On the archimedean Euler factors for spin L–functions

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ABSTRACT. The archimedean Euler factor in the completed spin L–function of a Siegel modular form is computed. Several formulas are obtained, relating this factor to the recursively defined factors of ANDRIANOV and to symmetric power L–factors for GL(2). The archimedean ε–factor is also computed. Finally, the critical points of certain motives in the sense of DELIGNE are determined.

Introduction

Let f be a classical holomorphic Siegel modular form of weight k and degree n, assumed to be an eigenform of the Hecke algebra. ANDRIANOV [An] has associated to f an L–function $Z_f(s)$ as an Euler product over all finite primes, where the Euler factor at p is a polynomial in p^{-s} of degree 2^n. This is called the *spin* L–function of f since it is attached to the 2^n–dimensional spin representation of the L–group $\mathrm{Spin}(2n+1,\mathbb{C})$ of the underlying group $\mathrm{PGSp}(2n)$, see [AS].

A serious problem is that of obtaining the analytic continuation and a functional equation for $Z_f(s)$. The case $n=1$ is the classical HECKE theory, the case $n=2$ was done by ANDRIANOV in [An]. Beyond that, very little is presently known.

To obtain smooth functional equations, the partial L–function $Z_f(s)$ has to be completed with an Euler factor at the archimedean prime. For example, for $n=1$, the function $L(s,f)=(2\pi)^{-s}\Gamma(s)Z_f(s)$ has the functional equation $L(s,f)=(-1)^{k/2}L(k-s,f)$. For $n=2$, the definition $L(s,f)=(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_f(s)$ leads to the functional equation $L(s,f)=(-1)^kL(2k-2-s)$ proved in [An].

The purpose of this paper is to give formulas for the archimedean Euler factor in any degree. Automorphic representation theory provides the recipe to compute this factor. Let Π_k be the archimedean component of the automorphic representation of $\mathrm{PGSp}(2n,\mathbb{A})$ attached to the eigenform f (see [AS]). By the local Langlands correspondence over $\mathbb{R}$, there is an associated *local parameter* $\varphi: W_\mathbb{R}\to\mathrm{Spin}(2n+1,\mathbb{C})$, where $W_\mathbb{R}$ is the real Weil group. Let ρ be the spin representation. Then the factor we are looking for is

$$L(s,\,\Pi_k,\,\rho)=L(s,\,\rho\circ\varphi),$$

where on the right we have the L–factor attached to a finite-dimensional representation of the Weil group.

The formulas we will thus obtain coincide for $n=1$ and $n=2$ with the Γ–factors from above, except for some constants. Note however that we are working with the automorphic normalization that is designed to yield a functional equation relating s and $1-s$. To compare with the classical formulas, we have to make a shift in the argument s.

At the end of the paper [An] ANDRIANOV gives another definition of an archimedean Euler factor by a recursion formula. It turns out that for $n\geq 3$ this definition leads to a factor that is different from ours.

However, the difference is not too serious. One factor can be obtained from the other by multiplying with a quotient of *reflection polynomials*, where we call a polynomial δ a reflection polynomial if its zeros are integers or half-integers and if it has a "functional equation" $\delta(s) = \pm\delta(1-s)$. Therefore, for questions of meromorphic continuation and functional equation, either factor is suitable. The difference becomes however very relevant if one is interested in controlling poles in a global L–function.

If the degree n is *even* we shall establish a formula relating $L(s,\,\Pi_k,\,\rho)$ to symmetric power L–factors for GL(2). To be precise, we shall prove for $n = 2n'$ that

$$L(s,\,\Pi_{k+n'},\,\rho) = \delta(s)\prod_{j=0}^{n'}\prod_{r\in\mathbb{Z}} L\big(s-r/2,\,\mathcal{D}(2k-1),\,\mathrm{Sym}^{n'-j}\big)^{\beta(r,j,n')}, \tag{1}$$

where $\beta(r,j,n')$ are certain combinatorially defined exponents, $\mathcal{D}(2k-1)$ is the discrete series representation of $\mathrm{PGL}(2,\mathbb{R})$ of lowest weight $2k$, and $\delta(s)$ is a reflection polynomial. The existence of such a formula is already a hint that there is a "lifting" from elliptic modular forms of weight $2k$ to Siegel modular forms of weight $k+n'$ and degree $2n'$. This is indeed the case, as proved by IKEDA [Ik]. The above formula is needed to express the completed spin L–function of an Ikeda lift in terms of symmetric power L–functions, see [Sch].

As part of a global L–function, the reflection polynomial $\delta(s)$ in formula (1) should make a contribution of $\delta = \delta(s)/\delta(1-s)$ to the global ε–factor. This is indeed the case; we shall prove the analogous formula

$$\varepsilon(s,\,\Pi_{k+n'},\,\rho,\,\psi) = \delta\prod_{j=0}^{n'}\prod_{r\in\mathbb{Z}} \varepsilon\big(s-r/2,\,\mathcal{D}(2k-1),\,\mathrm{Sym}^{n'-j},\,\psi\big)^{\beta(r,j,n')}. \tag{2}$$

We shall also compute the ε–factor in odd degree. In each case (except $n=1$) it is just a sign. It is important to know this sign because it appears in the global functional equation.

DELIGNE has defined *critical points* of motives and made conjectures about the values of the L–function of the motive at these points. The critical points do only depend on the archimedean Euler factor. Having these factors for spin L–functions explicitly at hand, we will compute the critical points (of motives corresponding to Siegel modular forms) in the final part of this paper.

The first section of this paper contains background material on Weil group representations and their L–factors. In the second section we shall compute the archimedean L–factors for symmetric powers on GL(2), both for later use and as an illustration of our method. The next section contains the basic formula for $L(s,\,\Pi_k,\,\rho)$. We shall continue by establishing a recursion formula that will enable us to compare our factors with ANDRIANOV's. This is followed by a section in which the formula (1) is established. In the next section we use similar methods to compute the archimedean ε–factor. The final section is devoted to critical points.

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1 Preparations

1.1 Representations of the Weil group

We recall some basic facts about representations of the real Weil group. References are [Ta] and [Kn].

The real Weil group $W_{\mathbb{R}}$ is a semidirect product $W_{\mathbb{R}} = \mathbb{C}^* \rtimes \langle j \rangle$, where j is an element with $j^2 = -1$ which acts on $\mathbb{C}^*$ by $jzj^{-1} = \bar{z}$. A representation of $W_{\mathbb{R}}$ in some finite-dimensional complex vector space is called semisimple if its image consists entirely of semisimple elements. Every such representation is fully reducible. An irreducible semisimple representation is either one- or two-dimensional. Here is a complete list.

One-dimensional representations:

$$\tau_{+,t} : z \longmapsto |z|^t, \qquad j \longmapsto 1, \tag{3}$$

$$\tau_{-,t} : z \longmapsto |z|^t, \qquad j \longmapsto -1. \tag{4}$$

Here we have $t \in \mathbb{C}$ and $|\,|$ is the usual absolute value on $\mathbb{C}$ (not its square).

Two-dimensional representations:

$$\tau_{l,t} :\ re^{i\theta} \longmapsto \begin{pmatrix} r^{2t}e^{il\theta} & \\ & r^{2t}e^{-il\theta} \end{pmatrix}, \qquad j \longmapsto \begin{pmatrix} & (-1)^l \\ 1 & \end{pmatrix}. \tag{5}$$

Here the parameters are a positive integer l and some $t \in \mathbb{C}$. An L–factor is attached to a semisimple representation of $W_{\mathbb{R}}$. For the irreducible representations the L–factors are given as follows (see [Kn] (3.6)).

$$L(s, \tau_{+,t}) = \pi^{-(s+t)/2}\,\Gamma\Big(\frac{s+t}{2}\Big), \tag{6}$$

$$L(s, \tau_{-,t}) = \pi^{-(s+t+1)/2}\,\Gamma\Big(\frac{s+t+1}{2}\Big), \tag{7}$$

$$L(s, \tau_{l,t}) = 2(2\pi)^{-(s+t+l/2)}\,\Gamma\Big(s+t+\frac{l}{2}\Big). \tag{8}$$

For an arbitrary semisimple representation the associated L–factor is the product of the L–factors of its irreducible components.

The local Langlands correspondence is a parametrization of the infinitesimal equivalence classes of irreducible admissible representations of a real reductive group $G = G(\mathbb{R})$ by admissible homomorphisms $W_{\mathbb{R}} \to {}^LG$ into the L–group of G. If G is split over $\mathbb{R}$, then ${}^LG$ may be replaced by its identity component, which is a complex Lie group.

If π is an irreducible, admissible representation of G and ρ is a finite-dimensional representation of ${}^LG$, then an L–factor $L(s,\pi,\rho)$ is defined as follows. Let $\varphi :\ W_{\mathbb{R}} \to {}^LG$ be the local parameter attached to π. Define a semisimple representation of $W_{\mathbb{R}}$ by $\tau := \rho \circ \varphi$. Then the L–factor associated to π and ρ is

$$L(s,\pi,\rho) := L(s,\tau).$$

In this paper we shall be concerned with the following situation. G is the group $\mathrm{PGSp}(2n,\mathbb{R})$ (rank n) and π is a certain holomorphic discrete series representation (corresponding to a Siegel modular form of weight k). The identity component of the L–group is $\mathrm{Spin}(2n+1)$. As ρ we take the smallest genuine representation of this group, the 2^n–dimensional spin representation. The resulting factor $L(s,\pi,\rho)$ is of interest because it is the "correct" archimedean Euler factor of the spin L–function of a Siegel modular form, see [AS].

There are also ε–factors attached to semisimple representations of $W_{\mathbb{R}}$. In the archimedean case they do not depend on the complex variable s. They do however depend on the choice of an additive character ψ of $\mathbb{R}$. Throughout we will fix the character $\psi(x)=e^{2\pi i x}$. Then the ε–factors are given on irreducible representations as follows.

$$\varepsilon(s,\,\tau_{+,t},\,\psi)=1, \tag{9}$$

$$\varepsilon(s,\,\tau_{-,t},\,\psi)=i, \tag{10}$$

$$\varepsilon(s,\,\tau_{l,t},\,\psi)=i^{l+1}. \tag{11}$$

The ε–factor of an arbitrary semisimple representation is again the product of the factors of the irreducible components. Via the local Langlands correspondence we have ε–factors attached to irreducible, admissible representations of a reductive group G.

