Representation of Quadratic Forms

Rainer Schulze-Pillot

Universität des Saarlandes, Saarbrücken, Germany

October 6, 2009

Outline

Representation of sufficiently large numbers

Theorem (Kloosterman 1924, Tartakovskii 1929)

Let $A \in M_m^{\mathrm{sym}}(\mathbb{Z})$ be a positive definite integral symmetric $m \times m$-matrix with $m \geq 5$. Then for every sufficiently large integer t for which

$$ {}^t\mathbf{x}_p A \mathbf{x}_p = t $$

is solvable with $\mathbf{x}_p \in \mathbb{Z}_p^m$ for all primes p the equation

$$ Q_A(\mathbf{x}) := {}^t\mathbf{x} A \mathbf{x} = t $$

is solvable with $\mathbf{x} \in \mathbb{Z}^m$
In other words:
Every sufficiently large integer t which is representable by the quadratic form Q_A locally everywhere is representable by Q_A globally.

Proof.

The original proof uses the Hardy-Littlewood circle method. An alternative proof uses modular forms instead; we'll come back to that. □

Representation of matrices, simplest case

Theorem (Ellenberg, Venkatesh 2006)

Let $A \in M_m^{\mathrm{sym}}(\mathbb{Z})$ be a positive definite integral symmetric $m \times m$-matrix, let $n \leq m-5$.
Then there is a constant C such that for each positive definite matrix $T \in M_n^{\mathrm{sym}}(\mathbb{Z})$ with $\det(T)$ square free the equation

$$ {}^t X A X = T $$

is solvable with $X \in M_{m,n}(\mathbb{Z})$
provided T satisfies:

1. *For each prime p the equation*

$$ {}^t X_p A X_p = T $$

   is solvable with $X_p \in M_{m,n}(\mathbb{Z}_p)$.
2. $\min(T) := \min\{{}^t\mathbf{y} T \mathbf{y} \mid \mathbf{0} \neq \mathbf{y} \in \mathbb{Z}^n\} > C$

Outline of talk

- Analytic approach
- Arithmetic and ergodic approach
- Analytic problems

Asymptotic formula

Always: $A \in M_m^{\mathrm{sym}}(\mathbb{Z})$ and $T \in M_n^{\mathrm{sym}}(\mathbb{Z})$, A positive definite, T positive semidefinite.

Idea: Prove the existence of a solution of ${}^tXAX = T$ (a *representation of T by A*) by proving more, namely an
asymptotic formula
for the *representation number*

$$r(A, T) := |\{X \in M_{m,n}(\mathbb{Z}) \mid {}^tXAX = T\}|,$$

i. e., a formula of the type

$$r(A, T) = \text{main term}(T) + \text{error term}(T)$$

where the main term grows faster than the error term if T grows in a suitable sense, e.g., $\min(T) := \min\{{}^t\mathbf{y}T\mathbf{y} \mid \mathbf{0} \neq \mathbf{y} \in \mathbb{Z}^n\}$ tends to ∞.

Theta series

A as always (symmetric of size m, positive definite). In addition: A has even diagonal.

Definition

The theta series (of degree 1) of A is

$$\vartheta(A, z) := \sum_{t=0}^{\infty} r(A, t) \exp(\pi i t z), \quad z \in H = \{z \in \mathbb{C} \mid \Re(z) > 0\}.$$

It is a modular form of weight $k = \frac{m}{2}$ for the group $\Gamma_0(N)$ where NA^{-1} is integral with even diagonal:
$\vartheta(A, \cdot) \in M_k(\Gamma_0(N), \chi)$ with χ depending on $\det(A)$.

Siegel theta series extend this to the theta series of degree n, encoding representation numbers of $n \times n$-matrices.

Siegel theta series

Write $\mathfrak{H}_n = \{Z = X + iY \in M_n^{\mathrm{sym}}(\mathbb{C}) \mid X, Y \text{ real}, Y > 0\}$ (Siegel's upper half space).

