Global Gross-Prasad conjecture for $SO(5)$ and Yoshida's theta lifting

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Gross-Prasad-conjecture: Local or global situation

$SO(n) = G, \quad SO(n-1) = G' \subseteq G, \quad H = G' \overset{\triangle}{\hookrightarrow} G' \times G$

π, π' irreducible representations of G, G' resp. (cuspidal automorphic in global case, admissible in local case).

Local Problem: When is $\mathrm{Hom}_H(\pi \otimes \pi', \mathbf{C}) \neq 0$?
(Does there exist an H- invariant linear functional on $\pi \times \pi'$?)
equivalently: is $\mathrm{Hom}_H(\pi, \check{\pi}') \neq 0$?

If existent, the invariant linear functional is unique.

Global problem: Is the invariant functional

$$I_H(\varphi) = \int_{H(\mathbf{Q}) \backslash H(\mathbf{A})} \varphi(h) dh$$

nonzero?

In this talk: Global situation over $\mathbf{Q}$.

Example: $G = SO(4)$ split:
$G \cong \{(g_1, g_2) \in GL_2 \times GL_2 \mid \det(g_1) = \det(g_2)\}/Z(GL_2)_\Delta$,

(g_1, g_2) acting on $(M(2 \times 2), \det)$ by $A \mapsto g_1 A g_2^{-1}$.

$G' \cong PGL_2 \overset{\Delta}{\hookrightarrow} G$ acting on trace 0 matrices

π_1, π_2, π_3 irreducible cuspidal automorphic representations of GL_2, central character of $\pi_1 \otimes \pi_2$ and of π_3 trivial,
$\pi := \pi_1 \otimes \pi_2$ (as rep. of $G = SO(4)$), $\pi' = \pi_3$ (as rep. of $G' = PGL_2$).

Similarly for D quaternion algebra over $\mathbf{Q}$ with reduced norm n.

$G^D = SO(D) \cong \{(g_1, g_2) \in D^\times \times D^\times \mid n(g_1) = n(g_2)\}/Z(D^\times)_\Delta$,,
acting on (D, n) as above,

$G'^D \cong PD^\times \overset{\Delta}{\hookrightarrow} G$ acting on trace 0 quaternions D^0

π_1, π_2, π_3 irreducible cuspidal automorphic representations of $GL_2(\mathbf{A})$,
central character of $\pi_1 \otimes \pi_2$ and of π_3 trivial,
$\pi := \pi_1 \otimes \pi_2$ (as rep. of $G(\mathbf{A}) = SO(4, \mathbf{A})$), $\pi' = \pi_3$ (as rep. of $G'(\mathbf{A}) = PGL_2(\mathbf{A})$).

S set of places, where all of π_1, π_2, π_3 are discrete series,
D a quaternion algebra over $\mathbf{Q}$ unramified outside S,
π_i^D the Jacquet-Langlands lift of π_i to $D^\times(\mathbf{A})$.

Jacquet conjecture: (Harris/Kudla)

$$L(\pi_1 \otimes \pi_2 \otimes \pi_3, 1/2) \neq 0 \Leftrightarrow \int\limits_{\mathbf{Q}_{\mathbf{A}}^\times Z(D_{\mathbf{A}}^\times) \backslash D_{\mathbf{A}}^\times} \varphi_1^D(x) \varphi_2^D(x) \varphi_3^D(x) dx \neq 0$$

for some quaternion algebra D as above and some $\varphi_i^D \in \pi_i^D$.
(Notice: Integral becomes a finite sum if D is ramified at ∞.)

R.H.S.:$\Leftrightarrow$ Invariant functional
$I((\varphi_1^D \otimes \varphi_2^D) \otimes \varphi_3^D)$ on $\underbrace{(\pi_1^D \otimes \pi_2^D)}_{\pi^D \text{ on } G^D = SO(D)} \otimes \underbrace{\pi_3^D}_{\pi'^D \text{ on } G'^D = SO(D^0)}$ is nonzero for some D.

Prasad: Such a functional exists locally at v for precisely one D_v.

