# ON EXPLICIT VERSIONS OF TARTAKOVSKI'S THEOREM

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Abstract. We give an explicit bound for representation of numbers by quaternary integral positive definite quadratic forms

## Introduction

Let $q(x_1, \ldots, x_m) = \sum_{i,j=1}^{m} a_{ij} x_i x_j$ be an integral positive definite quadratic form in $m \geq 4$ variables. A famous theorem of Tartakovski states that any sufficiently large positive integer $a$ for which the congruence $q(x_1, \ldots, x_m) \equiv a \bmod c$ is solvable for all $c \in \mathbb{Z}$ with integral coprime $x_1, \ldots, x_m$ is represented (primitively) by the form $q$. This can be proved using the Hardy-Littlewood circle method or using modular forms; an arithmetic proof has been given in [7], [6].

It is well-known that the result is effective, i.e., that one can specify a constant $C(q)$ depending on $q$ such that all $a \geq C(q)$ are sufficiently large in this sense. Such explicit versions have (for $m \geq 5$) been discussed by Watson [9] and by Hsia/Icaza ([6]) and for $m = 4$ in [2, 3].

It is therefore in principle possible to decide in finitely many steps which numbers outside the finitely many arithmetic progressions that are excluded by the congruence conditions above are represented (primitively) by the form. Nevertheless it is in general not feasible to do this using a computer since the size of the constant $C(q)$ is either unknown or too large for practical purposes.

In this note we try therefore to get as good and as explicit as possible estimates on the size of the constant $C(q)$ using the modular forms method in the case $m = 4$. The approach is essentially that of [2, 3], taking care to evaluate constants explicitly.

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## 1. Cusp form estimates

Let $V$ be a vector space of dimension $m$ over $\mathbb{Q}$, equipped with the positive definite quadratic form $q : V \longrightarrow \mathbb{Q}$ with associated symmetric bilinear form $B(x,y) = q(x+y) - q(x) - q(y)$, $M \subseteq V$ a lattice of rank $m$ with $q(M) \subseteq \mathbb{Z}$. The $p$-adic completions $M \otimes_{\mathbb{Z}} \mathbb{Z}_p$ are also denoted by $M_p$, with $M_\infty = V \otimes_{\mathbb{Q}} \mathbb{R} = V_\infty$.

A positive integer $n$ admits (primitive) solutions to the congruence $q(x) \equiv n \bmod c$ (with $x \in M$) for all integers $c$ if and only if $n \in q(M_p)$ for all $p$ (respectively $n$ is represented primitively by all $M_p$); the set of these numbers is also the set of numbers represented (primitively) by some form in the genus of $M$ (for short: by the genus of $M$) and is denoted by $q$ (gen $M$) (resp. $q^*$ (gen $M$) for primitive representations).

The theta series of $(M,q)$ is given by

$$\vartheta(M,q,z) = \sum_{x \in M} \exp(2\pi i q(x) z),$$

the genus theta series by

$$\vartheta(\text{gen } M, q, z) = \left( \sum_{\{K\} \in \text{gen } M} \frac{\vartheta(K,q)}{|O(K,q)|} \right) \left( \sum_{\{K\} \in \text{gen } M} \frac{1}{|O(K,q)|} \right)^{-1},$$