1.2 Archimedean factors for symmetric power L–functions

Let l be a positive integer, and let $\mathcal{D}(l)$ be the discrete series representation of $\mathrm{SL}^{\pm}(2,\mathbb{R})=\{g\in\mathrm{GL}(2,\mathbb{R}):\det(g)\in\{\pm1\}\}$ with a lowest weight vector of weight $k=l+1$ and a highest weight vector of weight $-l-1$. For $t\in\mathbb{C}$ we consider the representation

$$\mathcal{D}(l,t)=\mathcal{D}(l)\otimes|\det|^t$$

of $\mathrm{GL}(2,\mathbb{R})$ which coincides with $\mathcal{D}(l)$ on $\mathrm{SL}^{\pm}(2,\mathbb{R})$ and with $|a|^{2t}$ on matrices $\begin{pmatrix} a & \\ & a\end{pmatrix}$, $a>0$. The connected component of the L–group of $\mathrm{GL}(2,\mathbb{R})$ is $\mathrm{GL}(2,\mathbb{C})$, and, by [Kn], the local parameter $W_{\mathbb{R}}\to\mathrm{GL}(2,\mathbb{C})$ attached to $\mathcal{D}(l,t)$ is the representation $\tau_{l,t}$ defined in (5).

For a positive integer n let Sym^{n-1} be the n–dimensional irreducible representation of $\mathrm{GL}(2,\mathbb{C})$ on the space of homogeneous polynomials in two variables of degree $n-1$ given by

$$\mathrm{Sym}^{n-1}(g)P(x,y)=P((x,y)g),\qquad g\in\mathrm{GL}(2,\mathbb{C}).$$

The L–functions $L(s,\pi,\mathrm{Sym}^{n-1})$, for a global representation π of $\mathrm{GL}(2,\mathbb{A})$, are called *symmetric power L–functions*, cf. [Sh]. It is easy to compute their archimedean Euler factors, and we shall do so for the representations $\mathcal{D}(l,t)$ of $\mathrm{GL}(2,\mathbb{R})$.

1.2.1 Lemma. *The archimedean symmetric power L–factors for the representation $\mathcal{D}(l,t)$ of $\mathrm{GL}(2,\mathbb{R})$ are given as follows.*

i) If n is even, then

$$L(s,\,\mathcal{D}(l,t),\,\mathrm{Sym}^{n-1})=2^{n/2}\,(2\pi)^{-sn/2}\,(2\pi)^{-tn(n-1)/2-ln^2/8}$$

$$\prod_{j=1}^{n/2} \Gamma\Big(s+t(n-1)+\frac{1}{2}l(n+1)-lj\Big).$$

ii) If n is odd, then

$$L(s,\,\mathcal{D}(l,t),\,\mathrm{Sym}^{n-1}) = 2^{(n-1)/2}\,(2\pi)^{-s(n-1)/2}\,(2\pi)^{-t(n-1)^2/2-l(n^2-1)/8}\,\pi^{-(s+\varepsilon)/2-t(n-1)}$$
$$\Big(\prod_{j=1}^{(n-1)/2} \Gamma\Big(s+t(n-1)+\frac{1}{2}l(n+1)-lj\Big)\Big)\Gamma\Big(\frac{s+\varepsilon}{2}+t(n-1)\Big),$$

where

$$\varepsilon = \begin{cases} 0 & \text{if } n\equiv 1 \bmod 4,\\ 1 & \text{if } n\equiv 3 \bmod 4.\end{cases}$$

Proof: We have to decompose the composition

$$\tau:\ W_{\mathbb{R}} \longrightarrow \mathrm{GL}(2,\mathbb{C}) \longrightarrow \mathrm{GL}(n,\mathbb{C})$$

into irreducibles, where the first map is $\tau_{l,t}$ and the second one is Sym^{n-1}. If we take the natural basis $x^{n-1-j}y^j$, $j\in\{0,\dots,n-1\}$, for the space of Sym^{n-1}, then this representation is explicitly given by

$$re^{i\theta} \longmapsto r^{2t(n-1)}\begin{pmatrix} e^{il(n-1)\theta} &&&& \\ & e^{il(n-3)\theta} &&& \\ && \ddots && \\ &&& e^{-il(n-3)\theta} & \\ &&&& e^{-il(n-1)\theta}\end{pmatrix},$$

$$j \longmapsto \begin{pmatrix} &&&& -1 \\ &&& 1 & \\ && \iddots && \\ & 1 &&& \\ -1 &&&& \end{pmatrix}.$$

From this the irreducible components are obvious. If n is even, we have

$$\tau = \bigoplus_{j=1}^{n/2} \tau_{l(n+1-2j),t(n-1)},$$

with $\tau_{l,t}$ as in (5), and if n is odd, then

$$\tau = \Big(\bigoplus_{j=1}^{(n-1)/2} \tau_{l(n+1-2j),t(n-1)}\Big) \oplus \tau_{*,2t(n-1)},$$

where $*$ means "+" if $n\equiv 1 \bmod 4$ and "−" if $n\equiv 3 \bmod 4$ (see (3) and (4)). Using the definitions (6) to (8), the assertion is now easily obtained. ∎

For later use we shall rewrite the formulas in the lemma slightly, assuming that $t=0$. If n is odd, then

$$\begin{aligned} L(s,\mathcal{D}(l),\mathrm{Sym}^n) &= \prod_{\substack{j=0\\ \text{(step 2)}}}^{n-1} L(s,\,\tau_{l(n-j),0}) \\ &= \big(2(2\pi)^{-s}\big)^{(n+1)/2}\,(2\pi)^{-l(n+1)^2/8} \prod_{\substack{j=0\\ \text{(step 2)}}}^{n-1} \Gamma\Big(s+\frac{1}{2}ln-\frac{1}{2}lj\Big). \end{aligned} \tag{12}$$

If n is even, then

$$\begin{aligned} L(s,\mathcal{D}(l),\mathrm{Sym}^n) &= \Bigg(\prod_{\substack{j=0\\ \text{(step 2)}}}^{n-2} L(s,\,\tau_{l(n-j),0})\Bigg) L(s,\tau_{*,0}) \\ &= \big(2(2\pi)^{-s}\big)^{n/2}\,(2\pi)^{-ln(n+2)/8}\,\pi^{-(s+\varepsilon)/2}\Bigg(\prod_{\substack{j=0\\ \text{(step 2)}}}^{n-2}\Gamma\Big(s+\frac{1}{2}ln-\frac{1}{2}lj\Big)\Bigg)\Gamma\Big(\frac{s+\varepsilon}{2}\Big) \end{aligned} \tag{13}$$

where

$$\varepsilon = \begin{cases} 0 & \text{if } n\equiv 0 \bmod 4,\\ 1 & \text{if } n\equiv 2 \bmod 4,\end{cases} \qquad * = \begin{cases} + & \text{if } n\equiv 0 \bmod 4,\\ - & \text{if } n\equiv 2 \bmod 4.\end{cases}$$

2 The Euler factor for the spin representation

2.1 Basic computation

Let X (resp. P, Q) denote the character lattice (resp. weight lattice, root lattice) and $X^\vee$ (resp. $P^\vee$, $Q^\vee$) the cocharacter lattice (resp. coweight lattice, coroot lattice) of the group $\mathrm{PGSp}(2n)$. We can choose a basis $e_1,\dots,e_n$ of $X\otimes_{\mathbb{Z}}\mathbb{Q}$ and a dual basis $f_1,\dots,f_n$ of $X^\vee\otimes_{\mathbb{Z}}\mathbb{Q}$ such that these lattices are given as follows:

$$\begin{array}{ll} P=\langle e_1,\dots,e_n\rangle & P^\vee=\{\sum c_if_i:\ c_i\in\mathbb{Z}\ \forall i \text{ or } c_i\in\mathbb{Z}+\frac{1}{2}\ \forall i\}\\ {}_2| & \|\\ X=\{\sum c_ie_i:\sum c_i\in 2\mathbb{Z}\} & X^\vee\\ \| & {}_2|\\ Q=\langle e_1-e_2,\dots,e_{n-1}-e_n,\,2e_n\rangle & Q^\vee=\langle f_1-f_2,\dots,f_{n-1}-f_n,\,f_n\rangle \end{array}$$

The root datum of $\mathrm{Spin}(2n+1)$ is dual to this one, meaning that characters (cocharacters) become cocharacters (characters). Thus the $e_1,\dots,e_n$ identify with cocharacters of $\mathrm{Spin}(2n+1)$ and $f_1,\dots,f_n$ identify with characters of $\mathrm{Spin}(2n+1)$. See [Sp] for basic facts about root data.

Let us choose the standard set $e_1-e_2,\dots,e_{n-1}-e_n,2e_n$ of simple roots of $\mathrm{PGSp}(2n)$. The element

$$\nu=(k-1)e_1+\ldots+(k-n)e_n\in P$$

then lies in the fundamental Weyl chamber provided $k > n$. It is the Harish-Chandra parameter for a certain discrete series representation Π_k of $\mathrm{PGSp}(2n,\mathbb{R})$. This representation has a scalar minimal K–type with highest weight $ke_1+\ldots+ke_n$. It appears as the archimedean component of automorphic representations of $\mathrm{PGSp}(2n,\mathbb{A}_{\mathbb{Q}})$ attached to classical Siegel modular forms of weight k, see [AS]. If $k=n$ then we get a representation Π_k which is a limit of discrete series.

