{2} \tag{4.11}$$

if $p^2 \nmid N$,

$$C_p = N_p^2 \cdot (1 + \frac{1}{p})^{3/2} \tag{4.12}$$

if $p^2 \nmid dL$.

A different approach was taken by Hanke in [18]. There the densities are bounded for a given L by the recursive algorithm mentioned before, and for the cuspidal part enough Fourier coefficients are determined by computer to be able to express this cusp form explicitly as a linear combination of Hecke eigenforms, using W. Stein's database of modular forms.
The resulting estimates for individual cases give in these cases slightly sharper bounds for representation than in [58] but do still require additional considerations if one wants to determine $q(L)$ explicitly.

In the special case of the quadratic form $f(x, y, z, w) = x^2 + 3y^2 + 5z^2 + 7w^2$ e.g., Hanke proves first that all integers $\geq 10^{12}$ are represented by f; since this bound is still too large to examine the numbers below it exhaustively, he uses arguments involving ternary sublattices in order to answer (affirmatively) the classical question whether f represents all integers except the known exceptions 2 and 22.
Work on obtaining estimates for $m = 4$ that are practically usable continues.

The qualitative version of the Theorem of Kloostermann/Tartakovski, i.e. the statement that every sufficiently large integer that is represented by L locally everywhere (and is not divisible by a high power of a prime p with L_p anisotropic if $\mathrm{rk}(L) = 4$) is represented by L, was proved by a completely different method in the 1972 lecture notes of Kneser that underly [46]. His method is purely arithmetic and uses the strong approximation theorem for the Spin group to reduce the problem to the study of rational representations whose denominator is a bounded power of a single prime. It has been shown by Hsia and Icaza in [26] that the bound for sufficiently large integers can be made explicit with this method, with an example giving a bound of $\sim 10^{10}$ for a lattice of rank 5 and level 4.

The case $m = 3$ is more difficult; in particular, it is known that even with the restriction on a from Theorem 4.1 there exist L for which there are infinite families of numbers represented by gen L but not by L.

A result of Duke [11] gives an asymptotic formula for $r(L, a)$, when a is restricted to square free integers. For general a the problem arises that there are cusp forms of weight $3/2$ whose a-th Fourier coefficient grows as fast as $a^{1/2}$ (which is the order of growth of the main term $r(\mathrm{gen}L, a)$), but only for a tending to infinity in one of finitely many square classes. In order to deal with this behaviour one has to take into account the effects caused by the representation behaviour of spinor genera that was mentioned in Section 2.

More precisely, the situation is as follows:
If one denotes by $r(\mathrm{spn}\ L, a)$ the weighted average of the $r(L_i, a)$ for the lattices L_i in the spinor genus of L, then by [45, 65] one has

$$r(\mathrm{gen}\ L, a) = r(\mathrm{spn}\ L, a) \tag{4.13}$$

for all a outside a finite set of square classes

$$t_j\mathbb{Z}^2 = \{t_j n^2 \mid n \in \mathbb{Z} \setminus \{0\}\}, \tag{4.14}$$

these are called the spinor exceptional classes; the same is true (with the same square classes) for numbers of primitive representations. For each of these square classes, the set of numbers that are in $q(\mathrm{gen}\ L)$, but are not represented by all spinor genera in the genus (the spinor exceptions of the genus) has been determined in

[53], the same problem for primitive representations has been solved in [13], both of these exceptional sets are in general infinite.

Writing

$$\vartheta(\text{spn } L, z) := \sum_{a=0}^{\infty} r(\text{spn } L, a) \exp(2\pi i a z), \tag{4.15}$$

one has by [55] that

$$\vartheta(L, z) - \vartheta(\text{spn } L, z) \tag{4.16}$$

is a cusp form of weight $\frac{3}{2}$ whose Shimura lifting is cuspidal. For such cusp forms of weight 3/2 the growth of the Fourier coefficients can be estimated using [11] and the Shimura lifting, which gives an asymptotic formula of the same type as in Theorem 3.1 (with an error term $O(a^{\frac{1}{2}-\frac{1}{28}+\epsilon})$) for all a outside the exceptional square classes [12].