where the summation runs over a set of representatives of the (finitely many) isometry classes in the genus of $M$ and where $O(K,q) = \{\varphi \in \operatorname{Aut}(V) \mid \varphi(K) = K,\ q(\varphi(x)) = q(x) \text{ for all } x \in V\}$ is the group of units (integral automorphs) of $(K,q)$. It is known that $\vartheta(M,q,z)$ is a modular form of weight $k = \frac{m}{2}$ and character $\chi(d) = (\frac{(-1)^{m/2} \det M}{d})$ for the group $\Gamma_0(N)$. Here $N$ is the level of $M$, i.e., $N^{-1}\mathbb{Z} = q(M^{\#})\mathbb{Z}$, where $M^{\#} = \{y \in V \mid B(y,M) \subseteq \mathbb{Z}\}$ is the dual lattice of $M$, and $D := \det M$ is the determinant of the Gram matrix $A = (B(e_i, e_j))$ of $M$ relative to some basis $\{e_1, \ldots, e_m\}$ of $M$. It is also known (although not trivial for $N > 1$) that $\vartheta(\text{gen } M, q, z) - \vartheta(M,q,z) =: h(z)$ is a cusp form. By Siegel's theorem $g(z) := \vartheta(\text{gen } M, q, z)$ is a linear combination of Eisenstein series, and its Fourier coefficients $r(\text{gen } M, q, n) =: a_g(n)$ (the average numbers of representations of $n$ by gen $M$) can be computed as products of local densities measuring the number of congruence representations of $n$ modulo high powers of the primes $p$.

We specialize now to $m = 4$ and let $\{f_1, \ldots, f_r\}$ be an orthonormal basis of $S_2(\Gamma_0(N), \chi)$, where each $f_i$ is of the form $f_i = f_i'|_2(\left(\begin{smallmatrix} N'' & 0 \\ 0 & 1 \end{smallmatrix}\right))$, where $f_i'$ is a newform for $\Gamma_0(N')$ for some $N'|N$ and $N''$ is a divisor of $\frac{N}{N'}$. Orthonormality is understood with respect to the normalized Petersson inner product

$$\langle f, f\rangle = \frac{1}{(\Gamma(1):\Gamma_0(N))}\int_{\mathcal{F}_0(N)} f(z)\overline{g(z)}dxdy,$$

where $\mathcal{F}_0(N)$ is a fundamental domain for $\Gamma_0(N)$. The following Lemma is essentially due to Fomenko [2].

**Lemma 1.1.** *With the notations from above one has*

$$\langle h, h\rangle \leq 0.008777N^5$$

*Proof.* (See [2]). We notice that the Fourier coefficient $\alpha_F(n;x_0)$ at $\exp(\frac{2\pi i n z}{N})$ of the series $\vartheta(M, x_0, q; z) = \sum_{x\in x_0+M}\exp(2\pi i q(x)z)$ for $x_0 \in M^\#$ is not larger than $2^4\cdot(\frac{n}{N})^2\cdot\frac{1}{\sqrt{D}}$; this is most easily seen by comparison with the algorithm given in 3.11 of [8]; the same estimate does of course hold for the average of this quantity over a genus and hence for the Fourier coefficients of $h$. The Fourier coefficient at $\exp(2\pi i\frac{nz}{N})$ in the series denoted by $S(\tau|L^{(j)})$ $(j = 1, 2)$ in [2] is therefore estimated by $2^4(\frac{n}{N})^2\frac{1}{\sqrt{D}}$.

We split the sum

$$\sum_{n=1}^{\infty}\int_{\frac{\sqrt{3}}{2}}^{\infty}(\frac{n}{N})^4 e^{-4\pi ny/N}dy = \frac{N}{4\pi}\sum_{n=1}^{\infty}\frac{1}{N}(\frac{n}{N})^3\exp(-2\pi\sqrt{3}\frac{n}{N})$$

up as

$$\frac{N}{4\pi}\sum_{n=1}^{2N}\frac{1}{N}(\frac{n}{N})^3\exp(-2\pi\sqrt{3}\frac{n}{N}) + \frac{N}{4\pi}\sum_{n=2N+1}^{\infty}\frac{1}{N}(\frac{n}{N})^3\exp(-2\pi\sqrt{3}\frac{n}{N})$$

and interpret the term

$$\frac{1}{4\pi}\sum_{n=2N+1}^{\infty}\frac{1}{N}(\frac{n}{N})^3\exp(-2\pi\sqrt{3}\frac{n}{N})$$

as a Riemann sum which we estimate by

$$\frac{1}{4\pi}\int_2^{\infty}x^3\exp(-2\pi\sqrt{3}x)dx < 0.2375\cdot 10^{-10}.$$