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Gross/Kudla: Compute $L(\pi_1 \otimes \pi_2 \otimes \pi_3, 1/2)$ explicitly as multiple of $(I(\varphi_1^D \otimes \varphi_2^D \otimes \varphi_3^D))^2$ for hol. newforms of weight 2.
Böcherer/SP: same for triple of weights "balanced" (weights are sides of a triangle, quaternion algebra is ramified at ∞), Watson: Sum of weights 0.

Gross-Prasad conjecture:

$SO(n) = G, \quad SO(n-1) = G' \subseteq G, \quad H = G' \overset{\Delta}{\hookrightarrow} G' \times G,$
π, π' irred. rep. of G, G' resp. (cuspidal automorphic resp. admissible), central character of $\pi \times \pi'$ trivial.

Locally H-invariant functional exists for precisely one inner form $\tilde{G}, \tilde{G}'$ and $\tilde{\pi}, \tilde{\pi}'$ of $\tilde{G}, \tilde{G}'$ corresponding to π, π' (assume π, π' to have generic parameters), nonzero on spherical vector at places where π, π' are unramified.

Global nonvanishing of $I_{\tilde{H}}$ (on some inner form) depends on central value of L-function $L(\pi \otimes \pi', s)$.

Here: $G' = SO(4) \subseteq G = SO(5) \cong PGSp(2 \cdot 2)$,

in $G = PGSp(2 \cdot 2)$ we have $G' = \left\{ \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix} \right\}$.

Goal: Compute $I(\varphi)$ explicitly for φ corresponding to holomorphic modular forms!

F holomorphic Siegel modular form on the Siegel uper half plane

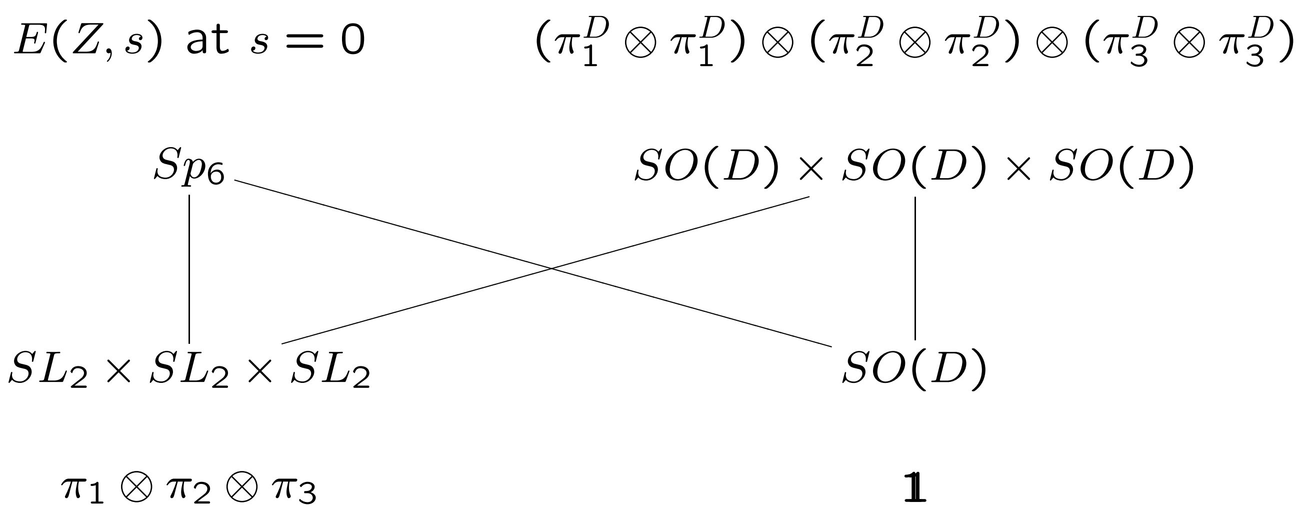
$$\mathbf{H}_2 = \{Z = X + iY \in M_2^{\mathsf{sym}}(\mathbf{C}) \mid Y > 0\}$$

(possibly vector valued), f_1, f_2 elliptic modular forms for Γ, compute

$$\int_{(\Gamma\backslash\mathbf{H})\times(\Gamma\backslash\mathbf{H})} F\left(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}\right) \overline{f_1(z_1) f_2(z_2)} d^* z_1 d^* z_2.$$

Should get connection to $L(\mathsf{S}pin(F), f_1, f_2, s)$.