If there is only one spinor genus in the genus, one is done at this point; since it is known [44, 23] that this is the case if the discriminant is not divisible by 2^7 and not by any p^3 for odd primes p, we are finished here for L of small discriminant.
If there are several spinor genera, a more detailed analysis [12, 54, 57] yields the following result inside the exceptional square classes:

Theorem 4.3. [57] *If* $\text{rk}(L) = 3$ *and* a *is restricted to numbers in* $q(\text{gen } L)$ *not divisible by* p^r *(*r *fixed) for the primes for which* L_p *is anisotropic, one has:*
If a *is sufficiently large, then* a *is represented by all lattices in the genus of* (L, q) *unless one of the following holds*

- a *is a spinor exception. In this case* a *is represented by exactly half the spinor genera in the genus of* L*, and it is represented by all the classes in these spinor genera.*
- a *is of the form* $a'p^2$*, where* a' *is a spinor exception and* p *is a prime that is inert in the imaginary quadratic extension*

$$E = \mathbb{Q}(-2a \det L) \tag{4.17}$$

of $\mathbb{Q}$*. In this case* a' *is represented by exactly half the spinor genera in the genus of* L *and* $a = a'p^2$ *is represented precisely by those classes in this half of the spinor genera that represent* a' *and by all lattices in the other half of the spinor genera.*

In particular, if there is a spinor exceptional integer a' *for the genus of* L *that is represented by* $\text{spn}(L)$ *but not by* L *(so* a' *is below the bound for being sufficiently large), then there are infinitely many integers* $a'p^2$ *with* p *prime that are not represented by* L*.*

An example for the behaviour of this theorem is the quadratic form

$$4x^2 + 48y^2 + 49z^2 + 48yz + 4xz \tag{4.18}$$

discussed in [57]; it does not represent any p^2 where $p \equiv -1 \bmod 3$ is a prime although the form $x^2 + 48y^2 + 144z^2$ in the same spinor genus represents all these numbers (but not primitively).

We will come back to the proof of this theorem in the next section where we discuss its generalization to the number field case. A different proof, using a refined analysis of the cusp form $\vartheta(L, z) - \vartheta(\text{spn } L, z)$ and of its Shimura lifting has recently been announced by Hanke [19].

Both methods of proof lead in fact to an asymptotic formula for the numbers not excluded by one of the conditions of the theorem; in fact one can show that all those numbers are represented by each spinor genus in the genus with not too bad imprimitivity (but not necessarily with bounded imprimitivity), which allows to give a lower bound for $r(\text{spn } L, a)$ using results of [54].

It should be mentioned that the bound for an integer to be sufficiently large in Theorem 4.2 can at present not be made effective without using the generalized Riemann hypothesis since the estimate of the main term involves an estimate from below on the value $L(1, \chi_D)$ of a Dirichlet series in dependence of the conductor D of χ_D.

Even if one admits the Riemann hypothesis and makes optimistic assumptions on bounds for the error term, the bound for "sufficiently large" in the above theorem that one obtains is not of a feasible size, as has been remarked in [51]. Ono and Soundararajan use in that article a different method (assuming additional Riemann hypotheses on L functions attached to elliptic curves) in order to prove that the numbers 3, 7, 21, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391, 679, 2719 are indeed the only numbers that are represented locally everywhere but not globally by the Ramanujan form $x^2+y^2+10z^2$. Their method has recently been generalized by Reinke ([52]) to give similar results for $x^2 + y^2 + 7z^2$ and some related forms.