2.1.1 Proposition. *Let n and k be positive integers with $k \geq n$, and let Π_k be the (limit of) discrete series representation of* $\mathrm{PGSp}(2n,\mathbb{R})$ *considered above. Let ρ be the 2^n–dimensional spin representation of the dual group* $\mathrm{Spin}(2n+1,\mathbb{C})$. *Then*

$$L(s,\,\Pi_k,\,\rho)=\Bigg(\prod_{\substack{\varepsilon_1,\ldots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_i>0}}2(2\pi)^{-s}(2\pi)^{-|k\sum\varepsilon_i-\sum i\varepsilon_i|/2}\,\Gamma\Big(s+\frac{|k\sum\varepsilon_i-\sum i\varepsilon_i|}{2}\Big)\Bigg)$$
$$\Bigg(\prod_{\substack{\varepsilon_1,\ldots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_i=0,\,\sum i\varepsilon_i>0}}2(2\pi)^{-s}(2\pi)^{-\sum i\varepsilon_i/2}\,\Gamma\Big(s+\frac{1}{2}\sum i\varepsilon_i\Big)\Bigg)\Big(2(2\pi)^{-s}\,\Gamma(s)\Big)^{N/2}. \tag{14}$$

Here $N=\#\{(\varepsilon_1,\ldots,\varepsilon_n)\in\{\pm1\}^n:\ \sum\varepsilon_i=\sum i\varepsilon_i=0\}$. The second line of the formula can be ommitted for odd n.

Proof: As before let $e_1,\ldots,e_n$ span the character lattice and $f_1,\ldots,f_n$ span the cocharacter lattice of $\mathrm{PGSp}(2n)$, in a way such that $e_i(f_i(z))=z$ and $e_i(f_j(z))=1$ for $i\neq j$. The $e_1,\ldots,e_n$ identify with cocharacters of $\mathrm{Spin}(2n+1)$, and the $f_1,\ldots,f_n$ identify with characters. Let

$$\nu=(k-1)e_1+\ldots+(k-n)e_n$$

be the Harish-Chandra parameter of our representation Π_k. The local parameter $\varphi: W_{\mathbb{R}}\to\mathrm{Spin}(2n+1,\mathbb{C})$ attached to Π_k is then given by

$$\varphi(z)=z^{\nu}\bar{z}^{-\nu}\qquad(z\in\mathbb{C}^*),\qquad\qquad\varphi(j)=w,$$

where w is a representative for the longest Weyl group element (sending e_i to $-e_i$ for each i, see [Bo] 10.5). We have to consider the decomposition of $\tau:=\rho\circ\varphi$ into irreducibles.

The weights of the spin representation ρ are well known; they are

$$\frac{\varepsilon_1f_1+\ldots+\varepsilon_nf_n}{2},\qquad\varepsilon_i\in\{\pm1\}.$$

Each weight space is one-dimensional, so that ρ is a 2^n–dimensional representation. Let $v_{\varepsilon_1,\ldots,\varepsilon_n}$ be vectors spanning the one-dimensional weight spaces. Then, writing $z=re^{i\theta}$,

$$\begin{aligned}\tau(z)v_{\varepsilon_1,\ldots,\varepsilon_n}&=\rho(z^{\nu}\bar{z}^{-\nu})v_{\varepsilon_1,\ldots,\varepsilon_n}=\rho((e^{i\theta})^{2\nu})v_{\varepsilon_1,\ldots,\varepsilon_n}\\&=(\varepsilon_1f_1+\ldots+\varepsilon_nf_n)((e^{i\theta})^{\nu})v_{\varepsilon_1,\ldots,\varepsilon_n}\\&=e^{i(\varepsilon_1(k-1)+\ldots+\varepsilon_n(k-n))\theta}v_{\varepsilon_1,\ldots,\varepsilon_n}=e^{i(k\sum\varepsilon_i-\sum i\varepsilon_i)\theta}v_{\varepsilon_1,\ldots,\varepsilon_n}.\end{aligned}\tag{15}$$

Since w represents the longest Weyl group element, we have that

$$\tau(j)v_{\varepsilon_1,\ldots,\varepsilon_n} \text{ is a multiple of } v_{-\varepsilon_1,\ldots,-\varepsilon_n}$$

It follows that the two-dimensional spaces $\langle v_{\varepsilon_1,\ldots,\varepsilon_n} v_{-\varepsilon_1,\ldots,-\varepsilon_n}\rangle$ are invariant for the action of $W_{\mathbb{R}}$. Let $\tau_{\varepsilon_1,\ldots,\varepsilon_n}$ be the representation on this two-dimensional space. If $l := k\sum\varepsilon_i - \sum i\varepsilon_i \neq 0$, then obviously $\tau_{\varepsilon_1,\ldots,\varepsilon_n} = \tau_{|l|,0}$ where $\tau_{l,t}$ was defined in (5). The associated L–factor is

$$L(s,\tau_{\varepsilon_1,\ldots,\varepsilon_n}) = 2(2\pi)^{-(s+|l|/2)}\Gamma(s+|l|/2), \qquad l = k\sum\varepsilon_i - \sum i\varepsilon_i. \tag{16}$$

On the other hand, if $l=0$, then $\tau_{\varepsilon_1,\ldots,\varepsilon_n} = \tau_{+,0}\oplus\tau_{-,0}$, with associated L–factor

$$L(s,\tau_{\varepsilon_1,\ldots,\varepsilon_n}) = \pi^{-s/2}\,\Gamma\Big(\frac{s}{2}\Big)\,\pi^{-(s+1)/2}\,\Gamma\Big(\frac{s+1}{2}\Big).$$

By Legendre's formula for the Γ–function, this is the same factor as if we put $l=0$ in (16). The formula (14) is obtained by taking the product of all these factors over a certain system of representatives of $\{\pm1\}^n/\langle -1\rangle$. ∎

Examples: For $\underline{n=1}$ we obtain

$$L(s,\Pi_k,\rho) = 2(2\pi)^{-(s+(k-1)/2)}\,\Gamma\Big(s+\frac{k-1}{2}\Big).$$

This is the archimedean Euler factor in the Jacquet–Langlands L–function for an automorphic form on $\mathrm{PGL}(2,\mathbb{A}_{\mathbb{Q}})$ of weight k. Up to a factor 2 and a shift by $\frac{k-1}{2}$ it is also the classical Γ–factor attached to an elliptic modular form of weight k.

For $\underline{n=2}$ and $k\geq 2$ our formula yields

$$L(s,\Pi_k,\rho) = 4(2\pi)^{-2s}(2\pi)^{1-k}\,\Gamma\Big(s+k-\frac{3}{2}\Big)\Gamma\Big(s+\frac{1}{2}\Big).$$

Replacing s with $s-k+\frac{3}{2}$ we get

$$4(2\pi)^{k-2}(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2).$$

Up to the constant $4(2\pi)^{k-2}$ this is the factor used by ANDRIANOV in [An] to complete the partial L–function for classical Siegel modular forms of degree 2.

The absolute values in formula (14) are inconvenient for computations, and we would like to remove them. Let us define $\tilde{L}(s,\Pi_k,\rho)$ by the same formula, but without the absolute values, i.e.,

$$\begin{aligned}\tilde{L}(s,\Pi_k,\rho) = &\Bigg(\prod_{\substack{\varepsilon_1,\ldots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_i>0}} 2(2\pi)^{-s}(2\pi)^{(\sum i\varepsilon_i - k\sum\varepsilon_i)/2}\,\Gamma\Big(s+\frac{k\sum\varepsilon_i-\sum i\varepsilon_i}{2}\Big)\Bigg)\\ &\Bigg(\prod_{\substack{\varepsilon_1,\ldots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_i=0,\ \sum i\varepsilon_i>0}} 2(2\pi)^{-s}(2\pi)^{-\sum i\varepsilon_i/2}\,\Gamma\Big(s+\frac{1}{2}\sum i\varepsilon_i\Big)\Bigg)\Big(2(2\pi)^{-s}\,\Gamma(s)\Big)^{N/2}.\end{aligned} \tag{17}$$

Then

$$L(s,\,\Pi_k,\,\rho)=\tilde L(s,\,\Pi_k,\,\rho)\prod_{\substack{\varepsilon_1,\dots,\varepsilon_n\in\{\pm1\}^n\\ \sum\varepsilon_i>0}}\frac{(2\pi)^{-|k\sum\varepsilon_i-\sum i\varepsilon_i|/2}\,\Gamma\Big(s+\frac{|k\sum\varepsilon_i-\sum i\varepsilon_i|}{2}\Big)}{(2\pi)^{-(k\sum\varepsilon_i-\sum i\varepsilon_i)/2}\,\Gamma\Big(s+\frac{k\sum\varepsilon_i-\sum i\varepsilon_i}{2}\Big)}.$$

If we put

$$\delta_m(s):=(2\pi)^{-m}\frac{\Gamma\big(s+\frac{m}{2}\big)}{\Gamma\big(s-\frac{m}{2}\big)} \tag{18}$$

then we can write $L(s,\,\Pi_k,\,\rho)=\delta(s)\tilde L(s,\,\Pi_k,\,\rho)$ with

$$\delta(s)=\prod_{\substack{\varepsilon_1,\dots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_i>0\\ k\sum\varepsilon_i-\sum i\varepsilon_i<0}}\delta_{\sum i\varepsilon_i-k\sum\varepsilon_i}(s). \tag{19}$$

Note that for non-negative integers m the function δ_m is a polynomial: using repeatedly the functional equation $s\Gamma(s)=\Gamma(s+1)$ we get

$$\delta_m(s)=\prod_{j=1}^{m}\Big(s+\frac{m}{2}-j\Big).$$

It is immediate from this description that

$$\delta_m(s)=(-1)^m\delta_m(1-s).$$

Let us call a polynomial p a *reflection polynomial* if $p(s)=\pm p(1-s)$ for some sign $\pm$, and if all the zeros of p are either integers or half-integers. Then all the δ_m, and therefore the function δ also, are reflection polynomials.

2.1.2 Proposition. *With notations as in Proposition 2.1.1 we have*

$$L(s,\,\Pi_k,\,\rho)=\delta(s)\tilde L(s,\,\Pi_k,\,\rho),$$

where $\tilde L(s,\,\Pi_k,\,\rho)$ is defined in (17) and where $\delta(s)$, defined in (19), is a reflection polynomial (depending on n and k). If k is large enough, specifically,

$$k\geq\begin{cases}\dfrac{1}{4}(n+1)^2 & \text{for } n \text{ odd,}\\[2mm] \dfrac{1}{8}(n^2+4n) & \text{for } n \text{ even,}\end{cases} \tag{20}$$

then $\delta(s)=1$.