Definition

The Siegel theta series of degree n of A is

$$\vartheta^{(n)}(A, Z) := \sum_T r(A, T) \exp(\pi i \mathrm{tr}(TZ)), \quad Z \in \mathfrak{H}_n,$$

where T runs over positive semidefinite symmetric matrices of size n with even diagonal.
It is a Siegel modular form of weight $k = \frac{m}{2}$ for the group $\Gamma_0^{(n)}(N)$ where NA^{-1} is integral with even diagonal:

$\vartheta^{(n)}(A, \cdot) \in M_k^n(\Gamma_0^{(n)}(N), \chi)$ with χ depending on $\det(A)$.

Genus theta series

Definition

The genus theta series of degree n of A is

$$\vartheta_{\mathrm{gen}}^{(n)}(A, Z) := \frac{\sum_{A'} \frac{\vartheta^{(n)}(A', Z)}{o(A')}}{\sum_{A'} \frac{1}{o(A')}},$$

where the summation runs over representatives A' of the integral equivalence classes in the genus of A and $o(A')$ denotes the number of automorphs (units) of A'.
We write

$$\vartheta_{\mathrm{gen}}^{(n)}(A, Z) = \sum_T r(\mathrm{gen}(A), T) \exp(\pi i \mathrm{tr}(TZ))$$

and call the $r(\mathrm{gen}(A), T)$ the average representation numbers for the genus.

Local densities and Siegel's theorem

Definition

The local density $\alpha_p(A,T)$ (for a prime p) is

$$\alpha_p(A,T) = p^{j\cdot(\frac{n\cdot(n+1)}{2}-mn)} \cdot |\{\bar{X} \in M_{m,n}(\mathbb{Z}/p^j\mathbb{Z}) \mid {}^tXAX \equiv T \bmod p^j M_n^{\text{sym}}(\mathbb{Z})\}|$$

for sufficiently large $j \in \mathbb{N}$.

Theorem (Siegel's theorem)

The genus theta series $\vartheta^{(n)}_{gen}(A,Z)$ is in the space of Eisenstein series.
One has for positive definite T:

$$r(gen(A),T) = c \cdot (\det T)^{\frac{m-n-1}{2}} (\det A)^{\frac{n}{2}} \prod_p \alpha_p(A,T)$$

with some constant c depending only on m,n.

Asymptotic formula for $n=1, m \geq 5$

Proof of the theorem of Kloosterman/Tartakovskii via modular forms.

The difference $\vartheta(A,z) - \vartheta_{\text{gen}}(A,z)$ is a cusp form of weight $m/2$.

We write

$$r(A,t) = r(\text{gen}(A),t)) + (r(A,t) - r(\text{gen}(A),t)).$$

The main term $r(\text{gen}(A),t)$ grows at least like $t^{\frac{m}{2}-1}$ for t that are represented locally everywhere
(estimate $\prod_p \alpha_p(A,t)$ from below by a constant or use an estimate for Fourier coefficients of Eisenstein series).

The error term $r(A,t) - r(\text{gen}(A),t)$ is the Fourier coefficient at t of a cusp form, hence grows at most like $t^{\frac{m}{4}}$. □

Hardy-Littlewood method

The original proof uses the circle method of Hardy and Littlewood.
The product $\prod_p \alpha_p(A, t)$ of local densities then appears as the singular series.

General n: Obstacles

For general n, the approach for $n = 1$ needs some modifications:

- An estimate of local densities from below shows: The main term grows like $\det(T)^{\frac{m-n-1}{2}}$ for $m \geq 2n + 3$; this is in general not valid for smaller m.
- for small m, obtaining the required estimate from below for the product of densities is impossible, particularly bad examples arise if one does not require the existence of local primitive representations (X_p has trivial elementary divisors).
- The difference $\vartheta^{(n)}(A, \cdot) - \vartheta^{(n)}_{\text{gen}}(A, \cdot)$ vanishes at all zero dimensional boundary components of $\mathfrak{H}_n$, but it is not a cusp form. Instead it is a sum of a cusp form and of Eisenstein series of Klingen type associated to cusp forms in degree $r < n$. This makes it more difficult to estimate the Fourier coefficients.

