Local Maaß Lifts

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Abstract. We call a Maass lift a prescription to associate an irreducible admissible resp. unitary representation of $G = \mathrm{PGSp}(4)$ to an irreducible admissible resp. unitary representation of the Jacobi group $G^J = \mathrm{SL}(2) \ltimes H$. Such a prescription is constructed in the general local case for the principal series representations $\pi^J_{m,\chi}$ of G^J and moreover in the real case for the discrete series representation $\pi^{J+}_{m,k}$ of $G^J(\mathbb{R})$. This prescription goes via induction from G^J to the maximal parabolic subgroup Q of G with non-abelian radical and uses an intertwining operator i_m relating the induced representation of Q to a restriction to Q of a representation of G. Moreover in the real case, some infinitesimal considerations of perhaps independent interest come in.

Introduction

For k an even positive integer consider the following spaces of various types of modular forms:

$S_k(\Gamma_2)$:	Siegel cuspforms of weight k and degree 2,
$S_{2k-2}(\Gamma_1)$:	elliptic cuspforms of weight $2k-2$,
$S_{k-1/2}(\Gamma_0(4))$:	cuspforms of half-integer weight $k-1/2$,
$J^{\mathrm{cusp}}_{k,1}$:	Jacobi cuspforms of weight k and index 1.

Then we have the following commutative diagram of "lifting maps" between these spaces:

$$\begin{tikzcd}
 & S_k(\Gamma_2) & \\
 & J^{\mathrm{cusp}}_{k,1} \arrow[u, "M"'] \arrow[dl, leftrightarrow, "\sim"] \arrow[dr, leftrightarrow, "\sim"] & \\
S_{k-1/2}(\Gamma_0(4))^+ \arrow[uur, "\theta"] \arrow[rr, leftrightarrow, "\mathrm{S}", "\sim"'] & & S_{2k-2}(\Gamma_1) \arrow[uul, "\mathrm{SK}"']
\end{tikzcd}$$

Here $S_{k-1/2}(\Gamma_0(4))^+$ is Kohnen's "+"-space, a certain subspace of $S_{k-1/2}(\Gamma_0(4))$ characterized by properties of Fourier coefficients; the lifting between this space and $J^{\mathrm{cusp}}_{k,1}$ is a more or less canonical isomorphism; S denotes the Shimura isomorphism; θ is a special theta-correspondence; M is the Maaß lift, constructed in the series of papers [Ma1-3] (see also [An] and [Za]); and SK is the Saito-Kurokawa-lift, whose existence was conjectured in [Ku], and which was eventually constructed as the composition of the other lifts, in particular utilizing the Maaß lift. A comprehensive reference for all these maps is [EZ].

The above diagram can be reinterpreted and generalized using the representation theory of the underlying groups. If G is an algebraic group defined over some number field, then denote by $\widehat{G}$ the

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space of cuspidal automorphic representations of $G(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of F. Then, in analogy to the above diagram of lifting maps between spaces of classical modular forms, we should have a commutative diagram as follows:

$$\begin{tikzcd}
 & \widehat{\mathrm{PGSp}(4)} & \\
 & \widehat{G^J} \arrow[u, "M"] \arrow[dr] \arrow[dl, leftrightarrow, "\sim"'] & \\
\widehat{\mathrm{Mp}} \arrow[uur, "\theta"] \arrow[rr, "\mathrm{W}"'] & & \widehat{\mathrm{PGL}(2)} \arrow[uul, "\mathrm{SK}"']
\end{tikzcd} \qquad (1)$$

Here G^J is the Jacobi group, a certain semidirect product of SL(2) with a three-dimensional Heisenberg group; Mp denotes the metaplectic group, the two-fold cover of SL(2) (this is not an algebraic group, but can nevertheless be globalized in a well-known way, and it makes sense to talk about its automorphic representations); θ is again a special instance of the theta-correspondence; W is the Waldspurger-lift, essentially also a version of theta-correspondence, see [Wa1] and [Wa2]; and the lift from the Jacobi group to PGL(2) is described in [Sch1]. The Maaß lift M and the Saito-Kurokawa-lift SK are not yet constructed. It is however explained in [Sch2] to what extend the Saito-Kurokawa-lift is an example for Langlands functoriality. (This diagram is somewhat oversimplified: θ and W depend on a parameter, and a choice of this parameter corresponds to considering only elements of $\widehat{G^J}$ with a certain fixed *index*, but we neglect these details for the purpose of this introduction.)

The purpose of this note is to get some insight into the way the still hypothetical Maaß lift on the level of representations ought to be constructed. We shall in particular obtain local versions (both archimedean and $\mathfrak{p}$-adic) of the Maaß lift for all principal series representations. These local lifts will be shown to be compatible with the classical map $J_{k,1}^{\mathrm{cusp}} \to S_k(\Gamma_2)$, and will provide an "explanation" for the classical construction.

Following Piatetski-Shapiro [PS2], let us call the image of the lifting maps to PGSp(4) *special representations*, both in a local or global context. Here is the fundamental idea of how to construct special representations from representations of G^J. Let Q be the maximal parabolic subgroup of GSp(4) with non-abelian radical. Of course Q contains the center C. Consider an irreducible representation π^J of G^J, and denote by

$$\sigma = C - \mathrm{Ind}_{G^J}^{Q}(\pi^J)$$

the subspace of C–invariant elements in $\mathrm{Ind}_{G^J}^{Q}(\pi^J)$. At least in the global context, we can extract from [PS1] that *this representation σ should now extend uniquely to a representation of* PGSp(4), and this extension will be a special representation, namely the lift of π^J.

We take the following approach to prove this fact for local principal series representations of the Jacobi group. Let F be a local field, which may be archimedean. For each character χ of F^* we have the principal series representation

$$\pi_{\chi,m}^J$$

of G^J, where $m \in F^*$ is a fixed *index*. To the same χ we will also attach a representation σ_χ of PGSp(4) as a subrepresentation of a certain induced representation. This is our candidate for the lift of $\pi^J_{\chi,m}$. We then define a Q-intertwining operator

$$i_m = i_{\chi,m} : \ \sigma_\chi \longrightarrow C - \mathrm{Ind}^Q_{G^J}(\pi^J_{\chi,m}).$$

We shall prove that $i_{\chi,m}$ *is an isomorphism.* As a corollary we obtain that $C - \mathrm{Ind}^Q_{G^J}(\pi^J_{\chi,m})$ indeed extends to a representation of PGSp(4), but also that σ_χ remains irreducible when restricted to Q.

In the real holomorphic case more details will be given in section 3. There are additional insights coming from Lie algebra combinatorics on the infinitesimal level. We shall also compute the image of the lowest weight vector on G under our intertwining operator i_m. When restricted to G^J, this image coincides with the lowest weight vector in holomorphic discrete series representations of G^J (Proposition 3.4). Thus distinguished vectors map to distinguished vectors.

By Frobenius reciprocity, an equivalent viewpoint is to say that the special representations σ of $G =$ PGSp(4) are the ones that have a Fourier-Jacobi model, i.e., the ones for which there exists a Jacobi representation π^J such that

$$\dim\Big(\mathrm{Hom}_G\big(\sigma, \mathrm{Ind}^G_{G^J}(\pi^J)\big)\Big) \geq 1.$$

This point of view has been pursued in the $\mathfrak{p}$-adic case by Baruch-Rallis [BR], proving here via distribution theory the other inequality, i.e., the uniqueness of the Fourier-Jacobi models. And in the real case there is a lot of information about the dimensions of these spaces coming from a method via solutions of systems of partial differential equations initiated by Yamashita [Ya] and Oda [O], and applied to this situation in Hirano [Hi]. In particular, part of [Hi] Theorem 6.3 can be looked at as another aproach to our result in section 3 below on the existence of a Maaß lifting of the holomorphic discrete series representation of the Jacobi group.

In a subsequent paper we are planning to give additional details in the $\mathfrak{p}$-adic case, mainly concerning unramified vectors. We shall also try to connect our local results with the classical global theory of the Maaß lift.