The remaining term

$$\frac{1}{4\pi}\sum_{n=1}^{2N}\frac{1}{N}(\frac{n}{N})^3\exp(-2\pi\sqrt{3}\frac{n}{N})$$

is $\leq 3.42848 \cdot 10^{-5}$ (and tends to $3.404\ldots \cdot 10^{-5}$ for $N$ tending to infinity), and we obtain for the integral (3) on p. 1374 of [2] the estimate $\frac{2^8}{D} \cdot 3.4285 \cdot 10^{-5} \cdot N^2 \leq 0.008777 \cdot \frac{N^2}{D}$. As in [2] this leads to

$$\begin{aligned}\langle h,h\rangle &\leq N^8 \cdot 0.008777 \cdot \textstyle\prod_{p|N}(1-\frac{1}{p^2})/N^3\prod_{p|N}(1-\frac{1}{p^2})\\ &\leq N^5 \cdot 0.008777\end{aligned}$$

(here $N^3\prod_{p|N}(1-\frac{1}{p^2})$ is the index $(\Gamma(1):\Gamma(N))$. □

We can do quite a bit better if the level $N$ happens to be square free:

**Lemma 1.2.** *Let the level $N$ of $(M,q)$ be square free. Then with the notations from above one has*

$$\langle h,h\rangle \leq 0.00877N \tag{1}$$

*Proof.* It is well known (see e.g. [1]) that a set of representatives of $\Gamma_0(N)\backslash\Gamma_0(\frac{N}{p})$ for $p \mid N$ can be chosen to consist of $1_2$ and the matrices

$$w_p\begin{pmatrix}1 & j\\ 0 & 1\end{pmatrix}$$

where $w_p$ is a matrix that is congruent to $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ modulo $p$ and congruent to $1_2$ modulo the other primes $\ell$ dividing $N$, and where $j$ runs through $0,\ldots,p-1$. Iterating this we obtain a set of representatives of $\Gamma_0(N)\backslash SL_2(\mathbf{Z})$ consisting of the matrices

$$w_{N'}\begin{pmatrix}1 & jN/N'\\ 0 & 1\end{pmatrix}$$

for the $N' \mid N$, where $w_{N'}$ is congruent to $1_2$ modulo $N/N'$ and congruent to $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ modulo $N'$ and where $j$ runs through a set of residues modulo $N'$. For $L$ in the genus of $M$ it is known that

$$\vartheta(L,q,\cdot)|_2 w_{N'} = c(N')\frac{1}{D_{N'}^{1/2}}\vartheta(L^{\#,N'},q,\cdot)$$

with a constant $c(N')$ of absolute value 1, with $L^{\#,N'} = L^{\#}\cap \mathbf{Z}[\frac{1}{N'}]L$ , and with $D_{N'}$ denoting the $N'$-part of the discriminant $D$ of $(L,q)$ (i.e., $D_{N'}$ is divisible only by the primes dividing $N'$ and $D = D_{N'}D_{N/N'}$). The Fourier coefficient at $\exp(2\pi i n z/N')$ of $h|_2 w_{N'}$ is estimated as above to be $\leq 2^4(n/N')^2\sqrt{D_{N'}}$, and the integral over a fundamental domain for $\Gamma_0(N)$ of $|h|^2$ becomes (in the same way as above) $\leq 0.008777\sigma_2(N)$ (where $\sigma_2(N) = \sum_{d|N} d^2$ as usual). Dividing by the index $\sigma_1(N)$ of $\Gamma_0(N)$ in $SL_2(\mathbf{Z})$ we obtain the asserted estimate.