Results of Raghavan and Kitaoka, I

Theorem (Raghavan, Annals 1959)

Assume $m \geq 2n+3$. Then for T running through positive definite integral $n \times n$-matrices with $\det(T) \to \infty$ *and* $\min(T^{-1}) \geq c \det(T)^{-\frac{1}{n}}$ *for some constant $c > 0$ one has with $m = 2k$*

$$r(A,T) = r(gen(A),t) + \mathrm{O}((\min(\mathrm{T}))^{\frac{n+1-k}{2}} \det(\mathrm{T})^{\frac{m-(n+1)}{2}}),$$

where $r(gen(A),T)$ grows like $\det(T)^{\frac{m-(n+1)}{2}}$*).*
In particular, all T which are represented over all $\mathbb{Z}_p$ by A, have sufficiently large minimum, and satisfy the condition on $\min(T^{-1})$ *above are represented by A over $\mathbb{Z}$.*

It should be noted that examples constructed with the help of the Leech lattice show that the error term can indeed grow as fast as the main term $r(gen(A),T)$ with respect to $\det(T)$ alone.

Results of Raghavan and Kitaoka, II

Theorem (Kitaoka 1982)

Assume $m \geq 2n+3$. Then for T running through positive definite integral $n \times n$-matrices with $\det(T) \to \infty$ *and* $\min(T) \geq c \det(T)^{\frac{1}{n}}$ *for some constant $c > 0$ one has with $m = 2k$*

$$r(A,T) = r(gen(A),t) + \mathrm{O}((\min(\mathrm{T}))^{\frac{n+1-k}{2}} \det(\mathrm{T})^{\frac{m-(n+1)}{2}}),$$

where $r(gen(A),T)$ grows like $\det(T)^{\frac{m-(n+1)}{2}}$*).*
In particular, all T which are represented over all $\mathbb{Z}_p$ by A, have sufficiently large determinant and satisfy the condition on $\min(T)$ *above are represented by A over $\mathbb{Z}$.*

Notice that by reduction theory one has $\min(T) = \mathrm{O}(\det(\mathrm{T})^{\frac{1}{n}}$.

Results of Raghavan and Kitaoka, III

Method of proof: Compute the Fourier coefficient $b(T)$ of $g(Z) := \vartheta^{(n)}(A, Z) - \vartheta^{(n)}_{\text{gen}}(A, Z)$ as

$$\int_{\mathfrak{E}} g(Z) \exp(-2\pi i \text{tr}(TZ)) dX,$$

where the variable $Z = X + iT^{-1}$ runs over a cube of sidelength 1 with one corner in T^{-1} and use a generalized Farey dissection of this cube introduced by Siegel in order to compute the integral.
Another method, due to Kitaoka, uses the decomposition of g into a cusp form and Klingen Eisenstein series. It gives a similar result with a better exponent at the minimum if $m > 4n + 4$ and A is even unimodular.

Representations in lattice notation

$(V, Q), (W, Q')$ quadratic spaces over $\mathbb{Q}$,
$(Q(x) = B(x, x)$, B symmetric bilinear form on $V)$,
$\dim(V) = m \geq \dim(W) = n$,
M a $\mathbb{Z}$-lattice on V, N a $\mathbb{Z}$-lattice on W.