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1 Groups

1.1 The symplectic group

The group containing all the other groups of current interest is

$$G = \mathrm{GSp}(4) = \{g \in \mathrm{GL}(4) : \ \exists\, \tilde\mu(g) \in \mathrm{GL}(1) \quad gJ\,{}^tg = \tilde\mu(g)J\}, \qquad\qquad J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}.$$

The function $\tilde{\mu}$ is a character of G, called the multiplier, and its kernel is the symplectic group $\mathrm{Sp}(4)$. As a Borel subgroup of G we choose $B = AN$ with the maximal torus

$$A = \left\{ t = d(u,a,a') := \mathrm{diag}(a, a', ua^{-1}, ua'^{-1}) : \ u, a, a' \in \mathrm{GL}(1) \right\}$$

and the unipotent radical

$$N = \left\{ n(x,\lambda,\mu,\kappa) := \begin{pmatrix} 1 & & x & \mu \\ & 1 & \mu & \kappa \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ \lambda & 1 & & \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix} : \ x, \lambda, \mu, \kappa \in \mathbb{G}_a \right\}.$$

The two standard maximal parabolics are then

$$P = MS,$$

where

$$M = \left\{ \begin{pmatrix} A & 0 \\ 0 & u\,{}^tA^{-1} \end{pmatrix} : \ A \in \mathrm{GL}(2), \ u \in F^* \right\} \simeq \mathrm{GL}(2) \times \mathrm{GL}(1),$$

$$S = \left\{ \begin{pmatrix} \mathbf{1} & s \\ 0 & \mathbf{1} \end{pmatrix} : \ s \text{ symmetric} \right\},$$

and

$$Q = FH,$$

where

$$F = \left\{ \begin{pmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & a' & & \\ & & u & \\ & & & ua'^{-1} \end{pmatrix} : \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2), \ u, a' \in \mathrm{GL}(1) \right\}, \tag{2}$$

$$H = \left\{ (\lambda,\mu,\kappa) = \begin{pmatrix} 1 & & & \mu \\ \lambda & 1 & \mu & \kappa \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix} : \ \lambda, \mu, \kappa \in \mathbb{G}_a \right\}. \tag{3}$$

The four positive roots are

$$\alpha_1(t) = a^2u^{-1}, \qquad \alpha_2(t) = a'a^{-1},$$
$$\alpha_3(t) = aa'u^{-1}, \qquad \alpha_4(t) = a'^2u^{-1}.$$

(α_1 is the long simple root, α_2 is the short simple root). The Weyl group of G is generated by

$$w_1 = \begin{pmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{pmatrix} \in Q, \qquad w_2 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \in P.$$

We have the relation $(w_1w_2)^4 = 1$. The other six elements of the Weyl group are the identity and

$$w_1w_2 = \begin{pmatrix} &&&1\\ 1&&&\\ &-1&&\\ &&1& \end{pmatrix}, \qquad w_2w_1 = \begin{pmatrix} &1&&\\ &&1&\\ &&&1\\ -1&&& \end{pmatrix},$$

$$w_1w_2w_1 = \begin{pmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{pmatrix}, \qquad w_2w_1w_2 = \begin{pmatrix} 1&&&\\ &&&1\\ &&1&\\ &-1&& \end{pmatrix},$$

$$w_1w_2w_1w_2 = w_2w_1w_2w_1 = \begin{pmatrix} &&1&\\ &&&1\\ -1&&&\\ &-1&& \end{pmatrix} \qquad \text{(longest element).}$$

The modular factors of the parabolics P, Q and $B = P \cap Q$ are easily computed as

$$\delta_B \begin{pmatrix} a&&&\\ &b&&\\ &&c&\\ &&&d \end{pmatrix} = |a^2bd^{-3}| \qquad (ac = bd), \tag{4}$$

$$\delta_P \begin{pmatrix} A & \\ & u\,{}^tA^{-1} \end{pmatrix} = |u^{-1}\det(A)|^3, \tag{5}$$

$$\delta_Q \begin{pmatrix} a&&b&\\ &1&&\\ c&&d&\\ &&&1 \end{pmatrix} = 1, \qquad \delta_Q \begin{pmatrix} 1&&&\\ &a'&&\\ &&u&\\ &&&ua'^{-1} \end{pmatrix} = |a'^4u^{-2}|. \tag{6}$$

Recall that there is an isomorphism $\mathrm{PGSp}(4) \simeq \mathrm{SO}(5)$, which can be described as follows. Let us realize the orthogonal groups $\mathrm{SO}(3)$ and $\mathrm{SO}(5)$ using the quadratic forms

$$\begin{pmatrix} &&1\\ &1&\\ 1&& \end{pmatrix} \qquad \text{resp.} \qquad \begin{pmatrix} &&&&1\\ &&&1&\\ &&1&&\\ &1&&&\\ 1&&&& \end{pmatrix}.$$

There is an isomorphism $\mathrm{PGL}(2) \simeq \mathrm{SO}(3)$ explicitly given by

$$\mathrm{GL}(2) \ni \begin{pmatrix} a & b\\ c & d \end{pmatrix} \overset{\varphi}{\longmapsto} \frac{1}{ad-bc} \begin{pmatrix} a^2 & -2ab & -2b^2\\ -ac & ad+bc & 2bd\\ -c^2/2 & cd & d^2 \end{pmatrix}.$$

Then the isomorphism $\mathrm{PGSp}(4) \simeq \mathrm{SO}(5)$ maps

$$\mathrm{GSp}(4) \ni \begin{pmatrix} A & 0\\ 0 & u(A^t)^{-1} \end{pmatrix} \longmapsto \begin{pmatrix} u^{-1}(ad-bc) && \\ & \varphi(A) & \\ && u(ad-bc)^{-1} \end{pmatrix}, \tag{7}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2)$. In particular, on the maximal torus the map is given by

$$\begin{pmatrix} a &&& \\ & b && \\ && c & \\ &&& d \end{pmatrix} \longmapsto \begin{pmatrix} ad^{-1} &&&& \\ & ab^{-1} &&& \\ && 1 && \\ &&& ba^{-1} & \\ &&&& da^{-1} \end{pmatrix}. \tag{8}$$

1.2 The parabolic Q and the Jacobi group

Let Q be the standard parabolic of $G = \mathrm{GSp}(4)$ described above. We shall consider several subgroups of Q.

Let C be the center of Q; it consists of the scalar matrices, and is also the center of G. Assume $\mathrm{GL}(2)$ is embedded in Q via

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a && b & \\ & 1 && \\ c && d & \\ &&& \det(A) \end{pmatrix}.$$

The *Jacobi group* is the subgroup

$$G^J := \mathrm{SL}(2) \ltimes H \subset Q$$

Here H is the Heisenberg group, the unipotent radical of Q. The group Q can be written in various ways as the direct product of the center and a subgroup, e.g.

$$Q = C \times (\{d(u,1,1)\} \ltimes G^J) \qquad \text{or} \qquad Q = C \times (\{d(v,1,v)\} \ltimes G^J). \tag{9}$$

It will be useful to keep this in mind, because we shall consider exclusively C-invariant functions, and are hence in effect working on one of these subgroups. The choice is by convenience. If we are over $\mathbb{R}$, being able to take square roots, there are even more possible choices, cf. section 3.5.

The center Z of the Jacobi group coincides with the center of the Heisenberg group; we denote its elements by

$$n_\kappa := (0,0,\kappa), \qquad \kappa \in \mathbb{G}_a$$

(see (3)). Obviously $Z \simeq \mathbb{G}_a$.

Now assume we are over a local or global field F. In the first case, let ψ be a fixed non-trivial character of F, in the second case a fixed non-trivial character of $\mathbb{A}_F/F$, where $\mathbb{A}_F$ is the ring of adeles of F. We also consider ψ a character of Z. For $m \in F$ let ψ^m be the shifted character $\psi^m(x) = \psi(mx)$. If a representation π^J of G^J has central character ψ^m, then we say that m is the *index* of π^J. We usually only consider representations of G^J with non-zero index. Information about the representation theory of G^J can be found in [BeS].

Let π^J be an irreducible representation of G^J with non-zero index m, and consider the induced representation

$$\mathrm{Ind}_{G^J}^{Q}(\pi^J).$$

Inside this induced representation we consider the invariant subspace consisting of C-invariant vectors, where C denotes as before the center of G. To have a short notation, we denote this subspace (and the Q-representation on it) by

$$\sigma := C - \mathrm{Ind}_{G^J}^{Q}(\pi^J). \tag{10}$$

We shall see that this Q-representations is in fact (at least for principal series representations) the restriction of a G-representation.

2 Special representations

In this section F is a local field of characteristic zero, possibly archimedean. We let $G = \mathrm{GSp}(4, F)$, and also B, T,... denote F-points.

2.1 Induced representations on GSp(4)

We define a class of induced representations on G. Let χ_1, χ_2 be characters of F^*. They define a character χ of the maximal torus T of G by

$$\chi\big(d(u, a, a')\big) = \chi_1(aa'u^{-1})\chi_2(aa'^{-1}).$$

Then χ extends to a character of B, and we let

$$\pi(\chi_1, \chi_2) = \mathrm{Ind}_B^G(\chi)$$

(normalized induction). The space of this induced representation is

$$I(\chi_1, \chi_2) = \Big\{ f : G \to \mathbb{C} \;\Big|\; f\big(d(u, a, a')n(x, \lambda, \mu, \kappa)g\big) = \chi_1(aa'u^{-1})\chi_2(aa'^{-1})|a^2a'^4u^{-3}|^{1/2}f(g)\Big\}.$$

Of course, there are also some regularity conditions imposed on the functions in $I(\chi_1, \chi_2)$. Those look slightly different in the archimedean and non-archimedean cases, and we therefore do not mention them for the sake of a simple and unified formulation.

Note that $\pi(\chi_1, \chi_2)$ has trivial central character, is thus a representation of PGSp(4). Equation (8) shows that we obtain each representation of PGSp(4) parabolically induced from the Borel subgroup in this way. From the paper [ST] one can get detailed information on the reducibility of these representations in the $\mathfrak{p}$-adic case.

