Paramodular theta series

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Outline

The paramodular group - definition

General n: Set $P = \begin{pmatrix} t_1 & 0 & \dots & 0 \\ & t_2 & & \\ & & \ddots & \\ 0 & \dots & 0 & t_n \end{pmatrix}$ with $t_i \mid t_{i+1}$,

$$\Gamma^{((2\cdot)n)}(P) = Sp_{(2)n}(\mathbb{Q}) \cap \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} M_{2n}(\mathbb{Z}) \begin{pmatrix} 1 & 0 \\ 0 & P^{-1} \end{pmatrix}$$

is the paramodular group of level P.

In particular, for $n = 2, P = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$:

$$\Gamma_t^{((2\cdot)2)} = \begin{pmatrix} * & t* & * & * \\ * & * & * & */t \\ * & t* & * & * \\ t* & t* & t* & * \end{pmatrix}.$$

Frequently: Instead: $*t$ in positions $(2,1), (3,1), (3,2), (3,4), (4,1)$ and $*/t$ in position $(1,3)$ (use $P = \left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$).

History

The group was considered by Conforto (1951)

The name paramodular appears in Shimura's article in Séminaire Cartan (1958)

Investigated by:

- Koecher: Math. Nachr. 13 (1955)
- Siegel: Nachr. Akad. Wiss. Göttingen (1960)
- Christian: Mathematische Annalen 168 (1967)
- Köhler: Nachr. Akad. Wiss. Göttingen (1967)
- Kappler: Doctoral thesis Freiburg 1977 (advisor: Köhler)
- Ibukiyama: Advanced Studies in Pure Mathematics 7 (1985)
- Gritsenko: Sém. DPP 1992/93, Math. Gottingensis 1995
- Delzeith: Doctoral thesis Heidelberg 1995 (advisor: Freitag)
- Runge: Abh. Math. Sem. Hamburg 66 (1996)
- Marschner: Doctoral thesis Aachen 2004 (advisor: Krieg)
- Roberts, Schmidt: Springer Leture Notes 1918 (2007)
- Poor, Yuen: Preprint 2009
- Brumer, Kramer: Preprint 2010

Sidetrack: The orthogonal model

Well known: $PGSp_{(2\cdot)2} \cong SO(3,2)$.
Via the operation of GL_4 on $\bigwedge^2(W)$, $\dim(W)=4$, with $\bigwedge^2(W)$ identified with the space alternating bilinear forms on W, the symplectic group being the stabilizer of the standard alternating form.

This leads to (several) ways to realize the paramodular group and the usual congruence subgroups of the integral symplectic group as (subgroups of) orthogonal groups of lattices on a (split) 5-dimensional quadratic space.

For this, denote by $P\Gamma_t^{((2\cdot)2)}$ the image of $\Gamma_t^{((2\cdot)2)}$ in the projective symplectic group $PGSp$ and by $P\Gamma_t^{((2\cdot)2)*}$ its extension by the Atkin-Lehner involutions.

Sidetrack: The orthogonal model, continued

Method 1: Set $M_t \cong H \perp H \perp \langle 2t \rangle$, i.e.,

M_t is a lattice with Gram matrix $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2t \end{pmatrix}$.

Then $P\Gamma_t^{((2\cdot)2)*} \cong SO(M_t)$.
Notice: In this picture, the involutions belonging to primes p dividing t to an odd power can be distinguished by a spinor norm condition.

Advantage: Natural inclusions via "Watson transformations":
$O(M_{tr^2}) \subseteq O(M_t)$, since $M_t^{\#} = \{x \in M_{tr^2}^{\#} \mid Q(x) \in Q(M_t^{\#})\}$ (and dual lattices have the same orthogonal group).
Disadvantage: The M_t don't live on the same quadratic space.

Sidetrack: The orthogonal model, continued

Modification: Set $M_t = Ze_1 + \mathbb{Z}f_1 + \mathbb{Z}e_2 + \mathbb{Z}f_2 + \mathbb{Z}e_0 \cong H \perp H \perp \langle 2t \rangle$.
Set $L'_t := {}^tM_t^\#$ (quadratic form multiplied by factor t),
let L_t be the sublattice of even integral vectors of L'_t.
Then L_t has Gram matrix

$$\begin{pmatrix} 0 & t & 0 & 0 & 0 \\ t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

and the same orthogonal group as M_t, all the L_t sit on the same quadratic space.