Another remarkable special result on representation numbers is the formula of S. Milne and K. Ono [49, 50] for the number of representations of a number n a sum of $4s^2$ or $4s^2 + 4s$ squares (for $s \in \mathbb{N}$). This uses totally different methods; Ono's proof relies on identities between particular modular forms.
There is also a vast literature proving formulae for representation numbers of individual quadratic forms by identifying the cuspidal part of the theta series explicitly as a cusp form whose coefficients one can express in some other way; we don't go into details on this here.

5. Results for $\text{rk}(K) = 1$, F a number field

Before we review the available results in the number field situation we make a few preliminary remarks:

- If the lattice K to be represented is not free, it is of the form $\mathfrak{a}x$ with a (fractional) ideal $\mathfrak{a}$ of $\mathfrak{o}_F$. Instead of considering $r(L, K)$ we can then consider $r(\mathfrak{a}^{-1}L, \mathfrak{o}_F x)$ and reduce to the situation of a free lattice K, i.e., of representation of numbers by a (not necessarily free) lattice L with $\mathfrak{n}(L) = q(L)\mathfrak{o}_F$ not necessarily equal to $\mathfrak{o}_F$.
- If $\mathfrak{n}(L) = q(L)\mathfrak{o}_F \neq \mathfrak{o}_F$, the theta series will not be a modular form for a group of type $\Gamma_0(N)$ but for a type of congruence group that is locally everywhere conjugate to this type. This doen't change the analytic arguments.
- We will call a number $t \in F$ square free relative to L if $t \cdot (\mathfrak{n}(L))^{-1}$ is not divisible by the square of an integral ideal.
  In particular, if $K = \mathfrak{o}_F x$ with $q(x) = t$ and t is square free relative to L then all representations of K by a lattice of the same norm as L are primitive.
- The fact that the difference of theta series of lattices in the same genus is a cusp form has been proved in the number field case by Walling [64].

It appears that the analogue of Theorem 4.1 for a totally definite integral quadratic form over the integers of a totally real number field has never been formally stated as a theorem.

The qualitative version has been extended to the number field situation in [27, 22] and appears to be the first definitive statement on the question in the literature. On the other hand, based on Kloosterman's investigation of theta series for the number field case [43] and on Siegel's theorem, Gundlach states in the introduction of [17] that the determination of the representation numbers of totally definite integral quadratic forms in the number field case will be accomplished if one can prove an estimate on the Fourier coefficients of cusp forms, notes that the trivial estimate $|a_\nu| = O((N(\nu))^{k/2})$ for forms of weight k will suffice if $\mathrm{rk}(L) > 4$ and proves the required estimate

$$|a_\nu| = O(N(\nu)^{3/4+\epsilon}) \tag{5.1}$$

for the case $k = 2$, i.e., $\mathrm{rk}(L) = 4$, so we may assume that he was aware of the validity of Theorem 4.1 for the number field case.

A possible explanation for the situation is that Kloosterman in [43] (still not having Siegel's main theorem at his disposal) stated that the evaluation (and estimation) of the singular series in the main term was very complicated and that after Siegel's work this question was considered as settled and as requiring no further formal statements.

Assuming this we state

Theorem 5.1. [43, 62, 17]
The analogue of Theorem 4.1 for $r(L, K)$, L totally positive definite integral of rank $m \geq 4$ over the integers $\mathfrak{o}_F$ of a totally real number field, K 1-dimensional over $\mathfrak{o}_F$, is true for all K with $N_{\mathbb{Q}}^F(\mathfrak{d}K)$ large enough (and satisfying the divisibility condition for the anisotropic primes if $\mathrm{rk}(L) = 4$*).*

The error term can here (by [36]) be taken to be $O(N_{\mathbb{Q}}^F(dK)^{\frac{m}{4}-\frac{7}{18}+\epsilon})$ for even m (which is $\frac{11}{18} + \epsilon$ for $m = 4$) and to be

$$O(N_{\mathbb{Q}}^F(dK)^{\frac{m}{4}-\frac{18}{65}+\epsilon}) \tag{5.2}$$

by the new estimate of Cogdell, Piatetski-Shapiro and Sarnak announced in [9] for odd $m \geq 5$ and K (or a) square free relative to L. Since for odd $m \geq 5$ the trivial estimate $O(N_{\mathbb{Q}}^F(dK)^{\frac{m}{4}})$ is sufficient, we don't go into detail how to use the Shimura correspondence in order to prove the better bound for square free K to be true for all K (there are some tricky points for dK highly divisible by primes dividing the level of L).