We now specialize and consider the representations $I(\chi, |\,|^{1/2})$ for an arbitrary character χ of F^* (it follows from the global considerations in [Sch2] that it is inside this space where we should be looking for the Maaß lifts). Its space consists of functions $f : G \to \mathbb{C}$ with

$$f\big(d(u, a, a')n(x, \lambda, \mu, \kappa)g\big) = \chi(aa'u^{-1})|aa'u^{-1}|^{3/2}f(g).$$

There is an obvious G-invariant subspace, namely

$$I_\chi := \left\{ f:\ G \to \mathbb{C} \;\Big|\; f\left(\begin{pmatrix} A & * \\ & u\,{}^tA^{-1} \end{pmatrix} g\right) = \chi(\det(A)u^{-1})\left|\det(A)u^{-1}\right|^{3/2} f(g) \right\}. \tag{11}$$

We let σ_χ be the representation of G (via right translation) on I_χ, and call (σ_χ, I_χ) the *special representation* of PGSp(4) attached to χ. We have

$$\sigma_\chi = \mathrm{Ind}_P^G(\tilde{\sigma}_\chi),$$

where $\tilde{\sigma}_\chi$ is the character of P given by

$$\tilde{\sigma}_\chi\begin{pmatrix} A & * \\ & u\,{}^tA^{-1} \end{pmatrix} = \chi(u^{-1}\det(A)).$$

2.2 Induced representations on G^J and on Q

Again let χ be a character of F^*. In 1.2 we also fixed an additive character ψ of F. Consider the "Borel subgroup" $B^J \subset G^J$ consisting of elements of the form

$$b^J = \begin{pmatrix} a & x \\ & a^{-1} \end{pmatrix}(0, \mu, \kappa) = n(ax, 0, a\mu, \kappa)d(1, a, 1).$$

Let $m \in F^*$, and consider the character of B^J given by

$$b^J \longmapsto \chi(a)\psi^m(\kappa). \tag{12}$$

Let $\pi^J_{\chi,m}$ be the representation of G^J induced from this character. Its space consists of functions $f:\ G^J \to \mathbb{C}$ with the transformation property

$$f\left(b^J g\right) = \chi(a)|a|^{3/2}\psi^m(\kappa)f(g) \qquad (g \in G^J).$$

In the real case we also require that

$$\int_{\mathbb{R}} |f(\lambda, 0, 0)|^2 \, d\lambda < \infty. \tag{13}$$

This means that f is square integrable on $B^J\backslash G^J$; see [BeS] 3.3.

2.1 Lemma. *If F is non-archimedean, then $\pi^J_{\chi,m}$ is irreducible unless $\chi^2 = |\,|^{\pm 1}$. If $F = \mathbb{R}$, then $\pi^J_{\chi,m}$ is irreducible unless $\chi = |\,|^{k-3/2}$ or $\chi = \mathrm{sgn}\,|\,|^{k-3/2}$. In each reducible case, the length of $\pi^J_{\chi,m}$ is 2, meaning there is exactly one proper, nontrivial invariant subspace.*

See [BeS] 3.1 and Theorem 5.4.4 for a proof of these statements. Now let π^J be any irreducible representation of G^J, and consider $C - \mathrm{Ind}_{G^J}^Q(\pi^J)$ as in (10). We get an equivalent representation if we first extend π^J trivially to the group CG^J, and then induce to Q. Because of

$$Q = T_1 \ltimes (CG^J), \qquad T_1 = \{d(u, 1, 1):\ u \in \mathrm{GL}(1)\},$$

see (9), and because a non-trivial element of T_1 conjugates π^J to a non-isomorphic representation (with different index), it follows from Mackey's theory that $C-\mathrm{Ind}_{G^J}^Q(\pi^J)$ is irreducible. Thus irreducibles go to irreducibles under the $C-\mathrm{Ind}$ operation.

It follows that

$$C-\mathrm{Ind}_{G^J}^Q\left(\pi_{\chi,m}^J\right)$$

decomposes the same way as it is described in Lemma 2.1 for $\pi_{\chi,m}^J$. In particular the two representations have the same length.

2.3 An intertwining operator

For $f \in I(\chi_1,\chi_2)$ and any $m \in F^*$, we define

$$(i_m f)(g) := \int\limits_F f\left(w_1 w_2 n_\kappa g\right)\psi^{-m}(\kappa)\,d\kappa, \qquad n_\kappa = \begin{pmatrix} 1 & & & \\ & 1 & & \kappa \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{14}$$

2.2 Lemma. *Let $|\chi_i| = |\,|^{s_i}$ with $s_i \in \mathbb{R}$. If $F \neq \mathbb{C}$, then the integral (14) converges absolutely for $s_1+s_2>0$. In particular, it converges for $(\chi_1,\chi_2) = (\chi, |\,|^{1/2})$ with $|\chi| = |\,|^s$, $-\frac{1}{2} < s$. If $F=\mathbb{C}$, then the integral converges for $s_1+s_2>1$.*

Proof: First let F be a $\mathfrak{p}$-adic field. Using

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & \kappa \\ & 1 \end{pmatrix} = \begin{pmatrix} -\kappa^{-1} & 1 \\ & -\kappa \end{pmatrix}\begin{pmatrix} 1 & \\ \kappa^{-1} & 1 \end{pmatrix},$$

one estimates for large enough compact sets $C \subset F$

$$\begin{aligned}
&\int\limits_{F\setminus C} \left| f\left(w_1 w_2 \begin{pmatrix} 1 & & & \\ & 1 & & \kappa \\ & & 1 & \\ & & & 1 \end{pmatrix} g \right) \right| d\kappa \\
&= \int\limits_{F\setminus C} \left| f\left(\left(\begin{pmatrix} -\kappa^{-1} & & 1 & \\ & 1 & & \\ & & -\kappa & \\ & & & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & & & \\ & 1 & & \\ \kappa^{-1} & & 1 & \\ & & & 1 \end{pmatrix} w_2 g \right) \right| d\kappa \\
&= \int\limits_{F\setminus C} \left| \chi_1(-\kappa^{-1})\chi_2(-\kappa^{-1})|\kappa|^{-1} f(g) \right| d\kappa \\
&= \int\limits_{F\setminus C} |\kappa|^{-(s_1+s_2+1)}\,d\kappa \,\cdot\, |f(g)|.
\end{aligned}$$

From this the assertion follows easily. Next assume that F is archimedean. First find a constant $c > 0$ such that

$$\big|f\,(rw_2g)\,\big| < c \qquad \text{for all } r \text{ in a maximal compact subgroup } K.$$

Then, using the Iwasawa decomposition, we have, with certain $r_\kappa \in K$,

$$\begin{aligned}
&\int\limits_F \left|f\,(w_1w_2n_\kappa g)\right| d\kappa \\
&\quad = \int\limits_{F^*} \left| f\left(\begin{pmatrix} 1/\sqrt{1+|\kappa|^2} & & \bar{\kappa}/\sqrt{1+|\kappa|^2} & \\ & 1 & & \\ & & \sqrt{1+|\kappa|^2} & \\ & & & 1 \end{pmatrix} r_\kappa w_2 g\right)\right| d\kappa \\
&\quad = \int\limits_{F^*} \left| f\left(\begin{pmatrix} 1/\sqrt{1+|\kappa|^2} & & & \\ & 1 & & \\ & & \sqrt{1+|\kappa|^2} & \\ & & & 1 \end{pmatrix} r_\kappa w_2 g\right)\right| d\kappa \\
&\quad = \int\limits_{F^*} \left|\chi_1\big(\sqrt{1+|\kappa|^2}\big)^{-1}\chi_2\big(\sqrt{1+|\kappa|^2}\big)^{-1}\sqrt{1+|\kappa|^2}^{\,-1} f\big(r_\kappa w_2 g\big)\right| d\kappa \\
&\quad < c\int\limits_F \big(1+|\kappa|^2\big)^{-(s_1+s_2+1)/2}\, d\kappa.
\end{aligned}$$

If $F = \mathbb{R}$, then this last integral converges for $s_1 + s_2 > 0$. If $F = \mathbb{C}$, then it converges for $s_1 + s_2 > 1$.■

From now on we consider the case

$$I(\chi_1, \chi_2) = I(\chi, |\,|^{1/2}),$$

and assume that the integral (14) converges. The following important formula is straightforward to check from the definitions.

2.3 Lemma. *For each $f \in I(\chi, |\,|^{1/2})$, we have*

$$(i_m f)\big(d(u,a,a')n(x,0,\mu,\kappa)g\big) = \chi(aa'^{-1})|a^3a'u^{-2}|^{1/2}\psi^{m'}(\kappa)(i_{m'}f)(g), \tag{15}$$

where $m' = ma'^2u^{-1}$, for every $x, \mu, \kappa \in F,\ u, a, a' \in F^*$.