For t square free the $SO(L_t)$ are maximal discrete subgroups of the real $SO(3,2)$, isomorphic to the maximal extensions of the projective paramodular group (by involutions normalizing the group).

Sidetrack: The orthogonal model, local picture

As above: L_t has Gram matrix $\begin{pmatrix} 0 & t & 0 & 0 & 0 \\ t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$.

Locally, $SO((L_1)_p), SO((L_p)_p)$ and $SO(M'_p)$,
with $M'_p = \mathbb{Z}_p e_1 + \mathbb{Z}_p f_1 + \mathbb{Z}_p(pe_2) + \mathbb{Z}_p f_2 + \mathbb{Z}_p e_0$ having Gram matrix
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 \\ 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$,
are the three conjugacy classes of maximal compact subgroups of the orthogonal group over $\mathbb{Q}_p$.
The last one is isomorphic to the extension of the projective local version of $\Gamma_0^{(2)}(p)$ by the local Fricke involution.

Theta constants

Various polynomials in classical theta constants are used to construct paramodular forms by Freitag, Kappler, Gritsenko, Ibukiyama/Onodera.

Runge defines for an elementary divisor matrix P as above "partial theta functions"

$$f_{\mathbf{a}}^{(m,P)}(\tau, z) = \sum_{\mathbf{x}\in\mathbb{Z}^g} e(m\tau[P\mathbf{x} + \frac{\mathbf{a}}{2m}] + \langle P\mathbf{x} + \frac{\mathbf{a}}{2m}, 2mz\rangle)$$

(with $\tau \in \mathfrak{H}_g, z \in \mathbb{C}^g, \mathbf{a} \in \mathbb{Z}^g, m \in \mathbb{Z}$) and obtains for $z = 0$ paramodular theta constants.

Oura's code construction

Oura (C. R. Acad. Sci. Paris 328 (1999)) uses codes over $\mathbb{Z}/\ell Z$ and Runge's partial theta functions $f_{\mathbf{a}}^{(m,P)}(\tau, 0)$:
Let $\ell_1 \mid \ell_2 \mid \dots \mid \ell_g$ be positive integers and $\mathcal{C}_i \subseteq (\mathbb{Z}/2\ell_i\mathbb{Z})^n (i = 1, \dots, g)$ be linear codes over $\mathbb{Z}/2\ell_i\mathbb{Z}$ of length n.
Call this sequence of codes $\mathcal{C}_i$ an $(\ell_1, \ell_2, \dots, \ell_g)$ -code of length n if it satisfies for $1 \le i < j \le g$:

- $x \in \mathcal{C}_j \Rightarrow x \bmod 2\ell_i \in \mathcal{C}_i$
- $y \in \mathcal{C}_i \Rightarrow \frac{\ell_j}{\ell_i} y \in \mathcal{C}_j$.

Denote by R the quotient of $(\mathbb{Z}/2\ell_1\mathbb{Z} \times \dots \times \mathbb{Z}/2\ell_g\mathbb{Z})$ by identification of a with $-a$, for $(c_1, \dots, c_g) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_g$ and $a \in R$ denote by $\nu_a(c_1, \dots, c_g)$ the number of places in which the entries of $c_1, \dots, c_g$ project to $a \in R$.
Put $W((X_a)_{a\in R}) = \sum_{(c_1,\dots,c_g)} \prod_{a\in R} X_a^{\nu_a(c_1,\dots,c_g)}$ (symmetrized weight enumerator of the sequence of codes).

Oura's code construction, continued

$f_{\mathbf{a}}^{(m,P)}(\tau, z) = \sum_{\mathbf{x}\in\mathbb{Z}^g} e(m\tau[P\mathbf{x} + \frac{\mathbf{a}}{2m}] + \langle P\mathbf{x} + \frac{\mathbf{a}}{2m}, 2mz\rangle)$

$W((X_a)_{a\in R}) = \sum_{(c_1,\dots,c_g)} \prod_{a\in R} X_a^{\nu_a(c_1,\dots,c_g)}$

Then one has for an elementary divisor sequence $1 = k_1 \mid \dots \mid k_g$, $P = \mathrm{diag}(k_1, \dots, k_g)$, $k \in \mathbb{Z}_{>0}$ and a $(kk_1, \dots, kk_g)$-code $(\mathcal{C}_1, \dots, \mathcal{C}_g)$ of length n, which is of type II (self dual doubly even):

$W((f_{\mathbf{a}}^{(k,P)}(\tau, 0))_{a\in R})$ is a modular form of weight $n/2$ for $\Gamma(P)$.