In the case $\mathrm{rk}(L) = 3$ there is the new result by Cogdell, Piatetski-Shapiro and Sarnak announced in [9] (which generalizes the result of Duke mentioned in Section 4):

Theorem 5.2. (Cogdell, Piatetski-Shapiro, Sarnak)
Let F be totally real, (L, q) an integral $\mathfrak{o}_F$-lattice of rank 3 with q totally positive definite. Then for all $a \in \mathfrak{o}_F$ square free relative to L that are represented locally everywhere by L one has the asymptotic formula (for all $\eta > 0$)

$$r(L, a) = r(\text{gen } L, a) + O_\eta(N_{\mathbb{Q}}^F(a)^{\frac{123}{260}+\eta}) \tag{5.3}$$

with

$$r(\text{gen } L, a) \geq C_\epsilon(N_{\mathbb{Q}}^F(a)^{\frac{1}{2}-\epsilon}) \quad \textit{for all } \epsilon > 0. \tag{5.4}$$

with some constant C_ϵ.
In particular, all sufficiently large a that are square free relative to L are represented by L if and only if they are represented by L locally everywhere

Proof. As in [9], this is obtained from the L-function estimate proved there by using the generalized Waldspurger formula

$$|a_\nu|^2 << N(\nu)^{\frac{1}{2}} L(\frac{1}{2}, \pi \otimes \chi_\nu) \tag{5.5}$$

of Baruch and Mao [1] for the squares of the Fourier coefficients with square free index of a Hilbert modular form of weight $\frac{3}{2}$.

Note that the adelic setting in which Baruch and Mao work implies that our notion of "square free relative to L" can not cause problems. □

In [9], particular attention is given to the classical problem of sums of three squares. That the result given there (all sufficiently large totally positive square free integers that are sums of three squares locally everywhere are sums of three squares globally) can not directly be generalized to arbitrary totally positive numbers can be seen from the following example:

Example 5.3. *Let $F = \mathbb{Q}(\sqrt{35})$. Then no number of the form $7p^2$ where p is a prime that is a quadratic residue modulo 7 is a sum of three integral squares in $\mathfrak{o}_F$, although 7 is a sum of three integral squares locally everywhere.*

Proof. Let L be the lattice corresponding to the sum of three squares. Since L is unimodular at all odd places we have to check whether 7 is represented by L_{v_2} over $\mathfrak{o}_{Fv_2}$, where v_2 is the (ramified) place of F over 2. This is clear because $7 = (\frac{7}{\sqrt{35}})^2 + (\frac{14}{\sqrt{35}})^2$, so 7 is represented locally everywhere.

The number of spinor genera in the genus of L is computed as in [44, 48]; notice that unlike in the case of ground ring $\mathbb{Z}$ the genus of a lattice L can have several spinor genera without dL being highly divisible by a prime.

We obtain from [44, 23] that the local spinor norm groups of L are

$$\Theta(SO(L_v)) = \mathfrak{o}_{F_v}^\times \cdot (F_v^\times)^2 \text{ for all nonarchimedean } v \tag{5.6}$$

which implies that the set of spinor genera in gen(L) is in bijection with the elements of the proper (or strict) ideal class group of F (i. e. the group of ideals modulo principal ideals generated by a totally positive number) modulo the subgroup generated by the squares of ideal classes.