Definition

W is represented by V if there is an isometric embedding $\varphi : W \to V$.
W is represented by V over $\mathbb{Q}_p$ if there is an isometric embedding $\varphi_p : W \otimes \mathbb{Q}_p \to V \otimes \mathbb{Q}_p$.
N is represented by M if there is an isometric embedding $\varphi : W \to V$ with $\varphi(N) \subseteq M$.
N is represented by M over $\mathbb{Z}_p$ if there is an isometric embedding $\varphi_p : W \otimes \mathbb{Q}_p \to V \otimes \mathbb{Q}_p$ with $\varphi(N \otimes \mathbb{Z}_p) \subseteq M \otimes \mathbb{Z}_p$.

The representation φ resp. φ_p of N by M is primitive if $M \cap \varphi(W) = \varphi(N)$.

Equivalence of notations

As above, $(V, Q), (W, Q')$ quadratic spaces over $\mathbb{Q}$,
$M = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_m$ a $\mathbb{Z}$-lattice with basis $(e_1, \ldots, e_m)$ on V,
$N = \mathbb{Z}f_1 + \cdots + \mathbb{Z}f_n$ a $\mathbb{Z}$-lattice with basis $(f_1, \ldots, f_n)$ on W.
$A = (B(e_i, e_j))$ the Gram matrix of Q with respect to the given basis of M,
$T = (B'(f_i, f_j))$ the Gram matrix of Q' with respect to the given basis of N.

Proposition

$N =: \langle T \rangle$ is represented (primitively) by $M = \langle A \rangle$ (over $\mathbb{Z}_p$) if and only if T is represented (primitively) by A (over $\mathbb{Z}_p$).

Number fields

Everything carries over with
$\mathbb{Q}, \mathbb{Q}_p$ replaced by F, F_v and
$\mathbb{Z}, \mathbb{Z}_p$ replaced by $\mathfrak{o}, \mathfrak{o}_v$,
where F is a number field, v a place of F.

Notice however: An $\mathfrak{o}$-lattice is allowed to be not free as an $\mathfrak{o}$-module, hence may not have a well defined Gram matrix.

Therefore: Don't use matrix notation in the number field case.

The HKK theorem

Theorem (Hsia, Kitaoka, Kneser 1978)

Let M be an $\mathfrak{o}$-lattice of rank $m \geq 2n+3$.
Then there is a constant $c(M)$ such that M represents all $\mathfrak{o}$-lattices N of rank n satisfying

1. *M_v represents N_v for all places v of F*
2. *If M is definite then it is positive and one has* $\min(N) \geq c(M)$

- The indefinite case is older (Kneser 1961, using strong approximation for the spin group)
- The constant $c(M)$ can in principle be effectively computed
- There is no statement about the number of representations

Spinor genus

Denote by O_V the group of isometries of the space (V, Q).

Definition

Consider lattices M, M_1 on V that are in the same genus, i. e., there is $u = (u_v)_v \in O_V(\mathbb{A})$ with $uM = M_1$.
Then M_1 is in the spinor genus of M if and only if

$$u \in O_V(F)\,\mathrm{Spin}_V(\mathbb{A}) O_M(\mathbb{A})$$

$\mathrm{Spin}_V(\mathbb{A})$ identified with its image in $O_V(\mathbb{A})$ under the covering map, $O_M(\mathbb{A})$ the stabilizer in $O_v(\mathbb{A})$ of M.

Theorem

If V is indefinite spinor genus and isometry class coincide.
If V is definite and w a place where $V_w = V \otimes F_w$ is isotropic (represents zero nontrivially) then a lattice M_1 in the spinor genus of M is isometric to a lattice M_1' with $M_1' \otimes \mathfrak{o}_v = M \otimes \mathfrak{o}_v$ for all places $v \neq w$.

Representations in the spinor genus

Lemma

Assume $m \geq n+3$. Let N (on W') be represented by M (on V) locally everywhere (primitively).
Then N is represented globally (primitively) by some lattice M_1 in the spinor genus of M.

Lemma

Assume $m \geq n+3$. Let M, N be as above, let w be a finite place of F with $V_w = V \otimes F_w$ isotropic.