□

**Lemma 1.3.** *With the notations from above let*

$$h = \sum_{j=1}^{r} \alpha_j f_j \qquad \text{with } \alpha_j \in \mathbb{C}.$$

*Then*

$$\sum_{j=1}^{r} |\alpha_j|^2 \leq 0.008777 \cdot N^5.$$

*If $N$ is square free we have*

$$\sum_{j=1}^{r} |\alpha_j|^2 \leq 0.008777 \cdot N.$$

*Proof.* Immediate from $\langle h, h\rangle = \sum_{j=1}^{r} |\alpha_j|^2$ and the above Lemmas. □

**Lemma 1.4.** *For the basis vectors $f_j \in S_2(\Gamma_0(N), \chi)$ let*

$$f_j(z) = \sum_{n=1}^{\infty} a_{f_j}(n) \ \exp(2\pi i n z).$$

*Then*

$$|a_{f_j}(n)| \leq n^{1/2}\sigma_0(n) \cdot 1898.27 \cdot N^{1/2} \sqrt{\prod_{p|N}(1+\frac{1}{p})}$$

*where $\sigma_0(n) = \sum_{1\leq d|n} 1$ is the number of positive divisors of $n$.*

*Proof.* If $f_j$ is a newform one has $|a_{f_j}(n)| \leq n^{1/2}\sigma_0(n)|a_{f_j}(1)|$ by Deligne's estimate, if $f_j = f_j'|\begin{pmatrix} N'' & 0 \\ 0 & 1\end{pmatrix}$ with a newform $f_j'$ of level $N'|N$ and $N'N'' \mid N$ we have

$$|a_{f_j}(n)| = |a_{f_j'}(\frac{n}{N''})| \leq (\frac{n}{N''})^{1/2}\sigma_0(\frac{n}{N''})|a_{f_j'}(1)|.$$

To prove our estimate it suffices hence to show

$$|a_{f_j}(1)| \leq 1898.27 N^{1/2} \cdot \sqrt{\prod_{p|N}(1+\frac{1}{p})}$$

for a newform $f_j$ with $\langle f_j, f_j\rangle = 1$ of level $N$. By the Main Theorem of [4] there is an effective constant $c_2$ with

$$|a_{f_j}(1)|^2 \leq c_2 \max(N^{1/2}, \log(2N+1)).$$

Since it seems to be rather difficult to make this constant explicit and since it appears to be quite large, we prefer (with applications to moderate $N$ in mind) to follow [2] (an ineffective estimate $|a_{f_j}(1)| \leq c_3(\epsilon)N^{\epsilon}$ for any $\epsilon > 0$ has been given in [3] and in [5, 4]).

As in Lemma 2 of [2] we see (keeping in mind that we use the normalized Petersson product)

$$\begin{aligned}|a_{f_j}(1)|^2 &\leq 4\pi e^{4\pi}\cdot N\prod_{p|N}(1+\frac{1}{p}), \text{ hence}\\ |a_{f_j}(1)| &\leq 1898.2685\cdot N^{1/2}\sqrt{\prod_{p|N}(1+\frac{1}{p})}.\end{aligned}$$

□

**Lemma 1.5.** *For the Fourier coefficients $a_h(n)$ of $h$ the estimate*

$$|a_h(n)| \leq 51.3382\ N^{7/2}\cdot n^{1/2}\sigma_0(n)\sqrt{\prod_{p|N}(1+\frac{1}{p})}$$

*holds.*
*If $N$ is square free, one has*

$$|a_h(n)| \leq 51.3382\ N^{3/2}\cdot n^{1/2}\sigma_0(n)\sqrt{\prod_{p|N}(1+\frac{1}{p})}$$

*Proof.* We have

$$\begin{aligned}|a_h(n)|^2 &= |\textstyle\sum_{j=1}^r \alpha_j a_{f_j}(n)|^2\\ &\leq \textstyle\sum_{j=1}^r |\alpha_j|^2 \sum_{j=1}^r |a_{f_j}(n)|^2\\ &\leq 0.008777\cdot N^5\cdot n\cdot \sigma_0(n)^2\cdot \frac{N^2}{12}\cdot \prod_{p|N}(1+\frac{1}{p})\cdot 4\pi e^{4\pi},\end{aligned}$$

hence

$$|a_h(n)| \leq 51.3382\cdot N^{7/2}\cdot n^{1/2}\sigma_0(n)\sqrt{\prod_{p|N}(1+\frac{1}{p})}.$$