Watson's talk: Jacquet conjecture and explicit formulae "explained" by see-saw identity:

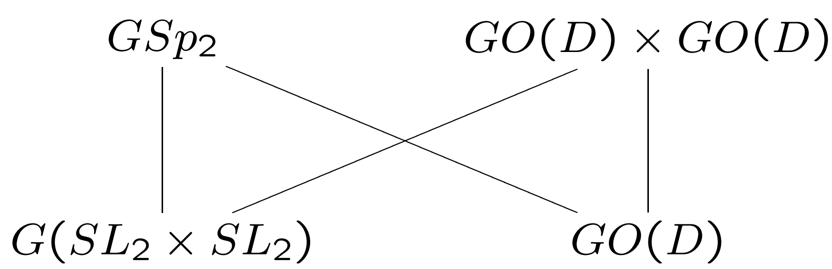


The slanted lines denote theta liftings, the vertical lines denote inclusions.

The line between Sp_6 and $SO(D)$ represents the Siegel-Weil formula.

We use a seesaw here too:

$$\begin{array}{cc} \Pi & (\pi_1^D \otimes \pi_1^D) \otimes (\pi_2^D \otimes \pi_2^D)\ . \\ \\ F & (\psi_1^D \otimes \psi_1^D) \otimes (\psi_2^D \otimes \psi_2^D) \end{array}$$



$$\begin{array}{cc} (f_1, f_2) & {\varphi'}_1^D \otimes {\varphi'}_2^D \\ \\ \pi_1 \otimes \pi_2 & {\pi'}_1^D \otimes {\pi'}_2^D \end{array}$$

Use D ramified at ∞ (so with positive definite norm form).

Theta liftings:

V over $\mathbf{Q}$ even dimensional quadratic space with form $Q(x) = \frac{1}{2}B(x,x)$.

The oscillator representation ω of $O(V,\mathbf{A}) \times Sp(2n,\mathbf{A})$ acts on $S(V(\mathbf{A})^n$, gives rise to the theta kernel

$$\theta^{(V,n)}(g,h;f) = \theta^n(g,h;f) := \sum_{\mathbf{x}\in V(\mathbf{Q})^n} \omega(g)f(h^{-1}\mathbf{x}).$$

For a cusp form φ on $O(V,\mathbf{A})$ and a test function $f \in S(V(\mathbf{A})^n)$ write

$$\Theta_f^{(n)}(\varphi)(g) := \int_{O(V,\mathbf{Q})\backslash O(V,\mathbf{A})} \varphi(h)\theta^n(g,h;f)dh,$$

this is an automorphic form on $Sp(2n,\mathbf{A})$.

For Q positive definite one has $O(V,\mathbf{A}) = \cup_{j=1}^t O(V,\mathbf{Q})h_j O(L,\mathbf{A})$ for a lattice $L \subseteq V$, $\Theta_f^{(n)}(\varphi)(g)$ is then a finite linear combination of classical theta series with spherical harmonic polynomials associated to a lattice $L \subseteq V$:

$$\vartheta_{L,\mathbf{y}}^{(n)}(P,z) = \sum_{\mathbf{x}\in\mathbf{y}+L^n} P(\mathbf{x}) \exp(2\pi i \operatorname{tr}(Q(\mathbf{x})Z))$$

where $\mathbf{y} \in V(\mathbf{Q})^n, Q(\mathbf{x}) = (\frac{1}{2}B(x_k,x_l)) \in M_n^{\mathrm{sym}}(\mathbf{Q})$ for $\mathbf{x} = (x_1,\ldots,x_n) \in V(\mathbf{Q})^n$.