Oura also gives an interpretation in terms of lattices $\Lambda(\mathcal{C}_i)$ (rescaled inverse image of $\mathcal{C}_i$ under $\mathbb{Z}^n \to (\mathbb{Z}/2kk_i\mathbb{Z})^n$); these lattices are k_i-scaled copies of unimodular lattices.

Theta series of lattices

Okazaki/Yamauchi (Math. Annalen 341 (2008)) construct paramodular Yoshida liftings, using suitable test functions in the framework of the oscillator (Weil) representation.

These test functions belong to lattices similar to Oura's $\Lambda(\mathcal{C}_i)$.

Both constructions generalize as follows:

Theta series of lattices, continued

Proposition

Let $L_1, \dots, L_g$ be lattices on the positive definite quadratic space V of dimension m over $\mathbb{Q}$ with quadratic form Q and associated symmetric bilinear form $b(x,y) = Q(x+y) - Q(x) - Q(y)$.
Put for $\tau \in \mathfrak{H}_g$

$$\vartheta(L_1, \dots, L_g; \tau) := \sum_{\mathbf{x}=(x_1,\dots,x_g)\in L_1\times\cdots\times L_g} \exp(2\pi i \operatorname{tr}(Q(\mathbf{x})\tau)),$$

where $2Q((x_1, \dots, x_g))$ is the matrix of the $b(x_i, x_j)$,
and let $P = \operatorname{diag}(t_1, \dots, t_g)$ be an elementary divisor matrix.

Then $\vartheta(L_1, \dots, L_g; \tau)$ is a modular form of weight $m/2$ for the paramodular group $\Gamma^{((2\cdot)g)}(P)$ if $L_{i+1} \subseteq L_i (1 \le i < g)$ and each L_i is a t_i-scaled copy of an even unimodular lattice (an even t_i-modular lattice).

Theta series of lattices, continued

Proof.

$\Gamma^{((2\cdot)g)}(P)$ is generated (Kappler) by

$$J_P = \begin{pmatrix} 0 & -P^{-1} \\ P & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & t_i^{-1}E_{ii} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & t_i^{-1}(E_{ij}+E_{ji}) \\ 0 & 1 \end{pmatrix} (1 \le i < j \le g).$$

ϑ is invariant under the translations above if
$t_i \mid b(x_i, x_j)(1 \le i < j \le g), t_i \mid Q(x_i)$,
this is satisfied if and only if one has
$L_j \subseteq t_i L_i^{\#}(i < j)$ and L_i is a t_i-scaled copy of an even integral lattice M_i.

J_P transforms ϑ into (a multiple of) $\vartheta(t_1 L_1^{\#}, \dots, t_g L_g^{\#})$ (modify the usual proof for the action of J_I on $\vartheta^{(g)}(L)$ or use the oscillator representation). Since $t_i L_i^{\#} \cong L_i$ if and only if M_i above is unimodular, we obtain the assertion.

□

An example

Example: Let $g = 2$ and $P = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Let M be any of the Niemeier lattices.

It is known that the Leech lattice Λ_{24} contains sublattices isometric to ${}^{(2)}M$ (quadratic form scaled by 2).

Let K be any such sublattice of Λ_{24}.

Then $\vartheta(\Lambda_{24}, K)$ is a modular form for the paramodular group of degree (genus) 2 and level 2.

Some remarks

Remark:

a) One can insert a harmonic form in the formulation of the theorem and obtain theta series of higher weight or vector valued theta series in the usual way.

b) If in degree (genus) 2 the level is p, the Fricke involution exchanges the unimodular lattices M_1, M_2 underlying the lattices L_1, L_2 in $\vartheta(L_1, L_2)$. For an arbitrary composite level t the situation with respect to the Atkin-Lehner involutions becomes more complicated; the full Fricke involution acts in the same way as above.

c) It would be interesting to investigate the transformation behaviour of an arbitrary $\vartheta(L_1, \ldots, L_g)$.