Proof: Only the last assertion remains to be proved. By definition, $\delta(s) = 1$ if $k\sum\varepsilon_i - \sum i\varepsilon_i \geq 0$ for each choice of $\varepsilon_i \in \{\pm 1\}$ such that $\sum\varepsilon_i > 0$. Let j be the number of indices i such that $\varepsilon_i = -1$. Then $0 \leq j \leq \frac{n-1}{2}$ (n odd) resp. $0 \leq j \leq \frac{n}{2} - 1$ (n even) and $\sum\varepsilon_i = n - 2j$. Since

$$\sum_{i=1}^{n} i\varepsilon_i \leq -\sum_{i=1}^{j} i + \sum_{i=j+1}^{n} i = \frac{1}{2}(n^2+n) - j^2 - j,$$

we have

$$k\sum\varepsilon_i - \sum i\varepsilon_i \geq k(n-2j) - \frac{1}{2}(n^2+n) + j^2 + j.$$

Thus we get $k\sum\varepsilon_i - \sum i\varepsilon_i \geq 0$ if

$$k \geq \frac{n^2+n}{2(n-2j)} - \frac{j^2+j}{n-2j} = \frac{1}{2}(n+2j+1) + \frac{j^2}{n-2j}.$$

The expression on the right grows with j, so we get an upper bound for it if we put $j = \frac{n-1}{2}$ (n odd) resp. $j = \frac{n}{2} - 1$ (n even). The upper bound is the one given on the right hand side of (20). ∎

Remark: Suppose our local factor $L(s, \Pi_k, \rho)$ appears as the archimedean factor in some global L–function $L(s, \Pi, \rho)$. Then, since $\delta(s)$ is a reflection polynomial, one can replace $L(s, \Pi_k, \rho)$ by $\tilde{L}(s, \Pi_k, \rho)$ if one is only concerned with questions of meromorphic continuation and functional equation. Of course, to know the precise sign in the functional equation, or to control poles, one has to take the "correct" $L(s, \Pi_k, \rho)$.

2.2 Recursion formulas

We are now seeking a relation between the factors $L(s, \Pi_k, \rho)$ in degree n and degree $n+1$ and shall therefore more precisely write $L(s, \Pi_{k,n}, \rho_n)$ for the factor in degree n.

2.2.1 Proposition. *We have the following recursion formulas for the factors $\tilde{L}(s, \Pi_k, \rho)$ defined in (17). Let $s_0 = (k-n-1)/2$.*

i) If n is even, then

$$\tilde{L}(s, \Pi_{k,n+1}, \rho_{n+1}) = \tilde{L}(s+s_0, \Pi_{k,n}, \rho_n)\tilde{L}(s-s_0, \Pi_{k,n}, \rho_n)\delta_{\text{even}}(s), \tag{21}$$

where $\delta_{\text{even}}(s)$ is a quotient of reflection polynomials given by

$$\delta_{\text{even}}(s) = \Bigg(\prod_{\substack{\varepsilon_1,\dots,\varepsilon_n \in \{\pm 1\} \\ \sum\varepsilon_i = 0,\ \sum i\varepsilon_i > 0}} \delta_{2s_0 - \sum i\varepsilon_i}(s)\Bigg)\delta_{2s_0}(s)^{N/2}.$$

Here $N = \#\{(\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^n : \sum\varepsilon_i = \sum i\varepsilon_i = 0\}$ and $\delta_m(s)$ is defined in (18).

ii) If n is odd, then

$$\tilde{L}(s, \Pi_{k,n+1}, \rho_{n+1}) = \tilde{L}(s+s_0, \Pi_{k,n}, \rho_n)\tilde{L}(s-s_0, \Pi_{k,n}, \rho_n)\delta_{\text{odd}}(s), \tag{22}$$

where $\delta_{\mathrm{odd}}(s)$ *is a quotient of reflection polynomials given by*

$$\delta_{\mathrm{odd}}(s) = \prod_{\substack{\varepsilon_1,\dots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_i=1,\ \sum i\varepsilon_i>n+1}} \delta_{\sum i\varepsilon_i-n-1}(s).$$

Proof: We shall prove the first statement; the manipulations for the second one are similar. The summations in the following formulas are from 1 to n, except in the first line, where they are from 1 to $n+1$.

$$\begin{aligned}
\tilde{L}(s,\,\Pi_{k,n+1},\,\rho_{n+1}) &= \prod_{\substack{\varepsilon_1,\dots,\varepsilon_{n+1}\in\{\pm1\}\\ \sum\varepsilon_i>0}} 2(2\pi)^{-s}(2\pi)^{(\sum i\varepsilon_i-k\sum\varepsilon_i)/2}\,\Gamma\Big(s+\frac{k\sum\varepsilon_i-\sum i\varepsilon_i}{2}\Big)\\
&= \prod_{\substack{\varepsilon_1,\dots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_i+1>0}} 2(2\pi)^{-s}(2\pi)^{(\sum i\varepsilon_i+(n+1)-k(\sum\varepsilon_i+1))/2}\,\Gamma\Big(s+\frac{k(\sum\varepsilon_i+1)-\sum i\varepsilon_i-(n+1)}{2}\Big)\\
&\quad\cdot\prod_{\substack{\varepsilon_1,\dots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_i-1>0}} 2(2\pi)^{-s}(2\pi)^{(\sum i\varepsilon_i-(n+1)-k(\sum\varepsilon_i-1))/2}\,\Gamma\Big(s+\frac{k(\sum\varepsilon_i-1)-\sum i\varepsilon_i+(n+1)}{2}\Big)\\
&= \prod_{\substack{\varepsilon_1,\dots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_i>0}} 2(2\pi)^{-(s+s_0)}(2\pi)^{(\sum i\varepsilon_i-k\sum\varepsilon_i)/2}\,\Gamma\Big(s+s_0+\frac{k\sum\varepsilon_i-\sum i\varepsilon_i}{2}\Big) &&(23)\\
&\quad\cdot\prod_{\substack{\varepsilon_1,\dots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_i=0}} 2(2\pi)^{-(s+s_0)}(2\pi)^{\sum i\varepsilon_i/2}\,\Gamma\Big(s+s_0-\frac{\sum i\varepsilon_i}{2}\Big) &&(24)\\
&\quad\cdot\prod_{\substack{\varepsilon_1,\dots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_i>0}} 2(2\pi)^{-(s-s_0)}(2\pi)^{(\sum i\varepsilon_i-k\sum\varepsilon_i)/2}\,\Gamma\Big(s-s_0+\frac{k\sum\varepsilon_i-\sum i\varepsilon_i}{2}\Big). &&(25)
\end{aligned}$$

For the last line note that $\sum\varepsilon_i>1$ is equivalent to $\sum\varepsilon_i>0$ since n is even. Now we are going to write the product (24) in the form

$$\begin{aligned}
(24) = &\prod_{\substack{\varepsilon_1,\dots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_i=0,\ \sum i\varepsilon_i>0}} 2(2\pi)^{-(s+s_0)}(2\pi)^{-\sum i\varepsilon_i/2}\,\Gamma\Big(s+s_0+\frac{\sum i\varepsilon_i}{2}\Big) &&(26)\\
&\cdot\prod_{\substack{\varepsilon_1,\dots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_i=0,\ \sum i\varepsilon_i>0}} 2(2\pi)^{-(s-s_0)}(2\pi)^{-\sum i\varepsilon_i/2}\,\Gamma\Big(s-s_0+\frac{\sum i\varepsilon_i}{2}\Big) &&(27)\\
&\cdot\prod_{\substack{\varepsilon_1,\dots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_i=0,\ \sum i\varepsilon_i>0}} \frac{2(2\pi)^{-(s+s_0)}(2\pi)^{\sum i\varepsilon_i/2}\,\Gamma\Big(s+s_0-\frac{\sum i\varepsilon_i}{2}\Big)}{2(2\pi)^{-(s-s_0)}(2\pi)^{-\sum i\varepsilon_i/2}\,\Gamma\Big(s-s_0+\frac{\sum i\varepsilon_i}{2}\Big)} &&(28)\\
&\cdot\Big(2(2\pi)^{-(s+s_0)}\Gamma(s+s_0)\Big)^{N/2} &&(29)
\end{aligned}$$

$$\cdot \Big(2(2\pi)^{-(s-s_0)}\Gamma(s-s_0)\Big)^{N/2} \tag{30}$$

$$\cdot \left(\frac{2(2\pi)^{-(s+s_0)}\Gamma(s+s_0)}{2(2\pi)^{-(s-s_0)}\Gamma(s-s_0)}\right)^{N/2}. \tag{31}$$

The terms (23), (26) and (29) combine to $\tilde{L}(s+s_0,\,\Pi_{k,n},\,\rho_n)$. The terms (25), (27) and (30) combine to $\tilde{L}(s-s_0,\,\Pi_{k,n},\,\rho_n)$. The terms (28) and (31) combine to the factor $\delta_{\text{even}}(s)$. This factor is indeed a quotient of reflection polynomials since $s_0 \in \frac{1}{2}\mathbb{Z}$ (we cannot conclude that it is itself a reflection polynomial since it may happen that $2s_0 - \sum i\varepsilon_i < 0$). ∎

At the end of the paper [An] ANDRIANOV defines an archimedean Euler factor for the spin L–function as

$$A_{n,k}(s) := (2\pi)^{-2^{n-1}s}\,\gamma_{n,k}(s) \tag{32}$$

(actually the exponent in ANDRIANOV's paper is $-2^n s$, but this is an obvious misprint), where the $\gamma_{n,k}$ are defined recursively by $\gamma_{1,k}(s) = \Gamma(s)$ and

$$\gamma_{n+1,k}(s) = \gamma_{n,k}(s)\,\gamma_{n,k}(s-k+n+1), \qquad n \geq 1. \tag{33}$$