Yoshida's lifting:

h_1, h_2 cuspidal newforms of weight 2, trivial character for $\Gamma_0(N)$ (N square-free) (for simplicity), having the same Atkin-Lehner eigenvalues for $p \mid N$, associated representations π'_1, π'_2.

D quaternion algebra $/\mathbf{Q}$, ramified at $\infty$and some places dividing N, R an order of level N in D
(maximal at the ramified places, intersection of two maximal orders with index p in both of them at unramified primes $p \mid N$).

h_1, h_2 have Jacquet-Langlands lift φ_1^D, φ_2^D to functions $D_{\mathbf{A}}^\times \to \mathbf{C}$ (new vectors in the representations ${\pi'_1}^D, {\pi'_2}^D$).

With $R_{\mathbf{A}}^\times = D_\infty^\times \times \prod_p R_p^\times$ we have the double coset decomposition

$$D_{\mathbf{A}}^\times = \cup_{i=1}^r D^\times y_i R_{\mathbf{A}}^\times$$

(r the number of classes of left ideals of the order R).
$e_i = \mid (y_i R y_i^{-1})^\times \mid$ is the number of units of the order $R_i = y_i R y_i^{-1}$ of D.

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The Yoshida lifting of degree 2 of φ_1^D, φ_2^D is:

$$Y^{(2)}(\varphi_1^D, \varphi_2^D)(Z) = \sum_{i,j=1}^r \frac{1}{e_i e_j} \varphi_1^D(y_i) \varphi_2^D(y_j) \vartheta^2(y_i R y_j^{-1}, Z)$$

where $\vartheta^2(y_i R y_j^{-1}, Z) = \sum_{(x_1,x_2)\in(y_i R y_j^{-1})^2} \exp(2\pi i \mathrm{tr}\left(\begin{pmatrix} n(x_1) & tr(\overline{x_1}x_2) \\ tr(\overline{x_1}x_2) & n(x_2) \end{pmatrix} Z\right)$.

It is nonzero (Bö-SP), cuspidal if $h_1 \neq h_2$.
It is the theta lifting of $\varphi_1^D \otimes \varphi_2^D$ from $SO(D)$ to $Sp(2 \cdot 2)$.

$$Y^{(2)}(\varphi_1^D,\varphi_2^D)(Z)=\sum_{i,j=1}^{r}\frac{1}{e_ie_j}\varphi_1^D(y_i)\varphi_2^D(y_j)\vartheta^2(y_iRy_j^{-1},Z)$$

where $\vartheta^2(y_iRy_j^{-1},Z)=\sum_{(x_1,x_2)\in(y_iRy_j^{-1})^2}\exp(2\pi i\mathrm{tr}\left(\begin{pmatrix} n(x_1) & tr(\overline{x_1}x_2) \\ tr(\overline{x_1}x_2) & n(x_2)\end{pmatrix}Z\right)$.

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If h_1,h_2 have even weights $2+2\nu_1,2+2\nu_2$ get similar construction, φ_i^D take values in harmonic polynomials of degree ν_i on $D^0(\mathbf{R})$.

$Y^{(2)}(\varphi_1^D,\varphi_2^D)(Z)$ is then a vector valued modular form of type $(\nu_1+\nu_2,\nu_1-\nu_2)$, a linear combination of theta series with harmonic polynomials on $D(\mathbf{R})$.

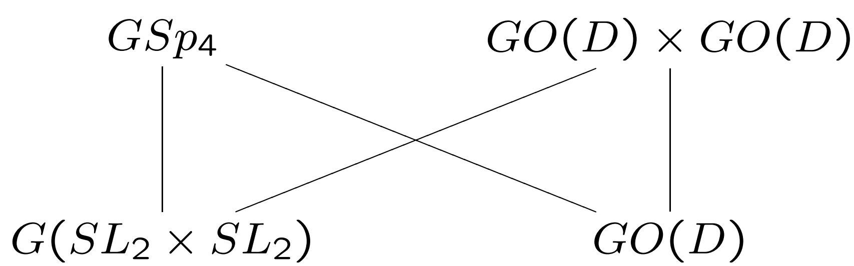
The same construction with

$$\vartheta^1(y_iRy_j^{-1},z)=\sum_{x\in(y_iRy_j^{-1})}\exp(2\pi i n(x)z)$$

gives zero for $h_1\neq h_2$, the Jacquet-Langlands lift as constructed by Eichler if $h_1=h_2$.