2.4 Lemma. *The restriction of $i_m f$ to Q is an element of*

$$C - \mathrm{Ind}_{G^J}^{Q}(\pi^J_{\chi,m}),$$

provided $|\chi| = |\,|^s$ with $s > -\frac{1}{4}$.

Proof: First of all the hypothesis ensures the convergence by Lemma 2.2. The Jacobi representation $\pi^J_{\chi,m}$ is itself induced from the character

$$G^J \ni b^J \longmapsto \chi(a)\psi^m(\kappa)$$

of the "Borel subgroup" of the Jacobi group, see (12). By induction in stages, the space of the representation $C-\mathrm{Ind}_{G^J}^Q(\pi^J_{\chi,m})$ may be realized as functions $F:\, Q\to\mathbb{C}$ with the property

$$F\big(d(1,a,1)n(x,0,\mu,\kappa)cq\big)=\chi(a)|a|^{3/2}\psi^m(\kappa)F(q)$$

for each $a\in F^*$, $x,\mu,\kappa\in F$, $c\in C$, $q\in Q$. But $i_mf|_Q$ has this property, see (15). What remains to be checked is the integrability condition (13) in the real case, i.e.,

$$\int\limits_{\mathbb{R}}\big|(i_mf)(\lambda,0,0)\big|^2\,d\lambda<\infty.$$

The estimates are similar to the ones in the convergence lemma 2.2. We have

$$\begin{aligned}(i_mf)(\lambda,0,0)&=\int\limits_{\mathbb{R}} f\big(w_1w_2(\lambda,0,\kappa)\big)e^{-2\pi i m\kappa}\,d\kappa\\ &=\int\limits_{\mathbb{R}} f\left(\begin{pmatrix}1&&&\\&1&&\\-\kappa&-\lambda&1&\\-\lambda&&&1\end{pmatrix}w_1w_2\right)e^{-2\pi i m\kappa}\,d\kappa.\end{aligned}\tag{16}$$

Let $g(\kappa,\lambda)$ be this matrix. We write down an Iwasawa decomposition

$$g(\kappa,\lambda)=g'(\kappa,\lambda)c(\kappa,\lambda)\qquad\text{with } c(\kappa,\lambda)\in K\tag{17}$$

($K\simeq\mathrm{U}(2)$ the maximal compact subgroup of $\mathrm{Sp}(4,\mathbb{R})$), and with $g'(\kappa,\lambda)$ of the form

$$g'(\kappa,\lambda)=n\big(x(\kappa,\lambda),\lambda'(\kappa,\lambda),\mu'(\kappa,\lambda),\kappa'(\kappa,\lambda)\big)d\big(1,a(\kappa,\lambda),a'(\kappa,\lambda)\big).$$

By acting with both sides of (17) on the element $I=\begin{pmatrix}i&\\&i\end{pmatrix}$ of the Siegel upper half space, one determines the parameters in $g'(\kappa,\lambda)$. We shall only need

$$a(\kappa,\lambda)^2=\frac{1+\lambda^2}{(1+\lambda^2)^2+\kappa^2},\qquad a'(\kappa,\lambda)^2=\frac{1}{1+\lambda^2}.\tag{18}$$

Plugging this into (16), we get, with a suitable constant c,

$$\begin{aligned}\big|(i_mf)(\lambda,0,0)\big|&\le c\int\limits_{\mathbb{R}}\big|a(\kappa)a'(\kappa)\big)|^{s+3/2}\,d\kappa\\ &=c\int\limits_{\mathbb{R}}\Big(\frac{1}{(1+\lambda^2)^2+\kappa^2}\Big)^{(s+3/2)/2}\,d\kappa\end{aligned}$$

$$= c\left(\frac{1}{1+\lambda^2}\right)^{s+1/2}\int\limits_{\mathbb{R}}\left(\frac{1}{1+\kappa^2}\right)^{(s+3/2)/2} d\kappa.$$

The integral on the right is convergent and independent of λ. As a function of λ, the whole expression is square integrable by our hypothesis. ∎

2.5 Lemma. *For each $f \in I(\chi, |\,|^{1/2})$, we have*

$$f(w_1 w_2 g) = |m| \int\limits_{F^*} (i_m f)\,(d(u,1,1)g)\, d^*u \qquad \textit{for all } g \in G$$

($d^*u = du/|u|$ is the multiplicative Haar measure).

Proof: For fixed $g \in G$, consider the function

$$F(\kappa) := f\,(w_1 w_2 n_\kappa g)\,.$$

Then $\hat{F}(m) = (i_m f)(g)$, where $\hat{F}$ denotes Fourier transformation. Fourier inversion thus yields

$$F(\kappa) = \int\limits_F (i_m f)(g)\psi^m(\kappa)\, dm,$$

which for $\kappa = 0$ becomes

$$f(w_1 w_2 g) = \int\limits_F (i_m f)(g)\, dm.$$

Now from (15) it follows that

$$(i_m f)\,(d(u,1,1)g) = |u|^{-1}(i_{mu^{-1}} f)(g) \qquad \text{for all } m, u \in F^*. \tag{19}$$

Using this, one can write

$$\begin{aligned} f(w_1 w_2 g) &= \int\limits_{F^*} (i_u f)(g)|u|\,\frac{du}{|u|} = \int\limits_{F^*} (i_{mu^{-1}} f)(g)|mu^{-1}|\, d^*u \\ &= |m| \int\limits_{F^*} (i_m f)\,(d(u,1,1)g)\ d^*u \end{aligned}$$

∎

Now assume that $f \in I_\chi$, the space of the special representation σ_χ; see (11). Suppose that $i_m f\big|_Q = 0$. Then from Lemma 2.5 we see that f vanishes on $w_1 w_2 Q$. By the transformation properties of f, it vanishes even on $P w_1 w_2 Q$. But this double coset contains $B w_2 w_1 w_2 w_1 B$, which is the big cell and therefore dense in G. This forces f to be zero. We have proved the following.

2.6 Proposition. *The restriction of the Q-intertwining map $f \mapsto i_m f\big|_Q$ to the space I_χ of the special representation σ_χ is injective.*

2.4 Local Maaß lifts

Recall that $\pi^J_{\chi,m}$ is usually irreducible, and in any case contains a unique non-zero irreducible, invariant subspace (Lemma 2.1). Our results thus far yield the following theorem.

2.7 Theorem. *If $\pi^J_{\chi,m}$ is irreducible, then $C-\mathrm{Ind}^Q_{G^J}(\pi^J_{\chi,m})$ extends to an irreducible G-representation, which is isomorphic to the special representation σ_χ. If $\pi^J_{\chi,m}$ is not irreducible, then the same is true provided σ_χ is not G-irreducible.*

Proof: The map $f \mapsto i_m f\big|_Q$ defines a Q-intertwining operator

$$\sigma_\chi \longrightarrow C - \mathrm{Ind}^Q_{G^J}(\pi^J_{\chi,m}).$$

We know it is injective by Proposition 2.6. If $\pi^J_{\chi,m}$ is irreducible, then so is $C - \mathrm{Ind}^Q_{G^J}(\pi^J_{\chi,m})$, and we get an isomorphism. If neither $\pi^J_{\chi,m}$ nor σ_χ is irreducible, then both representations must have length 2 by Lemma 2.1, and we also get an isomorphism. ■

Definition: If π^J is an irreducible representation of G^J, and if $C - \mathrm{Ind}^Q_{G^J}(\pi^J)$ extends to a representation σ of G, then we say that σ is a Maaß lift of π^J, and we also call σ a special representation of PGSp(4).

The theorem says that every principal series representation of G^J has a Maaß lift.

Conjecture: Every irreducible, admissible representation of G^J has a unique Maaß lift.

This conjecture is supported by the global situation, see diagram (1). In fact, it should be possible to lift every automorphic representation π^J of $G^J(\mathbb{A})$ to an automorphic representation σ of $\mathrm{PGSp}(4,\mathbb{A})$. The constructions in [PS1] show that σ, when restricted to the $Q(\mathbb{A})$-action, should be isomorphic to $C - \mathrm{Ind}^{Q(\mathbb{A})}_{G^J(\mathbb{A})}(\sigma)$.

Obviously a Maaß lift in the above sense of an irreducible G^J-representation is not only G-irreducible, but even Q-irreducible. In fact, from our results we see the following:

2.8 Corollary. *If $\pi^J_{\chi,m}$ is irreducible, then σ_χ is irreducible when restricted to Q. If σ_χ is not G-irreducible, then it has length 2, and the unique proper, nontrivial invariant subspace is Q-irreducible.*

3 Maaß lifts for real representations

The conjecture above is also supported by the following result which is most natural in view of the fact that the classical Maaß lift is a map

$$J_{k,1} \longrightarrow M_k(\Gamma_2),$$

and that holomorphic forms $f \in J_{k,m}$ and $F \in M_k(\Gamma_2)$ are "lowest weight vectors" for holomorphic discrete series representations $\pi^{J+}_{m,k}$ of $G^J(\mathbb{R})$ (see [BeS] 4.1) resp. σ^+_k of $G(\mathbb{R})$ (see [Na] or [AS]).