The proper class group of F is easily seen to be of order 4, generated by the ideals $\mathfrak{p}_5, \mathfrak{p}_7$ over (5) resp. over (7), the factor group by the group generated by the squares of ideal classes has hence order 4 as well, and we see that we have four spinor genera in the genus. (A computation of the genus with the help of the neighbouring lattice algorithm shows that the spinor genus of L consists of 3 isometry classes, while the other spinor genera have $3, 4, 4$ isometry classes). The spinor exceptional square classes are determined as in [45, 53], we find that there is precisely one such class, the class $7 \cdot (\mathfrak{o}_{F_v})^2$ associated to the extension $E = F(\sqrt{-7})$ of F, which is unramified at all places, with $\mathfrak{p}_5$ inert in E/F and $\mathfrak{p}_2, \mathfrak{p}_7$ split in E/F. It follows from [53] that 7 is a spinor exception for gen L.

In order to determine whether 7 is represented by that half of the spinor genera containing the spinor genus of L or by the other half of the spinor genera in the genus we consider the rational representation $7 = (\frac{7}{\sqrt{35}})^2 + (\frac{14}{\sqrt{35}})^2$ from above. The corresponding vector x is in all L_v except for $v = v_5$ and $v = v_7$.

The results of [10, 69] imply (since at v_7 the local norm group is all of $F_{v_7}^\times$ and at the place $v = v_5$ that is inert in E/F the index

$$[\mathfrak{o}_{F_v} x : \mathfrak{o}_{F_v} x \cap L_v] \tag{5.7}$$

is equal to 5) indeed 7 is not represented by the spinor genus of L. Since all places v over a prime p with $(\frac{p}{7}) = +1$ are split in E/F, the local norm group of E at these places is again all of $F_v^\times$, and by [53], $7p^2$ is spinor exceptional too and of course represented by the same spinor genera as 7, hence not by any lattice in the the spinor genus of L. □

On the positive side, the arguments necessary for the transition from square free numbers a to arbitrary a have already been carried out in [56], we have

Lemma 5.4. *Let L, F be as in Theorem 5.2, write*

$$\vartheta(L,z) - \vartheta(\text{spn } L, z) = \sum_{\nu>>0} a_\nu \exp(2\pi i \text{tr}(\nu z)) \tag{5.8}$$

$forz \in \mathfrak{H}_F$, the upper half plane for F.
Then for all $\eta > 0$ and $\nu, n \in \mathfrak{o}_F$ with ν square free relative to L one has

$$|a_{\nu n^2}| = O((N(n))^{\frac{11}{18}+\eta}), \tag{5.9}$$

where the implied constant does not depend on ν.

Proof. This is obtained from the theorem of [56] by inserting the Kim-Shahidi bound [36] for Fourier coefficients of Hilbert modular forms of integral weight.

Notice that the dependence of the implied constant on ν from [56] can be removed since it does only depend on the divisibility of ν by primes dividing the level of L and can hence be replaced by a uniform constant if one assumes ν to be square free relative to L.

Notice that in [56] a slightly modified version of the Shimura lifting is used; this helps to avoid difficulties that arise at primes dividing the level if one tries to use the Shimura correspondence directly. Notice also that in this way a problem in the proof of [12] for numbers divisible by a large power of 2 can be avoided; this has recently been remarked by Blomer [3]. □

Corollary 5.5. *Let F be a totally real number field such that the factor group of the proper ideal class group of F modulo the subgroup generated by the classes of those prime ideals $\mathfrak{p}_v$ dividing 2, for which*

$$\Theta(SO(L_v)) = F_v^\times$$

holds, has odd order. Fix $r \in \mathbb{N}$. Then every totally positive number $a \in \mathfrak{o}_F$ with $N_{\mathbb{Q}}^F(a)$ sufficiently large is represented by L provided it satisfies

- *$a \in q(L_\mathfrak{p})$ for all $\mathfrak{p}|2$*
- *a is totally positive*
- *a is not divisible by $\mathfrak{p}^r$ for primes $\mathfrak{p}|2$ with $L_\mathfrak{p}$ anisotropic.*