Back to our seesaw (D ramified at ∞ as above)

$$F = Y^2(\varphi_1^D, \varphi_2^D) \qquad (\psi_1^D \otimes \psi_1^D) \otimes (\psi_2^D \otimes \psi_2^D)\,.$$



(f_1, f_2) $\qquad$ $\varphi_1^D \otimes \varphi_2^D$

(h_1, h_2)

Sketch of computation (Weights 2, put $e_i = |(y_i R y_i^{-1})^\times| = 1$). Compute the period integral

$$\int Y^{(2)}(\varphi_1^D, \varphi_2^D)(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}) f(z_1) f(z_2) dz_1 dz_2$$

$$= \int \sum_{i,j=1}^{r} \varphi_1^D(y_i)\varphi_2^D(y_j)\vartheta^2(y_i R y_j^{-1}, \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix})) f(z_1) f(z_2) dz_1 dz_2$$

We have:

$$\int \vartheta^2(y_i R y_j^{-1}, \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}) f_1(z_1) f_2(z_2) dz_1 dz_2 =$$

$$= (\int \vartheta^1(y_i R y_j^{-1}, z_1) f_1(z_1) dz_1)(\int \vartheta^1(y_i R y_j^{-1}, z_2) f_2(z_2) dz_2)$$

$$= c\langle f_1, f_1\rangle\langle f_2, f_2\rangle \psi_1^D(y_i)\psi_1^D(y_j)\psi_2^D(y_i)\psi_2^D(y_j)$$ $c \neq 0$ explicit, hence

$$\int Y^{(2)}(\varphi_1^D, \varphi_2^D)(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}) f(z_1) f(z_2) dz_1 dz_2$$

$$= c\langle f_1, f_1\rangle\langle f_2, f_2\rangle \sum_{i,j} \varphi_1^D(y_i)\varphi_2^D(y_j)\psi_1^D(y_i)\psi_1^D(y_j)\psi_2^D(y_i)\psi_2^D(y_j)$$

which gives

$$\int Y^{(2)}(\varphi_1^D,\varphi_2^D)(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix})f_1(z_1)f_2(z_2)dz_1dz_2$$
$$= c\langle f_1, f_1\rangle\langle f_2, f_2\rangle \sum_{i,j} \varphi_1^D(y_i)\varphi_2^D(y_j)\psi_1^D(y_i)\psi_1^D(y_j)\psi_2^D(y_i)\psi_2^D(y_j)$$
$$= c\langle f_1, f_1\rangle\langle f_2, f_2\rangle (\sum_i \varphi_1^D(y_i)\psi_1^D(y_i)\psi_2^D(y_i))(\sum_j \varphi_2^D(y_j)\psi_1^D(y_j)\psi_2^D(y_j))$$

The square of $(\sum_i \varphi_1^D(y_i)\psi_1^D(y_i)\psi_2^D(y_i))$ is (Bö-SP) (up to an explicit constant) equal to

$$\frac{L(h_1, f_1, f_2, 1/2)}{\langle h_1, h_1\rangle\langle f_1, f_1\rangle\langle f_2, f_2\rangle}$$

(Aside: The version of this formula for harmonic polynomials leads (in a smilar way as in Watson's talk to a result about equidistribution of spherical functions on the 2-sphere that are Hecke eigenforms, Böcherer/Sarnak/SP).