3.1 Theorem. *The holomorphic discrete series representation σ_k^+ of $\mathrm{GSp}(4,\mathbb{R})$ with Blattner parameter $\lambda = (k,k)$ is the Maaß lift of the representation $\pi_{m,k}^{J+}$ of $G^J(\mathbb{R})$.*

This theorem will be proved in section 3.4. As the representations of the groups in the real case $F = \mathbb{R}$ may be characterized (at least up to infinitesimal equivalence) by the representations of the Lie algebras, we have to introduce some more notation.

3.1 The Lie algebras

Following [BeS], p. 12, and our preliminary text on the Maaß lift [Be2], we write for $\mathfrak{g}^J = \mathrm{Lie}(G^J(\mathbb{R}))$

$$\mathfrak{g}^J_{\mathbb{C}} = \mathfrak{g}^J \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}(2) + \mathfrak{h}_{\mathbb{C}}, \qquad \mathfrak{sl}(2) = \langle Z, X_\pm \rangle,\ \mathfrak{h}_{\mathbb{C}} = \langle Y_\pm, Z_0 \rangle$$

with the relations $[Z_0, \mathfrak{g}^J_{\mathbb{C}}] = 0$ and

$$\begin{array}{lll} [Z, X_\pm] = \pm 2X_\pm, & [X_\pm, Y_\mp] = -Y_\pm, & [X_+, X_-] = Z, \\ [Z, Y_\pm] = \pm Y_\pm, & [X_\pm, Y_\pm] = 0, & [Y_+, Y_-] = Z_0. \end{array}$$

Moreover, we will use the abelian subalgebras

$$\mathfrak{k}^J_{\mathbb{C}} = \langle Z, Z_0 \rangle \qquad \text{and} \qquad \mathfrak{p}^J_\pm = \langle X_\pm, Y_\pm \rangle$$

For $\mathfrak{g}_{\mathbb{C}} := \mathfrak{sp}(4) \times_{\mathbb{R}} \mathbb{C}$ we choose a basis

$$\mathfrak{g}_{\mathbb{C}} = \langle Z, Z', N_\pm, P_{0\pm}, P_{1\pm}, X_\pm \rangle.$$

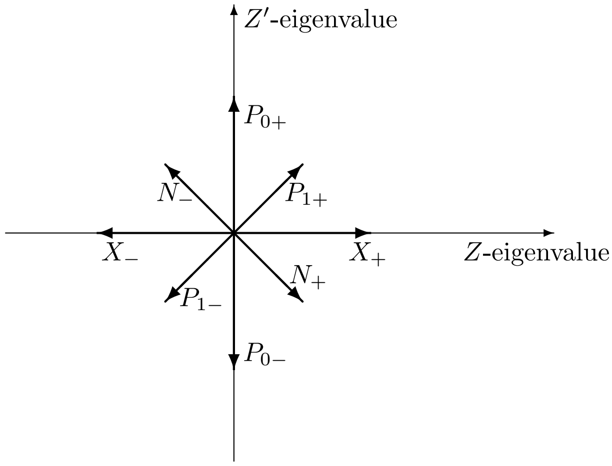
Here $\mathfrak{k}_{\mathbb{C}} = \langle Z, Z', N_\pm \rangle$ is the complexification of $k = \mathrm{Lie}(K)$, the subalgebra $\langle Z, Z' \rangle$ is a compact Cartan, and

$$\mathfrak{p}_\pm = \langle P_{0\pm}, P_{1\pm}, X_\pm \rangle$$

are maximal abelian subalgebras. Among others we have the relations

$$\begin{array}{ll} [Z, P_{0\pm}] = 0, & [Z', P_{0\pm}] = \pm 2P_{0\pm}, \\ [Z, P_{1\pm}] = \pm P_{1\pm}, & [Z', P_{1\pm}] = \pm P_{1\pm}, \\ [Z, X_\pm] = \pm 2X_\pm, & [Z', X_\pm] = 0, \\ [Z, N_\pm] = \pm N_\pm, & [Z', N_\pm] = \mp N_\pm. \end{array}$$

symbolized by the root diagram



$\mathfrak{g}_{\mathbb{C}}^J$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with

$$Y_\pm = \frac{1}{2}(P_{1\pm} - N_\pm) \qquad \text{and} \qquad Z_0 = \frac{1}{2}(P_{0-} - P_{0+} + Z').$$

Another subalgebra is given by

$$\tilde{\mathfrak{q}}_{1,\mathbb{C}} = \langle \mathfrak{g}_{\mathbb{C}}^J, U \rangle \qquad \text{with} \qquad U = P_{0+} + P_{0-};$$

the relations are

$$[U, Z] = [U, X_\pm] = 0, \qquad [U, Z_0] = 2Z_0, \qquad [U, Y_\pm] = Y_\pm.$$

3.2 Infinitesimal representations

Now we construct representations of the (complex) Lie algebras by the usual procedure.

The representations $\hat{\sigma}_k^+$ of $\mathfrak{g}_{\mathbb{C}}$

We take the one-dimensional complex representation $\hat{\rho}_k$ of $\mathfrak{k}_{\mathbb{C}} = \langle Z, Z', N_+, N_- \rangle$ given by

$$Z1 = Z'1 = k, \quad N_\pm 1 = 0,$$

extend $\hat{\rho}_k$ trivially to $\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_-$, and induce from here to $\mathfrak{g}_{\mathbb{C}}$ to get a lowest weight representation $\hat{\sigma}_k^+$ with space

$$S_k = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_-)} \mathbb{C} = \sum_{i,j,l \in \mathbb{N}_0} \mathbb{C}\, P_{0+}^i P_{1+}^j X_+^l w_0,$$

where w_0 denotes a lowest weight or "vacuum" vector, here fixed by

$$X_- w_0 = P_{1-} w_0 = P_{0-} w_0 = N_\pm w_0 = 0, \qquad Z w_0 = Z' w_0 = k w_0. \tag{20}$$

This is the infinitesimal representation belonging to the holomorphic discrete series representation σ_k^+ with Harish-Chandra parameter $\Lambda = (k-1, k-2)$ resp. Blattner parameter $\lambda = (k, k)$ (corresponding to the minimal K-type (k, k)).

The representation $\hat{\pi}^{J+}_{m,k}$ of $\mathfrak{g}^J_{\mathbb{C}}$

As in [BeS] 3.1 we take the one-dimensional complex representation $\hat{\rho}^J_{m,k}$ of $k^J_{\mathbb{C}} = \langle Z, Z_0 \rangle$ given by

$$Z1 = k, \quad Z_0 1 = 2\pi m =: \mu,$$

extend it trivially to $\mathfrak{k}^J_{\mathbb{C}} + \mathfrak{p}^J_{-}$, and induce form here to $\mathfrak{g}^J_{\mathbb{C}}$ to get a "lowest weight" representation $\hat{\pi}^+_{m,k}$ with space

$$V^+_{m,k} = U(\mathfrak{g}^J_{\mathbb{C}}) \otimes_{U(\mathfrak{k}^J_{\mathbb{C}} + \mathfrak{p}^J_{-})} \mathbb{C} = \sum_{j,l \in \mathbb{N}_0} \mathbb{C}\, X^l_+ Y^j_+ v_0,$$

where v_0 denotes a vacuum vector fixed by

$$X_- v_0 = Y_- v_0 = 0, \qquad Z v_0 = k v_0, \qquad Z_0 v_0 = \mu v_0. \tag{21}$$

Though not all of it will be needed for the construction of the Maaß lift, we will assemble some material perhaps of independent interest.

Remark 1: The fundamental fact (see [BeS] 2.8) that each representation π^J of G^J with index $m \neq 0$ is a tensor product of a projective standard representation π^m_{SW} of G^J and a projective representation π_0 of SL(2) reflects in the decomposition

$$V^+_{m,k} = V^{1/2}_m \otimes W_{k-1/2} = \langle v_j \otimes w_l \rangle_{j \in \mathbb{N},\, l \in 2\mathbb{N}_0},$$

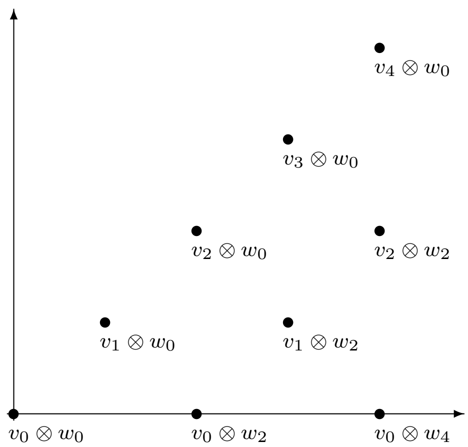
where $V^{1/2}_m$ belongs to $\hat{\pi_{SW}}^m$ (of weight 1/2) and $W_{k-1/2}$ is the space of a representation $\hat{\pi}_{k-1/2}$ of $\mathfrak{sl}(2)_{\mathbb{C}}$. Then, for the subspaces of Z-weights λ

$$V^{(\lambda)} = \left\{ v \in V^+_{m,k} :\ Zv = (k+\lambda)v \right\}$$

we have

$$\dim V^{(\lambda)} = \sharp\{(j,l) \in \mathbb{N}_0^2 :\ j + 2l = \lambda\},$$

i.e. $\dim V^{(0)} = \dim V^{(1)} = 1$, $\dim V^{(2)} = \dim V^{(3)} = 2$, $\ldots$. One might visualize the space $V^+_{m,k}$ as follows:



(elements along vertical lines have the same Z-weight).