This Euler factor is designed to fit into a global L–function that conjecturally has analytic continuation and a functional equation with center point

$$s = \frac{nk}{2} - \frac{n(n+1)}{4} + \frac{1}{2}.$$

To compare $A_{n,k}(s)$ with our factor $L(s,\,\Pi_{k,n},\,\rho_n)$ we shall make a shift in the argument to make $1/2$ the center point, and consider

$$\tilde{A}_{n,k}(s) := A_{n,k}\Big(s + \frac{nk}{2} - \frac{n(n+1)}{4}\Big), \qquad \tilde{\gamma}_{n,k}(s) := \gamma_{n,k}\Big(s + \frac{nk}{2} - \frac{n(n+1)}{4}\Big). \tag{34}$$

We see that both $\tilde{A}_{n,k}(s)$ and $L(s,\,\Pi_{k,n},\rho_n)$ contain a factor $(2\pi)^{-2^{n-1}s}$. Disregarding constant factors, what we really have to compare is $\tilde{\gamma}_{n,k}(s)$ and the Γ–functions in our formula (14). It follows from (33) that

$$\tilde{\gamma}_{n+1,k}(s) = \tilde{\gamma}_{n,k}(s+s_0)\,\tilde{\gamma}_{n,k}(s-s_0), \qquad s_0 = \frac{k-n-1}{2}, \tag{35}$$

a recursion formula very similar to (21) and (22), except that the δ–factors are missing. Thus we can say that, except for $n = 1$ and $n = 2$, ANDRIANOV*'s factor $A_{n,k}(s)$ is not the standard automorphic archimedean spin Euler factor, but differs from $L(s,\,\Pi_{k,n},\,\rho_n)$ only by a quotient of reflection polynomials.* As mentioned before, this does not make a big difference for questions of meromorphic continuation and functional equation.

2.3 Connection with symmetric power L–functions

In this section we shall show that if the degree n is even, then the L–factor $L(s,\Pi_k,\,\rho)$ can be expressed as a reflection polynomial times a product of symmetric power L–factors for GL(2). This was in principle already done in [Sch] and used to express the spin L–functions of the lifts constructed by IKEDA in [Ik] in terms of symmetric power L–functions for GL(2). Here we will present a more direct computation, but at one point will also use a result of [Sch].

Throughout we will assume that n is even and write $n = 2n'$. For an integer l let us abbreviate

$$f(l) := 2(2\pi)^{-(s+l/2)}\Gamma\Big(s+\frac{l}{2}\Big).$$

Our L–factor from Proposition 2.1.1 may then be written as

$$L(s,\,\Pi_k,\,\rho) = \prod_{(\varepsilon_1,\ldots,\varepsilon_n)\in\{\pm1\}^n/\langle -1\rangle} f\Big(\big|k\sum\varepsilon_i - \sum i\varepsilon_i\big|\Big).$$

We shall work with the modified factor

$$\tilde L(s,\,\Pi_k,\,\rho) = \Bigg(\prod_{\substack{\varepsilon_1,\ldots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_i>0}} f\Big(k\sum\varepsilon_i - \sum i\varepsilon_i\Big)\Bigg)\Bigg(\prod_{\substack{\varepsilon_1,\ldots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_i=0,\ \sum i\varepsilon_i>0}} f\Big(\sum i\varepsilon_i\Big)\Bigg) f(0)^{N/2} \tag{36}$$

defined in (17), which differs from $L(s,\,\Pi_k,\,\rho)$ at most by a reflection polynomial. For $\varepsilon_1,\ldots,\varepsilon_n\in\{\pm1\}$ let j be the number of negative ε_i's. Then $\sum\varepsilon_i = n-2j$, and the condition $\sum\varepsilon_i>0$ allows j to run between 0 and $n'-1$. With

$$\tilde\alpha(r,j,n) := \#\Big\{(\varepsilon_1,\ldots,\varepsilon_n\in\{\pm1\}^n:\ \sum\varepsilon_i = n-2j,\ \sum i\varepsilon_i = r\Big\}$$

we may then write

$$\tilde L(s,\,\Pi_k,\,\rho) = \Bigg(\prod_{j=0}^{n'-1}\prod_{r\in\mathbb{Z}} f\big(k(n-2j)-r\big)^{\tilde\alpha(r,j,n)}\Bigg)\Bigg(\prod_{r>0} f(r)^{\tilde\alpha(r,n',n)}\Bigg) f(0)^{\tilde\alpha(0,n',n)/2}. \tag{37}$$

Actually our interest is in $\tilde L(s,\,\Pi_{k+n'},\,\rho)$. The numbers $\tilde\alpha(r,j,n)$ can be described as follows. Let

$$\begin{aligned}\alpha(k,j,n') = {} & \text{number of possibilities to choose } j \text{ numbers from the set}\\ & \{1-2n', 3-2n', \ldots, 2n'-1\} \text{ such that their sum equals } k.\end{aligned} \tag{38}$$

This is the dimension of the weight–k space of the representation $\bigwedge^j V_n$ of $\mathrm{SL}(2,\mathbb{C})$, where V_n is the n–dimensional irreducible representation of this group ($n=2n'$). It is an easy exercise to show that

$$\tilde\alpha(r,j,n) = \alpha\big(r+(n+1)(j-n'),j,n'\big).$$

We therefore have

$$\tilde L(s,\,\Pi_{k+n'},\,\rho) = \Bigg(\prod_{j=0}^{n'-1}\prod_{r\in\mathbb{Z}} f\big((2k-1)(n'-j)-r\big)^{\alpha(r,j,n')}\Bigg)\Bigg(\prod_{r>0} f(r)^{\alpha(r,n',n')}\Bigg) f(0)^{\alpha(0,n',n')/2}. \tag{39}$$

Note that for a function of integers $g(j)$ and integers α_j we have

$$\prod_{j=0}^{m} g(j)^{\alpha_j} = \prod_{j_0=0}^{m}\ \prod_{\substack{j=j_0\\ j\equiv j_0 \bmod 2}}^{m} g(j)^{\alpha_{j_0}-\alpha_{j_0-2}}, \tag{40}$$

provided $\alpha_{-2} = \alpha_{-1} = 0$. Applying this to the above equation we get

$$\prod_{j=0}^{n'-1} f\big((2k-1)(n'-j)-r\big)^{\alpha(r,j,n)} = \prod_{j_0=0}^{n'-1} \prod_{\substack{j=j_0 \\ (\text{step } 2)}}^{n'-1-\varepsilon'(j_0)} f\big((2k-1)(n'-j)-r\big)^{\beta(r,j_0,n')}, \tag{41}$$

where

$$\varepsilon'(j_0) = \begin{cases} 0 & \text{if } n'-1 \equiv j_0 \bmod 2, \\ 1 & \text{if } n' \equiv j_0 \bmod 2. \end{cases}$$

and where $\beta(r,j,n') = \alpha(r,j,n') - \alpha(r,j-2,n')$. The following properties of the numbers α and β are all easy to see.

2.3.1 Lemma. *Let r, j, n' be integers with $0 \le j \le n'$.*

i) $\alpha(r,j,n') = 0$ *unless* $j(j-2n') \le r \le j(2n'-j)$ *and* $r \equiv j \bmod 2$. *Similar for* $\beta(r,j,n')$.

ii) $\alpha(r,j,n') = \alpha(-r,j,n')$. *Similar for* $\beta(r,j,n')$.

iii) $$\sum_{\substack{r=j(j-2n') \\ (\text{step } 2)}}^{j(2n'-j)} \alpha(r,j,n') = \binom{2n'}{j}, \qquad \sum_{\substack{r=j(j-2n') \\ (\text{step } 2)}}^{j(2n'-j)} \beta(r,j,n') = \binom{2n'}{j} - \binom{2n'}{j-2}.$$

iv) $$\sum_{\substack{j=0 \\ j\equiv n' \bmod 2}}^{n'} \beta(r,j,n') = \alpha(r,n',n').$$

■

If we consider a fixed j_0 and put $m = n' - j_0$, then

$$\prod_{\substack{j=j_0 \\ (\text{step } 2)}}^{n'-1-\varepsilon'(j_0)} f\big((2k-1)(n'-j)-r\big) = \prod_{\substack{j=0 \\ (\text{step } 2)}}^{m-1-\varepsilon'(j_0)} f\big((2k-1)(n'-j_0-j)-r\big) \tag{42}$$

$$= \prod_{\substack{j=0 \\ (\text{step } 2)}}^{m-1-\varepsilon'(j_0)} L\Big(s-\frac{r}{2}, \tau_{(2k-1)(m-j),0}\Big). \tag{43}$$

If we assume $\varepsilon'(j_0) = 0$, or equivalently m odd, then this expression is precisely $L(s-r/2, \mathcal{D}(2k-1), \mathrm{Sym}^m)$, see (12). This is not quite the case for $\varepsilon'(j_0) = 1$, or m even. In this case we have (see (13))

$$\prod_{\substack{j=j_0 \\ (\text{step } 2)}}^{n'-1-\varepsilon'(j_0)} f\big((2k-1)(n'-j)-r\big) = L\Big(s-\frac{r}{2}, \mathcal{D}(2k-1), \mathrm{Sym}^m\Big) L\Big(s-\frac{r}{2}, \tau_{*(j_0),0}\Big)^{-1}$$

where

$$*(j_0) = \begin{cases} + & \text{if } n' - j_0 \equiv 0 \bmod 4, \\ - & \text{if } n' - j_0 \equiv 2 \bmod 4. \end{cases}$$