Then there is a lattice M' in the spinor genus of M with $M'_v = M_v$ for all finite places $v \neq w$ of F and $M'_w \in \mathrm{Spin}_V(F_w)M_w$ such that N is represented by M' (primitively).

So under rather weak conditions we have a representation of N not by M itself but by a lattice M' which is "very close" to M.

Sketch of proof, 1

The proof in HKK proceeds by constructing a representation of N by M itself starting from the given one by M', for this one has to first add another copy of N. This raises the bound on m from $n+3$ to $2n+3$.

EV work group theoretically:
We have N embedded into $M_1 = uM$ whith

$$u \in O_V(F) \prod_{v \neq w} O_L(\mathfrak{o}_v)\,\mathrm{Spin}_V(F_w),$$

we need the same with $u \in O_V(F)O_L(\mathbb{A})$ instead.

To achieve this, modify u by a suitable element of $O_{W_1}(F_w)$ where $W_1 = (FN)^{\perp}$.
Problem: When N varies (through a sequence of lattices with increasing minima), W_1 varies.

Sketch of proof, 2

Lemma

w a non-archimedean place of F, M_w an $\mathfrak{o}_w$-lattice of rank m on $V_w = V \otimes F_w$.
Let $\mathcal{W}$ be a set of regular subspaces of V_w, put $N_W := W \cap M_w$ for $W \in \mathcal{W}$
Assume that the (additive) w-adic valuation $\mathrm{ord}_w(disc(N_W))$ of the discriminants of the N_W is bounded by some $j \in \mathbb{N}$ independent of W.
Then the set $\mathcal{W}$ is contained in the union of finitely many orbits under the action of the compact open subgroup $\tilde{K}_w := \mathrm{Spin}_{M_w}(\mathfrak{o}_w) = \{\tau \in \mathrm{Spin}_V(F_w) \mid \tau(M_w) = M_w\}$ of $\mathrm{Spin}_V(F_w)$.

Sketch of proof, 3

Proposition

As before: $\tilde{K}_v = \mathrm{Spin}_{M_v}(\mathfrak{o}_v)$ for finite places v of F
Let w be a fixed finite place of F and T_w a regular isotropic subspace of $V_w = V \otimes F_w$ with $\dim(T_w) \geq 3$.
Let $G_w = \mathrm{Spin}_V(F_w)$, $H_w = \mathrm{Spin}_{T_w}(F_w)$ and

$$\Gamma := \mathrm{Spin}_V(F) \cap \mathrm{Spin}_V(F_w) \prod_{v \neq w} \tilde{K}_v.$$

Let a sequence $(W_i)_{i \in \mathbb{N}}$ of subspaces W_i of V (over the global field F) be given such that $(W_i)_w^{\perp} = \xi_i T_w$ for each i with elements ξ_i from a fixed compact subset of G_w.

Then one has: If no infinite subsequence of the W_i has a nonzero intersection, the sets $\Gamma \backslash \Gamma \xi_i H_w$ are becoming dense in $\Gamma \backslash G_w$ as $i \to \infty$, i. e., for every open subset U of G_w one has $U \cap \Gamma \xi_i H_w \neq \emptyset$ for sufficiently large i.

Sketch of proof, 4

The proposition is proved by Ellenberg and Venkatesh using ergodic methods, it is the heart of their proof.

The following proposition uses it to deduce a first result about existence of representations:

Proposition

Let $j \in \mathbb{N}$ and let w be a fixed finite place of F.
Let $(W_i)_{i\in\mathbb{N}}$ be a sequence of regular subspaces W_i of V of dimension $n \leq m-3$ with isotropic orthogonal complement in V_w, with $\mathrm{ord}_w(\mathrm{disc}((W_i)_w \cap M_w)) \leq j$ for all i, and such that no infinite subsequence has nonzero intersection.
Then $N_i = W_i \cap M$ is represented primitively by all lattices in the spinor genus $\mathrm{spn}(M)$ for sufficiently large i.