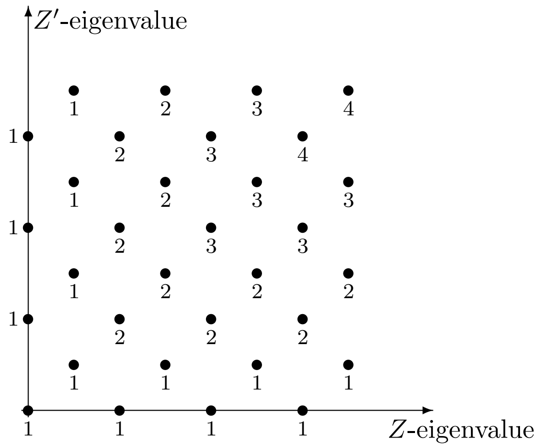
Remark 2: For the representation $\hat{\sigma}_k^+$ of $\mathfrak{g}_{\mathbb{C}}$ we have a decomposition

$$S_k = \bigoplus_{\lambda,\lambda'=0}^{\infty} S_{\lambda,\lambda'}, \qquad S_{\lambda,\lambda'} = \big\{ w \in S_k : \ Zw = (k+\lambda)w, \ Z'w = (k+\lambda')w \big\}.$$

The dimensions of the weight spaces are obviously given by

$$\dim S_{\lambda,\lambda'} = \sharp\big\{(i,j,l) \in \mathbb{N}_0^3 : \ 2i + j = \lambda, \ j + 2l = \lambda'\big\}.$$

(By the way, this may be recognized as a very special instant of the wonderful Blattner formula.) So we get the following picture of the space S_k (the numbers denote the above dimensions).



We remark that this picture may be obtained if infinitely many of the triangles from the previous picture are stacked together, translating each one by two units in the Z'-direction. Even after some efforts to interprete this, we still lack a nice algebraic explanation.

Remark 3: For $k = 1/2$ there is an irreducible representation $\hat{\sigma}_{1/2}^+$ with $\dim S_{\lambda,\lambda'} = 1$ for all (λ, λ'). This is the (infinitesimal version of the) *Weil representation* of $\mathfrak{sp}$ and may be characterized in this context as the unique lowest weight representation fulfilling the additional condition

$$4X_+ P_{0+} = P_{1+}^2.$$

This corresponds to the "heat equation" condition

$$2\mu X_+ = Y_+^2 \qquad (\mu = 2\pi m)$$

characterizing the Schrödinger–Weil representation of $\mathfrak{g}_{\mathbb{C}}^J$ (see [Be1]) among the lowest weight representations.

Remark 4: There is also a nice interpretation of $\hat{\sigma}_{1/2}^+$ by "doubling" the $\mathfrak{g}_{\mathbb{C}}^J$–representation $\hat{\pi}_{SW}^m$ acting on $V_m^{1/2} = \langle v_j \rangle_{j \in \mathbb{N}_0}$ in the following way. We take

$$V_m^{1/2} \otimes {V'}_m^{1/2} = \langle v_i \otimes v'_j \rangle_{i,j \in \mathbb{N}_0}$$

and define the operation of $\mathfrak{g}_\mathbb{C}$ by

$$\begin{aligned}
Z(v_i \otimes v'_j) &= (1/2+i)(v_i \otimes v'_j)\\
Z'(v_i \otimes v'_j) &= (1/2+j)(v_i \otimes v'_j)\\
X_+(v_i \otimes v'_j) &= X_+ v_i \otimes v'_j = (-1/(2\mu))v_{i+2} \otimes v'_j\\
P_{0+}(v_i \otimes v'_j) &= (-1/(2\mu))v_i \otimes v'_{j+2}\\
P_{1+}(v_i \otimes v'_j) &= (1/\mu)\, v_{i+1} \otimes v'_{j+1}\\
N_+(v_i \otimes v'_j) &= -j\, v_{i+1} \otimes v'_{j-1}\\
N_-(v_i \otimes v'_j) &= i\, v_{i-1} \otimes v_{j+1}\\
P_{1-}(v_i \otimes v'_j) &= -\mu ij\, v_{i-1} \otimes v_{j-1}
\end{aligned}$$

etc. Then

$$S^+_{1/2} = \langle v_i \otimes v'_j \rangle_{i+j \text{ even}}$$

is a copy of the lowest weight representation $\hat{\sigma}^+_{1/2}$.

Remark 5: An easy but somewhat lengthy calculation, similar to the one giving the proof of the previous remark, allows to verify the following description of the representation $\hat{\sigma}^+_k$.

3.2 Proposition. *$\hat{\pi}_k$ is a lowest weight representation, given on the $\mathbb{C}$-vector space*

$$S = \sum_{\lambda,\mu \ge 0} S_{\lambda\mu}$$

by the following formulae

$$\begin{aligned}
ZS_{\lambda\mu} &= (k+\lambda)S_{\lambda\mu}, & Z'S_{\lambda\mu} &= (k+\mu)S_{\lambda\mu}\\
S_{\lambda\mu} &= \sum_{\substack{2i+l=\mu\\ 2j+l=\lambda}} \mathbb{C}v_{i,j,l}, & v_{i,j,l} &:= P^i_{0+}X^j_+P^l_{1+}v,
\end{aligned}$$

with the lowest weight vector v fixed by

$$\begin{aligned}
Zv = Z'v &= kv,\\
N_+v = N_-v = P_{0-}v &= P_{1-}v = X_-v = 0
\end{aligned}$$

and the relations

$$\begin{aligned}
N_+v_{i,j,l} &= 2lv_{i,j+1,l-1} + iv_{i-1,j,l+1}\\
N_-v_{i,j,l} &= -2lv_{i+1,j,l-1} - jv_{i,j-1,l+1}\\
P_{0-}v_{i,j,l} &= -i(k+l+i-1)v_{i-1,j,l} - l(l-1)v_{i,j+1,l-2}\\
P_{1-}v_{i,j,l} &= -l(2k+2i+2j+l-1)v_{i,j,l-1} - ij\ v_{i-1,j-1,l+1}\\
X_-v_{i,j,l} &= -j(k+l+j-1)v_{i,j-1,l} - l(l-1)v_{i,j,l-2}.
\end{aligned}$$

3.3 Lowest weight vectors

We now come back to the problem of constructing the Maaß lift of the representation $\pi^{J+}_{m,k}$ of $G^J(\mathbb{R})$. To do so, we shall first exhibit distinguished (lowest weight) vectors in various induced representations.

Lowest weight vectors in Jacobi representations

We take over from [BeS] 3.3 that $\pi^{J+}_{m,k}$ can be realized as a subrepresentation of

$$\pi^J_{m,s} = \mathrm{Ind}^{G^J}_{B^J}(\chi\psi^m) \qquad \text{for } \chi = |\,|^{k-3/2}.$$

The lowest weight vector $v_0 = \phi^0$ is then given by

$$\phi^0(g^J) = y^{k/2} e(m(\kappa + pz))e^{ik\vartheta}$$

where g^J is fixed by the coordinates from [BeS]

$$g^J = (p, q, \kappa)n(x, 0, 0, 0)d(1, y^{1/2}, 1)r_1(\vartheta).$$

But if, as used here mostly,

$$g^J = n(x, \lambda, \mu, \kappa)d(1, a, 1)r_1(\vartheta), \qquad a \in \mathbb{R}^*,\ x, \lambda, \mu, \kappa, \vartheta \in \mathbb{R},$$

then

$$\varphi^0(g^J) = |a|^k e(m(\kappa + i(\lambda a)^2)e^{ik\vartheta}. \tag{22}$$

For (22) to be well-defined, we assume from now on that <u>k is even</u>.