Putting everything together we obtain

$$\begin{aligned}
\tilde{L}(s,\, \Pi_{k+n'},\, \rho) =& \Big(\prod_{j_0=0}^{n'-1} \prod_{r\in\mathbb{Z}} L\big(s - r/2,\, \mathcal{D}(2k-1),\, \mathrm{Sym}^{n'-j_0}\big)^{\beta(r,j_0,n')} \Big) \\
& \Big(\prod_{\substack{j_0=0\\ \varepsilon'(j)=1}}^{n'-1} \prod_{r\in\mathbb{Z}} L\Big(s - \frac{r}{2},\, \tau_{*(j_0),0}\Big)^{-\beta(r,j_0,n')} \Big) \Big(\prod_{r>0} f(r)^{\alpha(r,n',n')} \Big) f(0)^{\alpha(0,n',n')/2} \\
=& \ \delta(s) \prod_{j=0}^{n'} \prod_{r\in\mathbb{Z}} L\big(s - r/2,\, \mathcal{D}(2k-1),\, \mathrm{Sym}^{n'-j}\big)^{\beta(r,j,n')} \qquad (44)
\end{aligned}$$

with

$$\delta(s) = \Big(\prod_{\substack{j=0\\ \varepsilon'(j)=1}}^{n'} \prod_{r\in\mathbb{Z}} L\Big(s - \frac{r}{2},\, \tau_{*(j),0}\Big)^{-\beta(r,j,n')} \Big) \Big(\prod_{r>0} f(r)^{\alpha(r,n',n')} \Big) f(0)^{\alpha(0,n',n')/2}. \qquad (45)$$

Note that by Legendre's formula $\Gamma(s/2)\Gamma((s+1)/2) = 2^{1-s}\pi^{1/2}\Gamma(s)$ we have

$$f(r) = L\Big(s + \frac{r}{2}, \tau_{+,0}\Big) L\Big(s + \frac{r}{2}, \tau_{-,0}\Big) \qquad \text{for each } r \in \mathbb{Z}.$$

Applying Lemma 2.3.1 iv) we therefore have

$$\begin{aligned}
\delta(s) =& \prod_{\substack{j=0\\ \varepsilon'(j)=1}}^{n'} \prod_{r>0} \left(\frac{L(s + r/2, \tau_{+,0}) L(s + r/2, \tau_{-,0})}{L(s - r/2, \tau_{*(j),0}) L(s + r/2, \tau_{*(j),0})} \right)^{\beta(r,j,n')} \\
& \Big(\prod_{\substack{j=0\\ \varepsilon'(j)=1}}^{n'} L\big(s,\, \tau_{*(j),0}\big)^{-\beta(0,j,n')} \Big) \Big(L(s, \tau_{+,0}) L(s, \tau_{-,0}) \Big)^{\alpha(0,n',n')/2} \\
=& \Big(\prod_{\substack{j=0\\ \varepsilon'(j)=1}}^{n'} \prod_{r>0} \delta_{r,*(j)}(s)^{\beta(r,j,n')} \Big) \delta_{0,+}(s)^K,
\end{aligned}$$

where we put

$$\delta_{r,*}(s) = \frac{L(s + r/2, \tau_{+,0}) L(s + r/2, \tau_{-,0})}{L(s - r/2, \tau_{*,0}) L(s + r/2, \tau_{*,0})}$$

and

$$K = \frac{1}{2} \sum_{\substack{j=0\\ \varepsilon'(j)=1}}^{n'} (-1)^{(n'-j)/2} \beta(0, j, n'). \qquad (46)$$

First assume that $\underline{n' \text{ is odd}}$. Then $K = 0$ by Lemma 2.3.1 i), and our formula reads

$$\delta(s) = \prod_{\substack{j=1\\ \text{step } 2}}^{n'} \prod_{r \text{ odd}} \delta_{r,*(j)}(s)^{\beta(r,j,n')}. \tag{47}$$

It is easy to see that for odd r the factors $\delta_{r,*}$ are just reflection polynomials. Consequently $\delta(s)$ is itself a reflection polynomial.

Now assume that $\underline{n' \text{ is even}}$, so that

$$\delta(s) = \Bigg(\prod_{\substack{j=0\\ \text{step } 2}}^{n'} \prod_{r \text{ even}} \delta_{r,*(j)}(s)^{\beta(r,j,n')} \Bigg) \delta_{0,+}(s)^K. \tag{48}$$

In this case no $\delta_{r,*}$ is a polynomial. Nevertheless, it is still true that $\delta(s)$ is a reflection polynomial. This was proved in [Sch] using some finite-dimensional representation theory, and we are not going to reprove it here (in this paper we wrote $\delta_{r,0}$ for $\delta_{r,+}$ and $\delta_{r,1}$ for $\delta_{r,-}$). In [Sch] there is also more precise information on the location and order of zeros of the polynomial $\delta(s)$. To summarize:

2.3.2 Theorem. *The archimedean spin Euler factor in even degree $n = 2n'$ can be expressed as*

$$L(s,\,\Pi_{k+n'},\,\rho) = \delta(s) \prod_{j=0}^{n'} \prod_{r \in \mathbb{Z}} L\big(s - r/2,\, \mathcal{D}(2k-1),\, \mathrm{Sym}^{n'-j}\big)^{\beta(r,j,n')}$$

with a reflection polynomial $\delta(s)$.

Remark: The formula in this theorem is important for the following reason. Ikeda constructed a lifting map from elliptic cusp forms of weight $2k$ to Siegel modular forms of degree $n = 2n'$ and weight $k+n'$, see [Ik]. A group theoretic interpretation of this lifting was given in [Sch]. If π is a holomorphic cuspform of weight $2k$ on $\mathrm{PGL}(2)$, and Π is its Ikeda lift to $\mathrm{PGSp}(2n)$, then the spin L–function of Π is expressed through symmetric power L–functions of π. The Euler factor at a real place is handled by Proposition 2.3.2, while at unramified places a similar formula holds, but without the $\delta(s)$. In this way one reduces the analytic properties of the spin L–functions of the lifts to those of symmetric power L–functions for $\mathrm{GL}(2)$.

2.4 The ε–factor

We shall fix the additive character $\psi(x) = e^{2\pi i x}$ of $\mathbb{R}$ and give some information on the spin ε–factor $\varepsilon(s, \Pi_k, \rho, \psi)$. The basic computation is as in the proof of Proposition 2.1.1. With notation as in this proof, the two-dimensional $W_{\mathbb{R}}$–representation $\tau_{\varepsilon_1,\ldots,\varepsilon_n}$ has ε–factor

$$\varepsilon(s, \tau_{\varepsilon_1,\ldots,\varepsilon_n}, \psi) = i^{|l|+1}, \qquad l = k\sum_{j=1}^{n} \varepsilon_j - \sum_{j=1}^{n} j\varepsilon_j,$$

regardless of l being zero or not (see (9) – (11)). Consequently

$$\varepsilon(s,\Pi_k,\rho,\psi)=\prod_{(\varepsilon_1,\ldots,\varepsilon_n)\in\{\pm1\}^n/\langle -1\rangle} i^{|k\sum\varepsilon_j-\sum j\varepsilon_j|+1}$$
$$=i^{2^{n-1}}\Bigg(\prod_{\substack{\varepsilon_1,\ldots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_j>0}} i^{|k\sum\varepsilon_j-\sum j\varepsilon_j|}\Bigg)\Bigg(\prod_{\substack{\varepsilon_1,\ldots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_j=0,\ \sum j\varepsilon_j>0}} i^{\sum j\varepsilon_j}\Bigg). \tag{49}$$

For example, if $\underline{n=1}$ we get $\varepsilon(s,\Pi_k,\rho,\psi)=i^k$. This is the number appearing in the functional equation of the L–function of a classical elliptic modular form of weight k. If $\underline{n=2}$ and $k\geq 2$ (note our overall assumption $k\geq n$) then we get

$$\varepsilon(s,\Pi_k,\rho,\psi)=(-1)^k.$$

This is the sign in the functional equation of the spin L–function associated to a degree 2 Siegel modular form, see [An] Theorem 3.1.1.

2.4.1 Lemma. *For a positive integer n we have*

$$\sum_{\substack{\varepsilon_1,\ldots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_j>0}}\Big(\sum_{j=1}^n\varepsilon_j\Big)=\begin{cases}\dfrac{1}{2}n\dbinom{n}{n/2} & \text{if } n \text{ is even,}\\[2mm] n\dbinom{n-1}{(n-1)/2} & \text{if } n \text{ is odd.}\end{cases}$$

If n is even and ≥ 4, this number is divisible by 4.

Proof: Assume that n is even; the other case is similar. For $(\varepsilon_1,\ldots,\varepsilon_n)\in\{\pm1\}^n$ let l be the number of negative ε_j's. Then, for $\sum\varepsilon_j$ to be positive, l is allowed to run from 0 to $\frac{n}{2}-1$. Thus

$$\sum_{\substack{\varepsilon_1,\ldots,\varepsilon_n\in\{\pm1\}\\ \sum\varepsilon_j>0}}\Big(\sum_{j=1}^n\varepsilon_j\Big)=\sum_{l=0}^{(n-2)/2}(n-2l)\binom{n}{l}.$$

The first assertion now follows from the elementary formulas

$$\sum_{l=0}^{(n-2)/2}\binom{n}{l}=2^{n-1}-\frac{1}{2}\binom{n}{n/2}\qquad\text{and}\qquad\sum_{l=0}^{(n-2)/2}l\binom{n}{l}=n2^{n-2}-\frac{1}{2}n\binom{n}{n/2}.$$

The last statement is a consequence of the following lemma. ∎

2.4.2 Lemma. *i) For any positive integer n,*

$$\binom{2n}{n}\equiv\begin{cases}2 \mod 4 & \text{if } n \text{ is a power of } 2,\\ 0 \mod 4 & \text{otherwise.}\end{cases}$$

ii) For any odd positive integer $n \geq 3$,

$$\frac{1}{2}\binom{2n}{n} \equiv \begin{cases} 2 \mod 4 & \text{if } n-1 \text{ is a power of } 2, \\ 0 \mod 4 & \text{otherwise.} \end{cases}$$

Proof: i) This is an easy exercise; consider the 2–adic valuation of $n!$.
ii) We have $\frac{1}{2}\binom{2n}{n} = \frac{2n-1}{n}\binom{2n-2}{n-1}$. Since n is odd, the 2–adic valuation of the left hand side equals that of $\binom{2n-2}{n-1}$. Thus we are reduced to i). ∎

The case that n is odd

In this case the formula (49) simplifies to

$$\varepsilon(s,\Pi_k,\rho,\psi) = i^M \qquad \text{with} \qquad M = \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_n\in\{\pm 1\}\\ \sum\varepsilon_j>0}} \left|k\sum\varepsilon_j - \sum j\varepsilon_j\right|$$