Proof of the proposition, 1

The proof proceeds as follows:
Put $\tilde{K}_v = \mathrm{Spin}_{\Lambda_v}(\mathfrak{o}_v)$ for all finite places v of F and
$\Gamma := \mathrm{Spin}_V(F) \cap \mathrm{Spin}_V(F_w)\prod_{v\neq w}\tilde{K}_w$.

By the lemma on orbits $(N_i)_w$ (and hence the $(W_i)_w$) fall into finitely many orbits under the action of the compact open group $\tilde{K}_w$; we can assume that they all belong to the same orbit:
With $T_w = (W_1)_w^\perp$ we have $(W_i)_w^\perp = \xi_i T_w$ with $\xi_i \in \tilde{K}_w$ for all i.

Any isometry class in $\mathrm{spn}(M)$ has a representative $\tilde{M} \subseteq V$ with $\tilde{M}_v = M_v$ for all finite places $v \neq w$ of F and $\tilde{M}_w = g_w M_w$ for some $g_w \in G_w = \mathrm{Spin}_V(F_w)$.

Proof of the proposition, 2

Remember: $(W_i)_w^\perp = \xi_i T_w$ with $\xi_i \in \tilde{K}_w$ and $\tilde{M}_w = g_w M_w$.

By the previous proposition for every open set $U \subseteq G_w$ there is an i_0 with $U \cap \Gamma \xi_i H_w \neq \emptyset$ for $i \geq i_0$.

Take $U = g_w \tilde{K}_w \subseteq G_w$ and obtain i_0 such that for all $i \geq i_0$ one has elements $\gamma_i \in \Gamma, \eta_i \in H_w, \kappa_i \in \tilde{K}_w$ with $g_w \kappa_i = \gamma_i \xi_i \eta_i$.

The lattice $M_i' := \gamma_i^{-1} \tilde{M}$ is in the isometry class of $\tilde{M}$; it satisfies $(M_i')_v = M_v$ for all finite $v \neq w$ and
$(M_i')_w = \gamma_i^{-1} g_w M_w = \xi_i \eta_i M_w = \xi_i \eta_i \xi_i^{-1} M_w$.

From this and $\xi_i \eta_i \xi_i^{-1}|_{(W_i)_w} = \mathrm{Id}_{(W_i)_w}$ we see $N_i = W_i \cap M_i'$, i.e., we have the requested primitive representation by a lattice in the given isometry class.

Sequences of lattices with growing minima

We can now turn the “no infinite subsequence with nonzero intersection”-condition into a condition about lattices with growing minima:

Proposition

Let $(N_i)_{i \in \mathbb{N}}$ be a sequence of $\mathfrak{o}$-lattices of rank $n \leq m - 3$. Assume: We can fix a finite place w of F and a $j \in \mathbb{N}$ with:

1. *N_i is represented locally everywhere primitively by M with isotropic orthogonal complement at the place w for all i.*
2. $\mathrm{ord}_w(\mathrm{disc}((M_i)_w)) \leq j$ *for all i.*
3. *The sequence* $(\min(N_i))_{i \in \mathbb{N}}$ *of the minima of the N_i tends to infinity.*

Then there is an $i_0 \in \mathbb{N}$ such that N_i is represented primitively by all isometry classes in the genus of M for all $i \geq i_0$.

Proof

Proof of proposition.