Lowest weight vectors in special representations

As is well known (see for instance [Na]), we have, using the coordinates

$$g = n(x, \lambda, \mu, \kappa)d(u, a, a')r, \qquad r \in K,$$

a lowest weight vector

$$\Psi^0(g) = |aa'/u|^k \xi(r), \tag{23}$$

which fixes the holomorphic discrete series representation σ^+_k as a subrepresentation of $\sigma(\chi, |\,|^{1/2})$ for $\chi = |\,|^{k-3/2}$. Here ξ denotes the one-dimensional (k, k)-representation of $K \simeq U(2)$ (the identity component of the maximal compact subgroup of G), i.e.,

$$\xi(r) = e^{ik(\vartheta+\vartheta')},$$

where r is meant as the product of the elements

$$r_1(\vartheta) = \begin{pmatrix} \cos(\vartheta) & & \sin(\vartheta) & \\ & 1 & & \\ -\sin(\vartheta) & & \cos(\vartheta) & \\ & & & 1 \end{pmatrix}, \qquad r_2(\vartheta') = \begin{pmatrix} 1 & & & \\ & \cos(\vartheta') & & \sin(\vartheta') \\ & & 1 & \\ & -\sin(\vartheta') & & \cos(\vartheta') \end{pmatrix},$$

$$r_3(\varphi) = \begin{pmatrix} r(\varphi) & \\ & r(\varphi) \end{pmatrix}, \qquad r_4(\varphi') = \begin{pmatrix} \cos(\varphi') & & & \sin(\varphi') \\ & \cos(\varphi') & \sin(\varphi') & \\ & -\sin(\varphi') & \cos(\varphi') & \\ -\sin(\varphi') & & & \cos(\varphi') \end{pmatrix}.$$

We also have

$$\xi(r) = j(r, I)^{-k}, \qquad r \in K,\ I = \begin{pmatrix} i & \\ & i \end{pmatrix},$$

where $j(g, Z) = \det(CZ + D)$ is the usual automorphic factor, for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4)$ and Z an element of the Siegel upper half plane. It is immediate that

$$\Psi^0\left(\begin{pmatrix} A & * \\ & u\,{}^tA^{-1} \end{pmatrix} g\right) = |\det(A)u^{-1})|^k \Psi^0(g),$$

and therefore Ψ^0 is an element of the special representation σ_χ, see (11). It can be verified directly that Ψ^0 is indeed annihilated by the operators $P_{0-}, P_{1-}, X_-, N_\pm$. Thus, in the infinitesimal picture, Ψ^0 identifies with the vacuum vector w_0 from (20).

Special vectors on Q

By the recipe given in 2.4, we induce from the representation $\pi^J_{m,s}$ of G^J to a representation $\tau_{m,s}$ of Q with a space $\mathcal{B}^Q_{\chi,m} = \mathcal{B}^Q_{s,m}$. Then we search in this space for a distinguished "vacuum vector" $\tilde{v}_0$ spanning a representation $\tau^+_{m,k}$ equivalent to σ^+_k restricted to Q, via the intertwining operator i_m. It is not too hard to guess that $\tilde{v}_0 = \Phi^0_m$ does the trick, if we put

$$\text{for } m > 0: \qquad \Phi^0_m(q) = \begin{cases} |aa'/u|^k e(m\tau')e^{ik\vartheta} & \text{if } u > 0, \\ 0 & \text{if } u < 0. \end{cases} \tag{24}$$

$$\text{for } m < 0: \qquad \Phi^0_m(q) = \begin{cases} 0 & \text{if } u > 0, \\ |aa'/u|^k e(m\tau')e^{ik\vartheta} & \text{if } u < 0. \end{cases} \tag{25}$$

This defines a smooth function, since u is the multiplier, and its sign determines the connected component of G; we used the coordinates

$$q = n(x, \lambda, \mu, \kappa)d(u, a, a')r_1(\vartheta), \qquad \tau' = \kappa + i(\lambda^2 a^2 + a'^2)/u.$$

Note that

$$\Phi^0_m(d(-1, 1, 1)q) = \Phi^0_{-m}(q) \tag{26}$$

for all m and all q. Note also that Φ^0_m is invariant under the center. It is obviously an element of $C - \mathrm{Ind}^Q_{G^J}(\pi^{J+}_{m,k})$. In the following we shall give several explanations for this choice of Φ^0_m.

3.4 The Maaß lift of $\pi^{J+}_{m,k}$

Again let $\chi = |\,|^{k-3/2}$. Recall from Proposition 2.6 that we have an injective Q-intertwining operator

$$i_m:\ \sigma_\chi \longrightarrow C-\mathrm{Ind}_{G^J}^Q(\pi^J_{\chi,m}). \tag{27}$$

By Lemma 2.1 and the remarks following it, the representation on the right has Q-length 2. It follows that σ_χ has G-length no more than 2. We already know a subrepresentation of σ_χ, namely the (anti-) holomorphic discrete series representation σ_k^+ spanned by the lowest weight vector (23). But we can also write down the vector

$$\Psi^0(g) = |aa'/u|^k e^{i(k-2)(\vartheta+\vartheta')},$$

which is an element of I_χ, but not contained in σ_k^+, since the latter representation does not contain a vector of weight $(k-2, k-2)$. Hence we see that the G-length of σ_χ is in fact 2. It now follows that (27) is an isomorphism, and that the subspace σ_k^+ really maps Q-isomorphically onto the subspace $C-\mathrm{Ind}_{G^J}^Q(\pi^{J+}_{k,m})$. This proves Theorem 3.1.

We shall now show that Φ^0_m as in (24) resp. (25) is (up to scalars) the image of Ψ^0 under the intertwining operator i_m.

3.3 Lemma. *For each positive integer k the integral*

$$I_{k,m} = \int_{\mathbb{R}} (1-i\kappa)^{-k} e^{-2\pi i m\kappa}\, d\kappa, \qquad m \in \mathbb{R}^*,$$

converges, and has the value

$$I_{k,m} = 2\pi\frac{(2\pi m)^{k-1}}{(k-1)!}e^{-2\pi m} \qquad \text{for } m > 0,$$

and $I_{k,m} = 0$ for $m < 0$.

Proof: Partial integration yields the recursion formula

$$I_{k,m} = \frac{2\pi m}{k-1} I_{k-1,m}.$$

Thus we are reduced to the case $k = 1$. In this case one decomposes into real and imaginary parts, and obtains

$$I_{1,m} = \int \frac{1}{1+\kappa^2}\cos(2\pi m\kappa)\, d\kappa + \int \frac{\kappa}{1+\kappa^2}\sin(2\pi m\kappa)\, d\kappa.$$

The values of these integrals can be looked up in tables. The first one is $\pi e^{-2\pi|m|}$, the second one is $\mathrm{sgn}(m)\pi e^{-2\pi|m|}$. ■

3.4 Proposition. *Let Ψ^0 be as in (23) and Φ^0_m as in (24) resp. (25). If i_m denotes the intertwining operator (14), then*

$$i_m\Psi^0\Big|_Q = 2\pi\frac{(2\pi m)^{k-1}}{(k-1)!}\Phi^0_m. \tag{28}$$

Proof: It follows from (15) that

$$(i_m\Psi^0)(d(-1,1,1)q) = (i_{-m}\Psi^0)(q).$$

In view of (26) it is therefore enough to check equality of both sides of (28) on elements of positive multiplier. By the transformation properties of both sides (see (15) and (24), (25)) it is further enough to check that both sides are equal when evaluated at elements $(\lambda,0,0)$. In other words, we have to compute

$$\begin{aligned}(i_m\Psi^0)(\lambda,0,0) &= \int\limits_{\mathbb{R}} \Psi^0\big(w_1w_2(\lambda,0,\kappa)\big)e^{-2\pi im\kappa}\,d\kappa \\ &= \int\limits_{\mathbb{R}} \Psi^0\begin{pmatrix}1&&&\\&1&&\\-\kappa&-\lambda&1&\\-\lambda&&&1\end{pmatrix}e^{-2\pi im\kappa}\,d\kappa. \end{aligned}\tag{29}$$

We proceed as in the proof of Lemma 2.4. If $g(\kappa)$ denotes the matrix, then we write down an Iwasawa decomposition

$$g(\kappa) = g'(\kappa)c(\kappa) \qquad \text{with } c(\kappa)\in K, \tag{30}$$

and with $g'(\kappa)$ of the form

$$g'(\kappa) = n\big(x(\kappa),\lambda'(\kappa),\mu'(\kappa),\kappa'(\kappa)\big)d\big(1,a(\kappa),a'(\kappa)\big).$$

The parameters are as before, in particular

$$a(\kappa)^2 = \frac{1+\lambda^2}{(1+\lambda^2)^2+\kappa^2}, \qquad a'(\kappa)^2 = \frac{1}{1+\lambda^2}. \tag{31}$$

Furthermore, by (30) we get $j(g(\kappa),I) = j(g'(\kappa),I)j(c(\kappa),I)$, and thus

$$\xi(c(\kappa)) = j(c(\kappa),I)^{-k} = \left(\frac{j(g',I)}{j(g,I)}\right)^k = \left(\frac{1+\lambda^2-i\kappa}{\sqrt{(1+\lambda^2)^2+\kappa^2}}\right)^{-k}$$

(to compute $j(g',I)$ one uses (31)). Putting this into (29), we get

$$\begin{aligned}(i_m\Psi^0)(\lambda,0,0) &= \int\limits_{\mathbb{R}} \Psi^0\big(g'(\kappa)c(\kappa)\big)e^{-2\pi im\kappa}\,d\kappa \\ &= \int\limits_{\mathbb{R}} (a(\kappa)a'(\kappa))^k\xi(c(\kappa))e^{-2\pi im\kappa}\,d\kappa \\ &= \int\limits_{\mathbb{R}} (1+\lambda^2-i\kappa)^{-k}e^{-2\pi im\kappa}\,d\kappa \\ &= (1+\lambda^2)^{1-k}I_{k,m(1+\lambda^2)}\end{aligned}$$

with $I_{k,m(1+\lambda^2)}$ as in Lemma 3.3. Plugging in the value given in this lemma, the assertion follows. ■

Remark: Assume we would be working on Sp(4) instead of PGSp(4). Then we would have only one connected component, and no negative multipliers. In our definitions (24) and (25) we would just forget about $u < 0$. Proposition 3.4 would imply that $i_m\Psi^0$, the image of the holomorphic lowest weight vector, is non-vanishing if and only if m is positive. There is also an anti-holomorphic highest weight vector (given by $\sigma_k^+(d(-1,0,0))\Psi^0$), and its image would be non-zero if and only if $m < 0$. This is consistent with Theorems 6.3 – 6.6 in [Hi]; these imply that the holomorphic discrete series representations appear as lifts precisely for $m > 0$, and the antiholomorphic ones do so precisely for $m < 0$.