(we are assuming that $n \geq 3$). Let us assume that k is large enough, so that the absolute values can be omitted (see Proposition 2.1.2). Then

$$M = \Big(k - \frac{n+1}{2}\Big) \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_n\in\{\pm 1\}\\ \sum\varepsilon_j>0}} \Big(\sum\varepsilon_j\Big) - \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_n\in\{\pm 1\}\\ \sum\varepsilon_j>0}} \Big(j - \frac{n+1}{2}\Big)\varepsilon_j.$$

The second sum changes its sign if ε_j is replaced by $\varepsilon_j' := \varepsilon_{n+1-j}$ and therefore vanishes. The first sum was computed in Lemma 2.4.1, so we get

$$M = \Big(k - \frac{n+1}{2}\Big) n \binom{n-1}{(n-1)/2}. \tag{50}$$

Note that the representation Π_k, considered as a representation of $\mathrm{GSp}(2n,\mathbb{R})$, has trivial central character. The minimal K–type is the character

$$g \longmapsto \det(CZ+D)^k \qquad \text{for } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K,$$

where K is the standard maximal compact subgroup of $\mathrm{GSp}(2n)$ (its identity component is isomorphic to $U(n)$). Therefore, since n is odd, k must necessarily be even. But $\binom{n-1}{(n-1)/2}$ is also even, so we get

$$\begin{aligned} M &\equiv \frac{n(n+1)}{2}\binom{n-1}{(n-1)/2} \equiv \frac{n+1}{2}\binom{n-1}{(n-1)/2} \\ &= \frac{n-1}{2}\binom{n-1}{(n-1)/2} + \binom{n-1}{(n-1)/2} \equiv \binom{n-1}{(n-1)/2} \mod 4, \end{aligned}$$

the last congruence being true for $n \geq 5$ (see Lemma 2.4.1). In view of Lemma 2.4.2 we get the following result.

2.4.3 Proposition. *Assume that $n \geq 3$ is an odd integer. Then for the spin ε–factor in degree n we have*

$$\varepsilon(s, \Pi_k, \rho, \psi) = \begin{cases} -1 & \text{if } n = 2^m + 1 \text{ for some } m \geq 2, \\ 1 & \text{otherwise,} \end{cases}$$

for any even weight $k \geq \frac{1}{4}(n+1)^2$.

The case that n is even

If we assume that $n \geq 4$ is even and k is large enough (see (20)), then formula (49) gives $\varepsilon(s, \Pi_k, \rho, \psi) = i^M$ with

$$M = \sum_{\substack{\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\} \\ \sum \varepsilon_j > 0}} \Big(k \sum \varepsilon_j - \sum j\varepsilon_j\Big) + \sum_{\substack{\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\} \\ \sum \varepsilon_j = 0,\ \sum j\varepsilon_j > 0}} \Big(\sum j\varepsilon_j\Big).$$

If we replace k by $k+1$, then, by Lemma 2.4.1, M changes by a number that is divisible by 4. It follows that $\varepsilon(s, \Pi_k, \rho, \psi) = i^M$ is independent of k (note this is not true for $n = 2$).

If we put $f(l) = i^{l+1}$, then we can also proceed as in section 2.3. In analogy with formula (39) we obtain

$$\begin{aligned} \varepsilon(s,\, \Pi_{k+n'},\, \rho,\, \psi) = &\Bigg(\prod_{j=0}^{n'-1} \prod_{r \in \mathbb{Z}} f\big((2k-1)(n'-j) - r\big)^{\alpha(r,j,n')}\Bigg) \\ &\Bigg(\prod_{r>0} f(r)^{\alpha(r,n',n')}\Bigg) f(0)^{\alpha(0,n',n')/2}. \end{aligned} \tag{51}$$

Note that here we can omit the r from $f((2k-1)(n'-j) - r)$ since $\sum_{r \in \mathbb{Z}} r\alpha(r,j,n') = 0$ by Lemma 2.3.1 ii). We then apply (40) and get a formula similar to (41), namely

$$\prod_{j=0}^{n'-1} f\big((2k-1)(n'-j)\big)^{\alpha(r,j,n)} = \prod_{j_0=0}^{n'-1} \prod_{\substack{j=j_0 \\ (\text{step } 2)}}^{n'-1-\varepsilon'(j_0)} f\big((2k-1)(n'-j)\big)^{\beta(r,j_0,n')}, \tag{52}$$

If we consider a fixed j_0 and put $m = n' - j_0$, then we get the analogue of equation (42):

$$\prod_{\substack{j=j_0 \\ (\text{step } 2)}}^{n'-1-\varepsilon'(j_0)} f\big((2k-1)(n'-j)\big) = \prod_{\substack{j=0 \\ (\text{step } 2)}}^{m-1-\varepsilon'(j_0)} f\big((2k-1)(n'-j_0-j)\big) \tag{53}$$

$$= \prod_{\substack{j=0 \\ (\text{step } 2)}}^{m-1-\varepsilon'(j_0)} \varepsilon\Big(s - \frac{r}{2},\, \tau_{(2k-1)(m-j),0},\, \psi\Big). \tag{54}$$

Note that archimedean ε–factors are independent of s, so we are free to put the shift of $r/2$ here. If m is odd, this equals $\varepsilon(s - r/2,\, \mathcal{D}(2k-1),\, \mathrm{Sym}^m,\, \psi)$. If m is even, there is a correcting factor of

$\varepsilon(s, \tau_{*(j_0),0})$. Thus we get, as in (44) and (45),

$$\varepsilon(s,\, \Pi_{k+n'},\, \rho,\, \psi) = \delta \prod_{j=0}^{n'} \prod_{r\in\mathbb{Z}} \varepsilon\big(s-r/2,\, \mathcal{D}(2k-1),\, \mathrm{Sym}^{n'-j},\, \psi\big)^{\beta(r,j,n')} \tag{55}$$

with

$$\begin{aligned}
\delta &= \Bigg(\prod_{\substack{j=0\\ \varepsilon'(j)=1}}^{n'} \prod_{r\in\mathbb{Z}} \varepsilon\Big(s-\frac{r}{2},\, \tau_{*(j),0},\, \psi\Big)^{-\beta(r,j,n')}\Bigg) \Bigg(\prod_{r>0} f(r)^{\alpha(r,n',n')}\Bigg) f(0)^{\alpha(0,n',n')/2}\\
&= \prod_{\substack{j=0\\ \varepsilon'(j)=1}}^{n'} \prod_{r>0} \left(\frac{i^{r+1}}{\varepsilon\big(s-r/2,\, \tau_{*(j),0},\, \psi\big)\varepsilon\big(s+r/2,\, \tau_{*(j),0},\, \psi\big)}\right)^{\beta(r,j,n')}\\
&\qquad \Bigg(\prod_{\substack{j=0\\ \varepsilon'(j)=1}}^{n'} \varepsilon\big(s,\, \tau_{*(j),0},\, \psi\big)^{-\beta(r,j,n')}\Bigg) i^{\alpha(0,n',n')/2+1}\\
&= \Bigg(\prod_{\substack{j=0\\ \varepsilon'(j)=1}}^{n'} \prod_{r>0} \big(i^{r+1-2\varepsilon(j)}\big)^{\beta(r,j,n')}\Bigg) i^K,
\end{aligned} \tag{56}$$

where

$$\varepsilon(j) = \begin{cases} 0 & \text{if } n'-j \equiv 0 \bmod 4,\\ 1 & \text{if } n'-j\equiv 2 \bmod 4,\end{cases} \tag{57}$$

and K is as in (46). Consider first the case that $\underline{n' \text{ is odd}}$ and let us compare the formulas (47) and (56). One easily shows that for each of the reflection polynomials $\delta_{r,*}$ in (47) the "functional equation"

$$\frac{\delta_{r,*(j)}(s)}{\delta_{r,*(j)}(1-s)} = i^{r+1-2\varepsilon(j)}$$

holds (r is odd, so this is just a sign). It follows that

$$\delta = \frac{\delta(s)}{\delta(1-s)} \tag{58}$$

is the sign appearing in the functional equation of the polynomial $\delta(s)$. Now assume that $\underline{n' \text{ is even}}$. Using some standard formulas for the Γ–function, one shows that for each of the factors $\delta_{r,*}$ appearing in formula (48) one has

$$\frac{\delta_{r,*(j)}(s)}{\delta_{r,*(j)}(1-s)} = \tan\Big(\frac{\pi s}{2}\Big)^{\sigma}, \qquad \sigma = i^{r-2\varepsilon(j)}.$$

In other words, if $r-2\varepsilon(j)\equiv 0 \bmod 4$, then we get a contribution of $\tan(\pi s/2)$ to $\delta(s)/\delta(1-s)$, and if $r-2\varepsilon(j)\equiv 2 \bmod 4$ we get a contribution of $\tan(\pi s/2)^{-1}$. From the factors $\delta_{0,+}(s)$ we get a contribution of $\tan(\pi s/2)^K$. Furthermore, we know the total contribution is 1, i.e. $\delta(s) = \delta(1-s)$, because $\delta(s)$ is known to be a polynomial.

Comparing this to (56) we see that each factor $\tan(\pi s/2)^{\pm 1}$ corresponds to a factor $i^{\pm 1}$. Since we just saw that there are as many positive as negative exponents, it follows that $\delta = 1$. In particular, (58) holds in all cases. We summarize:

2.4.4 Proposition. *Assume $k > \frac{1}{8}n^2$ and let $\delta(s)$ be the reflection polynomial of Proposition 2.3.2. Then the archimedean spin epsilon factor in even degree $n = 2n'$ can be expressed as*

$$\varepsilon(s,\, \Pi_{k+n'},\, \rho,\, \psi) = \delta \prod_{j=0}^{n'} \prod_{r\in\mathbb{Z}} \varepsilon\big(s - r/2,\, \mathcal{D}(2k-1),\, \mathrm{Sym}^{n'-j},\, \psi\big)^{\beta(r,j,n')}$$

where $\delta = \delta(s)/\delta(1-s)$. If n' is even, then $\delta = 1$.