May consider only lattices in the spinor genus of M and assume $N_i \subseteq M$ primitive, let $W_i = FN_i$. By the previous proposition we must show: There is no infinite subsequence of the W_i with nonzero intersection. Otherwise:
Choose $\mathbf{0} \neq x \in M \cap \cap_{i \in I} W_i$ with I infinite. By prmitivity:
$x \in N_i = M \cap W_i$ for infinitely many i.
This contradicts the assumption iii) that the minima of the M_i tend to infinity. □

The main theorem

Theorem (Ellenberg and Venkatesh, slightly generalized)

Fix a finite place w of F and $j \in \mathbb{N}$.
Then there exists a constant $C := C(M, j, w)$ such that M primitively represents all $\mathfrak{o}$ - lattices N of rank $n \leq m - 3$ satisfying

1. *N is represented by M locally everywhere primitively with isotropic orthogonal complement at the place w.*
2. $\text{ord}_w(\text{disc}(N_w)) \leq j$
3. *The minimum of N is $\geq C$.*

The isotropy condition is satisfied automatically if $n \leq m - 5$ or if w is such that $\text{disc}(M_w)$ *and* $\text{disc}(N_w)$ *are units at w.*

The primitivity condition above may be replaced by bounded imprimitivity:
The representation φ of N by M has *imprimitivity bounded by* $c \in \mathfrak{o}$ if $cx \in \varphi(N)$ for all $x \in F\varphi(N) \cap M$.

Matrix version of the main result

Here is a matrix version of the main result:

Theorem

Let $A \in M_n^{\mathrm{sym}}(\mathbb{Z})$ be a positive definite integral symmetric $m \times m$-matrix, fix a prime q and positive integers j, c. Then there is a constant C such that a positive definite matrix $T \in M_n^{\mathrm{sym}}(\mathbb{Z})$ with $n \leq m-3$ is represented by S (i.e., $T = {}^tXAX$ with $X \in M_{m,n}(\mathbb{Z})$) provided it satisfies:

1. *For each prime p there exists a matrix $X_p \in M_{nm}(\mathbb{Z}_p)$ with ${}^tX_pAX_p = T$ such that the elementary divisors of X divide c and such that the equations ${}^tX_qA\mathbf{y} = \mathbf{0}$ and ${}^t\mathbf{y}A\mathbf{y} = 0$ have a nontrivial common solution $\mathbf{y} \in \mathbb{Z}_q^m$*
2. $q^j \nmid \det(T)$
3. $\min\{{}^t\mathbf{y}T\mathbf{y} \mid \mathbf{0} \neq \mathbf{y} \in \mathbb{Z}^n\} > C$

The matrix X may be chosen to have elementary divisors dividing c.

Corollaries,1

Corollary

Let $F = \mathbb{Q}$, fix a prime q and $j \in \mathbb{N}$. In the following cases there exists a constant $C := C(M, j, q)$ such that M represents all $\mathbb{Z}$-lattices N of rank n which are represented by M locally everywhere, have minimum $\geq C$ and satisfy $\mathrm{ord}_q(\mathrm{disc}(N)) \leq j$:

1. *$n \geq 6$ and $m \geq 2n$.*
2. *$n \geq 3$ and $m \geq 2n+1$, with the additional assumption that in the case $n = 3$ the orthogonal complement of the representation of N_q in M_q is isotropic.*
3. *$n = 2$ and $m \geq 6$, with the additional assumption that in the case $m = 6$ the orthogonal complement of the representation of N_q in M_q is isotropic.*

Notice that in these cases we have no primitivity condition. In fact, work of Kitaoka implies that N can be replaced by a lattice N' which is represented locally primitively and has roughly the same minimum.

Corollaries, 2

Corollary

Let a positive definite $\mathbb{Z}$-lattice N_0 of rank $n \leq m-3$ with Gram matrix T_0 be given. Let Σ be a finite set of primes with $q \in \Sigma$ such that one has

1. *M_p and N_p are unimodular for all primes $p \notin \Sigma$ and for $p = q$.*
2. *Each isometry class in the genus of M has a representative M' on V such that $M'_p = M_p$ for all primes $p \notin \Sigma$.*

Then there exists a constant $C := C(M, T_0, \Sigma)$ such that for all sufficiently large integers $a \in \mathbb{Z}$ which are not divisible by a prime in Σ, the $\mathbb{Z}$-lattice N with Gram matrix aT_0 is represented by M if it is represented by all completions M_p.