Proof. The spinor norms $\Theta(SO(L_v))$ for the nonarchimedean v not dividing 2 are computed as above. For the dyadic v, one has from [23] that

$$\Theta(SO(L_v)) = F_v^\times \quad \text{or } \Theta(SO(L_v)) = \mathfrak{o}_{F_v}^\times \cdot (F_v^\times)^2$$

holds, and the number of spinor genera is computed as in the previous example to be the order of the above factor group of the proper ideal class group modulo squares. Hence under the assumptions made the genus of L consists of a single spinor genus, and a satisfying the conditions of the corollary is represented by L locally everywhere.

The Lemma implies the truth of the assertion, given the estimate on $r(\text{gen } L, a)$ from Section 2. □

Corollary 5.6. *Let F be totally real, (L, q) a positive definite integral $\mathfrak{o}_F$-lattice, $r \in \mathbb{N}$ fixed. Then for sufficiently large integers $a \in q(\text{gen } L)$ that are not divisible by $\mathfrak{p}^r$ for $\mathfrak{p}$ with $L_\mathfrak{p}$ anisotropic, one has:*
a is represented by all classes in the genus of L, unless one of the following conditions holds:

- *a is a spinor exception. In this case a is represented by exactly half the spinor genera in the genus of L, and it is represented by all classes in these spinor genera.*
- *There is a prime $\mathfrak{p}$ inert in E/F, where $E = F(\sqrt{-2a \cdot \det(FL)})$, such that a is a spinor exception for the genus of $\mathfrak{p}L$.*
  In that case a is represented by exactly half the spinor genera in the genus of $\mathfrak{p}L$, and a is represented by $L_i \in \text{gen } L$ if and only if either
  - *a is not represented by the spinor genus of $\mathfrak{p}L_i$.*
  - *a is represented by $\mathfrak{p}L_i$*

  (Notice that if a is represented by the spinor genus of $\mathfrak{p}L_i$ and if the norm of $\mathfrak{p}$ is large, the norm of a need not be "sufficiently large" with respect to the genus of $\mathfrak{p}L_i$, hence a may be represented by some isometry classes in the spinor genus of $\mathfrak{p}L_i$ but not by the given $\mathfrak{p}L_i$ itself. The assertion implies that in that case a is not represented by L_i as well.)

Proof. As in [57] one can first establish a preliminary bound c_1 such that all numbers of norm $\geq c_1$ that are primitively represented by some spinor genus in the genus are indeed represented by all lattices in that spinor genus. To obtain this, we use for the error term $r(\text{spn } L_i, a) - r(L_i, a)$ the bound from Lemma 5.4, and for the main term $r(\text{spn } L_i, a)$ the fact that we can bound $r(\text{spn } L_i, a)$ from below (using the results from [54]) if a is primitively represented by the spinor genus of L_i. We notice here that Korollar 2 of [54] is stated only for $F = \mathbb{Q}$ but continues to be true without that restriction, in the slightly sharper form that replaces representation numbers $r^*(am^2, L_j)$ by $r^*(a, m^{-1}L_j)$ and studies $r^*(a, \mathfrak{m}^{-1}L_j)$ for arbitrary (non principal) ideals $\mathfrak{m}$.
We can then fix a larger bound $c > c_1$ such that all a of norm $\geq c$ in $q_r(\text{gen } (L))$ that do not satisfy one of the conditions given in the corollary are represented by all spinor genera in the genus of L with imprimitivity bounded by some c_2 with $c \geq c_1 c_2^2$. From this we get that a is represented by all classes in the genus of L by applying the result just established for numbers of norm $\geq c_1$ that are represented primitively by some spinor genus.
For the exceptional cases, the case covered by the first condition given is clear. If a satisfies the second condition above it has representations that are primitive at $\mathfrak{p}$ by lattices in precisely half the spinor genera in gen L, namely by those $\text{spn}(L_i)$ for which a is not represented by $\text{spn}(\mathfrak{p}L_i)$ (this follows from [13]), and the remaining imprimitivity of these representations can again be bounded by c_2 if this constant was chosen large enough. As in the beginning of this proof we can then show that a is represented by all lattices in those spinor genera.
On the other hand, for those spn L_i for which a is already represented by $\text{spn}(\mathfrak{p}L_i)$, all representations by lattices in the spinor genus of L_i are already in $\mathfrak{p}L_i$, hence in those spinor genera the representation behaviour for L_i is just the same as that for the smaller lattice $\mathfrak{p}L_i$. In particular, if the norm of $\mathfrak{p}$ was large enough it may happen that a is not represented by all lattices in the spinor genus of $\mathfrak{p}L_i$ and hence also not by all lattices in the spinor genus of L_i. □