Remark: This proposition makes sense, since, as part of a global L–function, the factor $\delta(s)$ obviously contributes the number $\delta(s)/\delta(1-s)$ to the global ε–factor appearing in the functional equation.

In fact, we can say more about the symmetric power part of the ε–factor.

2.4.5 Proposition. *Under the hypotheses of the previous proposition, we have*

$$\prod_{j=0}^{n'} \prod_{r\in\mathbb{Z}} \varepsilon\big(s - r/2,\, \mathcal{D}(2k-1),\, \mathrm{Sym}^{n'-j},\, \psi\big)^{\beta(r,j,n')} = \begin{cases} -1 & \text{if } n'-1 \text{ is a power of } 2, \\ 1 & \text{otherwise.} \end{cases}$$

Proof: Let

$$M = \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_n \in \{\pm 1\} \\ \sum \varepsilon_j > 0}} \Big(k \sum \varepsilon_j - \sum j\varepsilon_j + 1\Big),$$

so that i^M is the "first part" of our ε–factor, see (49). Similarly as in (50) we compute

$$\begin{aligned} M &= \Big(k - \frac{n+1}{2}\Big)\frac{n}{2}\binom{n}{n/2} + 2^{n-1} - \frac{1}{2}\binom{n}{n/2} \\ &\equiv \frac{(n+1)n}{4}\binom{n}{n/2} - \frac{1}{2}\binom{n}{n/2} \mod 4 \end{aligned} \tag{59}$$

(see Lemma 2.4.1). On the other hand, during the course of proving (55), we saw that

$$\begin{aligned} i^M = &\Big(\prod_{j=0}^{n'} \prod_{r\in\mathbb{Z}} \varepsilon\big(s - r/2,\, \mathcal{D}(2k-1),\, \mathrm{Sym}^{n'-j},\, \psi\big)^{\beta(r,j,n')}\Big) \\ &\Big(\prod_{\substack{j=0 \\ \varepsilon'(j)=1}}^{n'} \prod_{r\in\mathbb{Z}} \varepsilon\Big(s - \frac{r}{2},\, \tau_{*(j),0},\, \psi\Big)^{-\beta(r,j,n')}\Big) \end{aligned}$$

$$= \Big(\prod_{j=0}^{n'} \prod_{r \in \mathbb{Z}} \varepsilon\big(s - r/2,\, \mathcal{D}(2k-1),\, \mathrm{Sym}^{n'-j},\, \psi\big)^{\beta(r,j,n')} \Big) i^{-M'}$$

with

$$M' = \sum_{\substack{j=0 \\ \varepsilon'(j)=1}}^{n'} \sum_{r \in \mathbb{Z}} \varepsilon(j)\beta(r,j,n') = \sum_{\substack{j=0 \\ j \equiv n' \bmod 2}}^{n'} \varepsilon(j) \left(\binom{n}{j} - \binom{n}{j-2} \right)$$

(and $\varepsilon(j)$ as in (57)). Here we used Lemma 2.3.1 iii). It is an exercise to compute this number; one obtains

$$M' = \frac{1}{2}\binom{2n'}{n'} - 2^{n'-1}.$$

It follows that

$$M + M' \equiv \frac{(n+1)n}{4}\binom{n}{n/2} \mod 4.$$

This proves the assertion in view of Lemma 2.4.2. ∎

2.5 Critical points

We shall now work with the *geometric* normalization used in ANDRIANOV's paper [An] and put

$$L^g(s,\, \Pi_k,\, \rho) := L\Big(s - \frac{nk}{2} + \frac{n(n+1)}{4},\, \Pi_k,\, \rho\Big),$$

cf. (34). This L–factor fits into a global L–function that (conjecturally) admits a functional equation with center point

$$s_0 = \frac{nk}{2} - \frac{n(n+1)}{4} + \frac{1}{2}. \tag{60}$$

Assume f is a Siegel modular form of degree n and weight $k > n$, and let $M(f)$ be the (conjectural) associated motive such that

$$L(s,\, M(f)) = L(s,\, f,\, \rho),$$

where on the right we have the spin L–function of f. From the functional equation

$$L(s,\, f,\, \rho) = \varepsilon(s,\, f,\, \rho) L\Big(nk - \frac{n(n+1)}{2} + 1 - s,\, f,\, \rho\Big)$$

we conclude that $M(f)$ is of weight

$$w = nk - \frac{n(n+1)}{2}$$

(note that this is also the value of $k\sum \varepsilon_i - \sum i\varepsilon_i$ for the extreme case $\varepsilon_1 = \ldots = \varepsilon_n = 1$). The archimedean Euler factor of $L(s,\, f,\, \rho)$ (and of $L(s,\, M(f))$) is precisely our $L^g(s,\, \Pi_k,\, \rho)$. Following DELIGNE, an integer $n \in \mathbb{Z}$ is called *critical* for $M(f)$ if neither $L^g(s,\, \Pi_k,\, \rho)$ nor $L^g(nk - n(n+1)/2 + 1 - s,\, \Pi_k,\, \rho)$ has a pole at $s = n$.

2.5.1 Proposition. *Let f be a Siegel modular form of degree n and weight $k > n$, and let $M(f)$ be the motive whose L–function coincides with the spin L–function of f. Let $s_0 = \frac{nk}{2} - \frac{n(n+1)}{4} + \frac{1}{2}$ be the center point of the functional equation. Then the critical points of $M(f)$ are given as follows.*

i) Assume n is odd. If $k \geq \frac{(n+1)^2}{4}$, then there are $k - \frac{(n+1)^2}{4}$ critical points, namely the integer points in the interval

$$\Big[\frac{n-1}{2}k - \frac{n^2-1}{8} + 1, \ldots, \frac{n+1}{2}k - \frac{(n+1)(3n+1)}{8}\Big].$$

If $k < \frac{(n+1)^2}{4}$ and $n \equiv 1 \bmod 4$, then we have the single critical point s_0. If $k < \frac{(n+1)^2}{4}$ and $n \equiv 3 \bmod 4$, then there are no critical points.

ii) Assume n is even. If $n \equiv 0 \bmod 4$, then there are no critical points. If $n \equiv 2 \bmod 4$, then s_0 is the only critical point.

Proof: i) In the odd case we have

$$L^g(s, \Pi_k, \rho) := L\Big(s + \frac{1}{2} - s_0, \Pi_k, \rho\Big) = g(s) \prod_{\sum \varepsilon_i > 0} \Gamma(s - r(\varepsilon_1, \ldots, \varepsilon_n)),$$

where

$$r(\varepsilon_1, \ldots, \varepsilon_n) = s_0 - \frac{1}{2} - \frac{|k \sum \varepsilon_i - \sum i\varepsilon_i|}{2}$$

and where $g(s)$ is an entire function without zeros. The factor $\Gamma(s - r(\varepsilon_1, \ldots, \varepsilon_n))$ contributes poles at $s = r(\varepsilon_1, \ldots, \varepsilon_n) - \mathbb{N}_0$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Thus we need to find the maximal $r(\varepsilon_1, \ldots, \varepsilon_n)$, or, equivalently, the minimal $|k \sum \varepsilon_i - \sum i\varepsilon_i|$.

First assume that $k \geq (n+1)^2/4$, so that $k \sum \varepsilon_i - \sum i\varepsilon_i$ takes only non-negative values (see Proposition 2.1.2). Indeed, the minimum value of this expression is then $k - (n+1)^2/4$, so that the maximum value of $r(\varepsilon_1, \ldots, \varepsilon_n)$ is $\frac{n-1}{2}k - \frac{n^2-1}{8}$. Note this is indeed an integer. Thus all the integers greater than this number and less or equal s_0 are critical. The critical points lie symmetric about s_0, proving the first assertion.

Now assume $k < (n+1)^2/4$. Let j be the number of negative ε_i's. For fixed j let a_j be the minimum and b_j the maximum of the expression $k \sum \varepsilon_i - \sum i\varepsilon_i$. This expression takes every second integer value in $[a_j, b_j]$. It is easy to see that

$$a_{(n-1)/2} < 0 < b_{(n-1)/2} \qquad \text{for } k < \frac{(n+1)^2}{4}. \tag{61}$$

Furthermore, we have

$$k \sum \varepsilon_i - \sum i\varepsilon_i \equiv \begin{cases} 1 \mod 2, & \text{if } n \equiv 1 \mod 4, \\ 0 \mod 2, & \text{if } n \equiv 3 \mod 4 \end{cases} \tag{62}$$

(note that k is necessarily even in the odd case). It follows from (61) and (62) that the minimal value of $|k \sum \varepsilon_i - \sum i\varepsilon_i|$ is 1 if $n \equiv 1 \bmod 4$ and 0 if $n \equiv 3 \bmod 4$. Thus the maximum value of $r(\varepsilon_1, \ldots, \varepsilon_n)$

is $s_0 - 1$ resp. $s_0 - 1/2$. The assertion follows, since in the first case we have $s_0 \in \mathbb{Z}$, while in the second case $s_0 \in \frac{1}{2} + \mathbb{Z}$.

ii) These are similar considerations, and we omit the details. ■

Remark: The recursive definition (32) and (33) of archimedean Euler factors is not motivic in the following sense. As is easily seen, the $A_{n,k}(s)$ thus defined contains a factor $\Gamma(s-(n-1)k+n(n+1)/2-1)$. The point $(n-1)k-n(n+1)/2+1$ lies on the right of the center s_0 (given by (60)). But a motivic Euler factor contains only Γ–factors of the form $\Gamma(s-p)$ with $p < s_0 - \frac{1}{2}$ (coming from the Hodge types (p,q), (q,p) with $p < q$ and $p+q = w = 2s_0 - 1$) and of the form $\Gamma((s+\varepsilon-p)/2)$ with $2p = w$ and $\varepsilon \in \{\pm 1\}$ (coming from Hodge type (p,p)), see [De] 5.3. In particular, using ANDRIANOV's factors one would conclude that there are no critical points as soon as $n \geq 3$.

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