Again, work of Kitaoka implies that N can be replaced by a lattice N' which is represented locally primitively and has roughly the same minimum.

Remarks

- The main result also allows versions for representations with congruence conditions and for extensions of representations.
- The proof should also go through for hermitian forms.
- For applications it would sometimes be desirable to have a different condition than growing minimum of the lattices to be represented.
  It appears that at least the present method is not able to give such a result: If we consider an infinite sequence of lattices $N_i \subseteq M$ whose minimum is bounded, there must be infinite subsequences having a nonzero intersection, since there are only finitely many vectors of given length in M.

Jacobi forms and fixed minimum

For simplicity assume S to be even unimodular, $n = 2$, and let T run through matrices of fixed square free minimum μ and of fundamental discriminant.
Write $\vartheta^{(2)}(A, Z) - \vartheta^{(2)}_{\text{gen}}(A, Z) = g_0(Z) + g_1(Z)$,
where g_0 is a cusp form and $g_1(Z) = \sum b_1(T)\exp(2\pi i \text{tr}(TZ))$ is a Klingen Eisenstein series,
associated to the elliptic cusp form
$h(\tau) = \vartheta^{(1)}(A, \tau) - \vartheta^{(1)}_{\text{gen}}(A, \tau)$,
with Fourier Jacobi expansion

$$g_1(Z) = \sum_{t=0}^{\infty} \phi_t(\tau, z)\exp(2\pi i t\tau')$$

with $\phi_t(\tau, z) = \sum_{\nu, r} c_t(\nu, r)\exp(2\pi i(\nu\tau + rz))$.
Estimating the asymptotic behaviour of $b_1(T)$ is then equivalent to estimating the behaviour of $c_\mu(\nu, r)$ with $4\nu\mu - r^2 \to \infty$.

Analytic problems, cont'd

Estimating the asymptotic behaviour of $b_1(T)$ is then equivalent to estimating the behaviour of $c_\mu(\nu, r)$ with $4\nu\mu - r^2 \to \infty$.

To do this, we need the explicit decomposition of the Jacobi form ϕ_t into a Jacobi Eisenstein series and a Jacobi cusp form. This has been done in recent work of Böcherer.

One gets an asymptotic formula in this case, which also follows from previous work of Böcherer and Raghavan on extensions of representations.
A more general version of this decomposition may lead to asymptotic formulas not contained in that work, a PhD student (Thorsten Paul) in Saarbrücken is working on this.

Extensions

Corollary

Fix a finite place w of F and $j \in \mathbb{N}, c \in \mathfrak{o}$.
Let $R \subseteq M$ be a fixed $\mathfrak{o}$-lattice of rank r with R_w unimodular,
$\sigma : R \longrightarrow M$ a representation of R by M.
Then there exists a constant $C := C(M, R, j, w, c)$ such that one has:
If $N \supseteq R$ is an $\mathfrak{o}$-lattice of rank $n \leq m - 3$ and

1. *For each place v of F there is a representation $\tau_v : N_v \longrightarrow M_v$ with $\tau_v|_{R_v} = \sigma_v$ with imprimitivity bounded by c and with isotropic orthogonal complement in M_w*
2. $\mathrm{ord}_w(\mathrm{disc}(N_w)) \leq j$
3. *The minimum of $N \cap (FR)^{\perp}$ is $\geq C$,*

then there exists a representation $\tau : N \longrightarrow M$ with $\tau|_R = \sigma$.
The representation may be taken to be of imprimitivity bounded by c.
The isotropy condition is satisfied automatically if $n \leq m - 5$ or if w is such that $\mathrm{disc}(M_w)$ and $\mathrm{disc}(N_w)$ are units at w.