Theorem: Let newforms f_1, f_2 of even weights k_1, k_2 and h_1, h_2 of weights $k'_1 = 2 + 2\nu_1, k'_2 = 2 + 2\nu_2$, all of squarefree level N and trivial character, be given, h_1, h_2 with the same Atkin-Lehner eigenvalues ϵ'_p, f_i with Atkin-Lehner eigenvalues $\epsilon_{i,p}$, assume that $\prod_{p|N} \epsilon'_p \epsilon_{1,p} \epsilon_{2,p} = -1$ and that the triples $(k_1, k_2, k_1), (k_1, k_2, k'_2)$ are balanced.

Then there is a unique quaternion algebra D ramified at ∞, unramified outside N (depending on the Atkin-Lehner eigenvalues) such that the period integral

$$I(h_1, h_2, f_1, f_2, D) = \int Y^{(2)}(\varphi_1^D, \varphi_2^D)(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}) f_1(z_1) f_2(z_2) dz_1 dz_2$$

is not trivially zero.

For this choice of D one has with an explicit constant c:

$$I(h_1, h_2, f_1, f_2, D)^2 = \frac{c}{\langle h_1, h_1 \rangle \langle h_2, h_2 \rangle} L(h_1, f_1, f_2; \frac{1}{2}) L(h_2, f_1, f_2; \frac{1}{2}).$$

In particular the period integral is nonzero if and only if (with $F = Y^{(2)}(\varphi_1^D, \varphi_2^D)$) the central critical value of $L(\mathrm{S}pin(F), f_1, f_2, s) = L(h_1, f_1, f_2; s) L(h_2, f_1, f_2; s)$ is nonzero.

Remarks: a) If the triples of weights are balanced as stated but have $k_1+k_2 > 2\min(k_1', k_2')$ one has to modify the Yoshida-lifting by applying a differential operator that raises its weight (which amounts to changing the form in its representation space) in order to obtain the formula given in the Theorem.

b) The variation in D mimicks the variation of the inner form in the Gross-Prasad conjecture.

Recall: **Gross-Prasad conjecture:**
$SO(n) = G, \quad SO(n-1) = G' \subseteq G, \quad H = G' \overset{\Delta}{\hookrightarrow} G' \times G,$
π, π' irred. rep. of G, G' resp. (cuspidal automorphic resp. admissible), central character of $\pi \times \pi'$ trivial.

Locally H-invariant functional exists for precisely one inner form $\tilde{G}, \tilde{G}'$ and $\tilde{\pi}, \tilde{\pi}'$ of $\tilde{G}, \tilde{G}'$ corresponding to π, π' (assume π, π' to have generic parameters), nonzero on spherical vector at places where π, π' are unramified.

Global nonvanishing of $I_{\tilde{H}}$ (on some inner form) depends on central value of L-function $L(\pi \otimes \pi', s)$.

Remarks: c) Regrouping of terms changes, if $h_1 \neq h_2$, (again with $F = Y^{(2)}(\varphi_1^D, \varphi_2^D)$) the original expression

$$I(h_1, h_2, f_1, f_2, D)^2 = \frac{c}{\langle h_1, h_1 \rangle \langle h_2, h_2 \rangle} L(h_1, f_1, f_2; \frac{1}{2}) L(h_2, f_1, f_2; \frac{1}{2})$$

to

$$I(h_1, h_2, f_1, f_2, D)^2 = \frac{c\langle F, F \rangle}{L^{(N)}(F, \mathrm{Sym}^2, 1)} L(\mathrm{Spin}(F), f_1, f_2; \frac{1}{2}).$$

This formulation could be true for an arbitrary Siegel modular cusp form F.

d) There are versions for the degenerate cases that one of h_1, h_2 is the Eisenstein series of weight 2 (which gives Saito-Kurokawa cusp forms). It is not clear how to prove the result in the case of a Saito-Kurokawa cusp form of level 1.

The case of a group $G' = SO(4)$ which is split at infinity but not globally split leads to the integration of the restriction of F against a Hilbert modular cusp form f over an embedded Hilbert modular surface.

The result is similar, with $L(\text{Spin}(F), f_1, f_2)$ replaced by $L(\text{Spin}(F), \text{Asai}(f))$.