The estimate in the case of square free $N$ follows in the same way. □

## 2. Bounds for representation of numbers

By Siegel's mass formula (see [7]) we can write the Fourier coefficients of the Eisenstein series $g$ as

$$a_g(n) = \frac{4\pi^2}{6}\prod_p \alpha_p(n)$$

where

$$\alpha_p(n) = (2n)_p\cdot(1-(\frac{D}{p})p^{-2})\sum_{p^j|n}(\frac{D}{p^j})p^{-j} \quad (\text{with } (2n)_p = p^\mu \text{ if } p^\mu \| 2n)$$

holds for $p\nmid N$. In general we have (with $p^{r-1}\|N$) the primitive density

$\alpha_p^*(n) = (2n)_p p^{-3r}(D_p)^{1/2} \mid \{\overline{x}\in M_p/p^r\cdot M_p^\# \mid p^{-1}x\notin M_p,\ q(\overline{x})\equiv a \bmod p^r\}$

and
$$\alpha_p(n) = \sum_{p^{2j}|n} \alpha_p^*(\frac{n}{p^{2j}}).$$

**Lemma 2.1.** *We put*
$$c_4(n,q) := \prod_{p|N} \frac{\alpha_p(n)}{(2n)_p(1-p^{-2})}.$$
*Then one has*
$$\begin{aligned} a_g(n) &\geq 8nc_4(n,q)\prod_{\substack{p\nmid N\\ p|n}}(1+(\tfrac{D}{p})p^{-1}) \\ &\geq 8\varphi(n)c_4(n,q). \end{aligned}$$
*Moreover, one has*
$$\begin{aligned} c_4(n,q) &\geq N^{-1/2}\prod_{p|N}(1-\tfrac{1}{p})\prod_{\substack{p|N\\ p^2|D}}\tfrac{2\cdot(N)_p^{-1}}{(p^2-1)(1-\frac{1}{p})} \\ &\geq \max(N^{-3/2}\prod_{p|N}\tfrac{2}{(p^2-1)}, D^{-7/4}). \end{aligned}$$
*If $N$ is square free, one has*
$$c_4(n,q) \geq N^{-2}\prod_{p|N}\frac{2}{1+p}$$
*with*
$$c_4(n,q) \geq N^{-1/2}\prod_{p|N}\frac{1}{1+\frac{1}{p}}$$
*if $D$ is square free.*

**Remark 1.** a) The bound on $c_4(n,q)$ in Lemma 2.1 is extremely coarse. If one wants to compute the numbers represented by $q$ explicitly using a computer it is much better to compute the $\frac{\alpha_p(n)}{(2n)_p(1-p^{-2})}$ for the $p|n$ explicitly for all distributions of $p$-adic orders $\mathrm{ord}_p(n)$ and $p$-adic square classes of $n$ for the $p|N$; note that it will suffice to do this up to $\mathrm{ord}_p(n) \geq \mathrm{ord}_p(N)+1$.
b) In the special case that the level $N$ of $(M,q)$ is square free we have more precisely the following bounds for the $p$-factor of $c_{4,p}(n,q)$ of $c_4(n,q)$ :

- $$c_{4,p}(n,q) \geq p^{-1/2}(1+\frac{1}{p})^{-1}$$
  if $D_p = p$
- $$c_{4,p}(n,q) \geq 2p^{-3}(1+\frac{1}{p})^{-1}$$
  if $D_p = p^2$

- $$c_{4,p}(n,q) \geq p^{-3/2}(1+\frac{1}{p})^{-1}$$
  if $D_p = p^3$

In most cases the actual value of $c_{4,p}(n,q)$ is larger.