Not much has been obtained so far in the direction of effective results in the number field case. As a first step on the modular forms side of the problem, Wichelhaus [67] has given an effective and practically usable criterion for testing two Hilbert modular forms for equality. The work on computing tables of modular forms is still rudimentary. The computation of bounds for the local densities, on the other hand, should be possible essentially in the same way as over $\mathbb{Z}$.

6. The case $F = \mathbb{Q}$, $\mathrm{rk}(K) \geq 2$

In this case not too much has changed since Kitaoka's report [41]. One has an asymptotic formula for $\mathrm{rk}(L) \geq 2\mathrm{rk}(K) + 3$ with the minimum $m(K)$ of K going to infinity under the additional condition $m(K) > C(dK)^{\frac{1}{n}}$ for some constant C if $\mathrm{rk}(K) \geq 3$, and without this additional condition if $\mathrm{rk}(L) \geq 4\mathrm{rk}(K) + 4$. An asymptotic formula of the same type has also been proven for $\mathrm{rk}(L) = 6$, $\mathrm{rk}(K) = 2$ under the additional condition $m(K)^{32.2} < dK$ or upon restriction to K that are related to fixed K_0 by orthogonal similitudes of growing norm.

The crux of the matter in the analytic approach is that for $\mathrm{rk}(K) \geq 2$ it does not suffice to estimate Fourier coefficients of cusp forms, since the difference of theta series of degree ≥ 2 of lattices in the same genus is not cuspidal in general.
The estimates mentioned above have been obtained by giving a general estimate for Fourier coefficients of Siegel modular forms vanishing in all 0-dimensional cusps with a version of the circle method [38] and by using formulas for Fourier coefficients of Klingen-Eisenstein series [39, 4] in order to obtain estimates [37, 6]. It is unclear whether there is much improvement possible. If $\mathrm{rk}(L) \leq 2\mathrm{rk}(K) + 2$, the relevant Eisenstein series of Klingen type have to be defined by analytic continuation and matters become even more difficult. Bounds for Fourier coefficients of cusp forms are quite a bit better, see [7, 5] and the references given there.

The qualitative result of Hsia, Kitaoka, Kneser [27], whose proof is a purely arithmetic extension of Kneser's argument for the case $\mathrm{rk}(K) = 1$, guarantees that for $\mathrm{rk}(L) \geq 2\mathrm{rk}(K)+3$ all K of sufficiently large minimum are represented by L if they are represented locally everywhere. This has been extended to representations with congruence conditions by Jöchner and Kitaoka [32] and to $\mathrm{rk}(L) = 6$, $\mathrm{rk}(K) = 2$ by Jöchner. Chan, Estes and Jöchner [8] extended it to $\mathrm{rk}(L) \geq \mathrm{rk}(K) + 3$ under additional assumptions on fast growth of the successive minima of K and on primitive representation locally everywhere.
All these arithmetic proofs carry directly over to the number field situation, a generalization to hermitian forms has been proved by Hsia and Prieto-Cox [28].

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