*Proof of Lemma* 2.1. One has to calculate the local primitive densities $\alpha_p^*(n)$ in the possible cases and find a common estimate. For $p \not| N$ we estimate $\alpha_p(n)$ from below by

$$(2n)_p(1-(\frac{D}{p})p^{-2} \cdot \begin{cases} 1 & \text{if } p \not| n \\ 1+(\frac{D}{p})p^{-1} & \text{if } p|n \end{cases}$$

Bounding $\prod_{p \not| N}(1-(\frac{D}{p})p^{-2})\prod_{p|N}(1-p^{-2})$ from below by $\prod_p(1-p^{-2}) = (\zeta(2))^{-1} = \frac{6}{\pi^2}$ we obtain the estimate for $a_g(n)$ in terms of $c_4(n,q)$ as asserted.
For estimating the $p$-factor of $c_4(n,q)$ for $p|N$ the worst case turns out to be the case in which the unimodular component of $M$ is 1-dimensional. In this case we get a lower bound of

$$2 \cdot p^{-\frac{3}{2}r-2} \cdot \frac{1}{(1-p^{-2})},$$

which gives rise to the general bound asserted in the Lemma. The bounds given in the square free case are proved in the same way. □

**Lemma 2.2.** *One has*

$$\begin{array}{rcl} a_g(n) & \geq & 8\varphi(n)N^{-3/2}\prod_{p|N}\frac{2}{(p^2-1)} \\ & \geq & 8\frac{n}{\log\ n}N^{-3/2}\prod_{p|N}\frac{2}{(p^2-1)}, \end{array}$$

*the latter inequality being true for $n > 30$.*

*Proof.* The first inequality is obvious from Lemma 2.1, the second follows from $\varphi(n) \geq \frac{n}{\log\ n}$ for $n > 30$, which is easily proved.
A bound that is sharper for large $n$ can be obtained by deriving an explicit constant in the well known estimate $\varphi(n)/n \geq C \cdot (\log\log\ n)^{-1}$. □

**Theorem 2.3.** *Let $n$ be primitively represented by all $M_p$. Then $n$ is represented by $M$ if*

$$\frac{\varphi(n)}{n^{1/2}\sigma_0(n)} \geq 6.418 \cdot N^5 \prod_{p|N}\frac{p^2-1}{2} \cdot \sqrt{\prod_{p|N}(1+\frac{1}{p})}$$

*holds, and where we have* $\varphi(n) \geq \frac{n}{\log\ n}$ *for* $n > 30$.
*If the level* $N$ *is square free, one has the bound*

$$\frac{\varphi(n)}{n^{1/2}\sigma_0(n)} \geq 6.418 \cdot N^{9/2} \prod_{p|N} \frac{(1+\frac{1}{p})^{3/2}}{2}$$

*If* $D = \det\ M$ *is square free one has the even better bound*

$$\frac{\varphi(n)}{n^{1/2}\sigma_0(n)} \geq 6.418 \cdot N^2 \cdot \prod_{p|N}(1+\frac{1}{p})^{3/2}.$$

*Proof.* This follows from the lemmas given. □

**Remark 2.** a) The result obtained in the case of square free level is quite reasonable in the $N$-aspect for moderate $N$, if one takes into account the fact that the estimate for the local densities that has been used here is far from optimal (if the determinant $D$ is not square free) and can be replaced by a precise calculation for each individual case studied. The key point for further improvement in the case of not square free level appears to be Lemma 1.1., whereas an explicit version of the result of [4] promises only little improvement, in particular since it is likely to lead to a worse constant in the estimate.
b) The real bottleneck for practical applications is the number of divisors $\sigma_0(n)$ appearing in the theorem, even though it is asymptotically $O(n^\delta)$ for all $\delta > 0$. For $n$ around $10^{10}$ (e.g., $n = 8031343320 = 2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$) it can at least be as large as 2048, which gives a quotient $\sqrt{n}/\sigma_0(n)$ of about 50 (or 43.76 in the case mentioned). Even for numbers around $10^{12}$ this quotient can still be as small as 173.

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