3.5 Differential operators

As usual we realize the elements X of a Lie algebra as left-invariant differential operators $\mathcal{L}_X =: X$ acting on smooth functions ϕ living on the groups by

$$(X\phi)(g) = (\mathcal{L}_X\phi)(g) := \frac{d}{ds}\phi(g\exp(sX))\Big|_{s=0}.$$

We will now work under the assumption that $\underline{m > 0}$; the other case yields symmetric results. We shall again consider the function Φ_m^0 on Q defined by (24). Obviously this function is only interesting on the connected component Q^+ of the identity, which consists of all elements with positive multiplier. We shall therefore work on this connected component only, noting that (since we are over the reals)

$$Q^+ = C \times Q_1, \qquad Q_1 = T_1^+ \ltimes G^J,$$

where $T_1^+ = \{d(1,1,a') : \ a' \in \mathbb{R}_{>0}\}$. Since all our functions are invariant under the center, we can restrict further to Q_1. The Lie algebra $\mathfrak{q}_1$ of this group is generated by $\mathfrak{g}^J$ and the element

$$U = \mathrm{diag}(0,1,0,-1) = P_{0+} + P_{0-}$$

spanning the Lie algebra of T_1^+. It is easy to deduce from the Jacobi theory in [BeS] 3.2 the realization of the elements of $\mathfrak{q}_{1,\mathbb{C}}$ as differential operators acting on smooth functions on Q_1. If Q_1 is coordinatized by

$$q_1 = n(x,\lambda,\mu,\kappa)\, d\big(1, y^{1/2}, y'^{1/2}\big)\, r_1(\vartheta) = g^J\, d\big(1,1,y'^{1/2}\big),$$

such that with $I = \begin{pmatrix} i & \\ & i \end{pmatrix}$

$$q\langle I\rangle = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}, \qquad \tau = x + iy, \quad z = \mu + i\lambda y, \quad \tau' = \kappa + i(\lambda^2 y + y'),$$

then

$$\begin{aligned}
&\mathcal{L}_U = 2y'\partial_{y'}, \qquad \mathcal{L}_{Z_0} = -iy'\partial_\kappa, \qquad \mathcal{L}_Z = -i\partial_\vartheta,\\
&\mathcal{L}_{X_\pm} = \pm\frac{i}{2}e^{\pm 2i\vartheta}\Big(2y\big(\partial_x + \lambda\partial_\mu + \lambda^2\partial_\kappa \mp i\partial_y\big) - \partial_\vartheta\Big),\\
&\mathcal{L}_{Y_\pm} = \frac{1}{2}e^{\pm i\vartheta}(y'/y)^{1/2}\Big(\partial_\lambda \pm iy\partial_\mu \pm 2iy\lambda\partial_\kappa\Big).
\end{aligned}$$

By a small calculation, one can verify

3.5 Proposition. *The function Φ^0_m on Q_1 fulfills*

$$X_-\Phi^0_m = Y_-\Phi^0_m = 0, \qquad Z\Phi^0_m = (U + 2Z_0)\Phi^0_m = k\Phi^0_m,$$

and is (up to a constant factor) uniquely determined by these equations and the additional condition

$$\Phi^0_m(n_\kappa q) = e^{2\pi i m\kappa}\Phi^0_m(q).$$

This leads to the interpretation that $v_0 = \Phi^0_m$ is the vacuum vector for a representation $\hat{\tau}^+_{m,k}$ of $\mathfrak{q}_{1,\mathbb{C}}$ given on the space

$$V^+_{m,k} = \sum_{i,j,l\in\mathbb{N}_0} \mathbb{C}U^i Y^j_+ X^l_+ v_0,$$

and $\tau^+_{m,k}$ is fixed as the associated representation of Q_1 generated by Φ^0_m in the induced representation $\mathrm{Ind}^{Q_1}_{B^J}(\chi\psi^m)$, $\chi = |\,|^{k-3/2}$.

Remark: This representation space is isomorphic to

$$\mathcal{U}(\mathfrak{q}_{1,\mathbb{C}})/\langle X_-, Y_-, Z - k, U + 2Z_0 - k\rangle,$$

where the brackets here denote the left ideal generated by the elements in the brackets. By our previous results, this $\mathfrak{q}_{1,\mathbb{C}}$-module should extend to a representation of $\mathfrak{sp}(4)_\mathbb{C}$ which is isomorphic to the special representation we called $\hat{\sigma}^+_k$. The space of the latter can be realized as

$$\begin{aligned} S_k = &\sum_{i,j,l\in\mathbb{N}_0} \mathbb{C}P^i_{0+}P^j_{1+}X^l_+ w_0 \\ &\simeq \mathcal{U}(\mathfrak{sp}(4)_\mathbb{C})/\langle X_-, P_{1-}, P_{0-}, N_\pm, Z - k, Z' - k\rangle. \end{aligned}$$

In passing from v_0 to w_0 we are adding the conditions $N_\pm w_0 = P_{0-}w_0 = 0$. Furthermore, we have

$$P_{0+} = U - P_{0-}, \qquad P_{1\pm} = 2Y_\pm + N_\pm, \qquad Z' = U + 2Z_0 - P_{0-}.$$

Thus our condition $(U + 2Z_0)\Phi^0_m = k\Phi^0_m$ in Proposition 3.5 is forced by $P_{0-}w_0 = 0$ and $Z'w_0 = kw_0$. A bit more formally this may also be understood by realizing that the choice of the vacuum vector v_0 is the one such that its annihilator is just the intersection of the annihilator of w_0 with $\mathcal{U}(\mathfrak{q}_{1,\mathbb{C}})$.

3.6 The Fourier and Fourier-Jacobi developments

There is still another explanation for the special form (24), (25) for the vacuum vector Φ^0_m mediating the lift from $\pi^{J+}_{m,k}$ to σ^+_k. We simply take a Jacobi form $f \in J_{k,m}$ and a Siegel modular form $F \in M_k(\Gamma_2)$ and lift their Fourier developments

$$f(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ 4mn-r^2\geq 0}} c(n, r)e(n\tau + rz)$$

resp.

$$F(\tau, z, \tau') = \sum_{\substack{n,r,m\in\mathbb{Z} \\ 4mn-r^2\geq 0}} c(n, r, m)e(n\tau + rz + m\tau')$$

in a familiar way (see [BeS] 4.1 resp. [Na] 10) to functions on $G^J(\mathbb{R})$ resp. $\mathrm{Sp}(4,\mathbb{R})$ given by

$$\Phi_f(g^J) = \sum_{n,r} c(n,r) y^{k/2} e(n\tau + rz + m(\kappa + i\lambda^2 y)) e^{ik\vartheta} \tag{32}$$

resp.

$$\Psi_f(g) = \sum_{n,r,m} c(n,r,m)(yy')^{k/2} e(n\tau + rz + m\tau') e^{ik(\vartheta+\vartheta')}. \tag{33}$$

F also has a Fourier-Jacobi expansion

$$F(\tau, z, \tau') = \sum_m \tilde{f}_m(\tau, z, \tau'), \qquad \tilde{f}_m(\tau, z, \tau') = f_m(\tau, z) e(m\tau'),$$

where $f_m \in J_{k,m}$ by the definition of Jacobi forms. Here the functions $\tilde{f}_m$, as well as any $\tilde{f}$ of the form

$$\tilde{f}(\tau, z, \tau') = f(\tau, z) e(m\tau'), \qquad f \in J_{k,m}$$

(cf. [Gr]) can be lifted to functions $\tilde{\Phi}_f$ on Q_1 which are $\Gamma^Q = \Gamma_2 \cap Q_1$-leftinvariant, of $\mathrm{SO}(2)$-type k from the right, and given by

$$\tilde{\Phi}_f(q) = \sum_{n,r} c(n,r)(yy')^{k/2} e(n\tau + rz + m\tau') e^{ik\vartheta}. \tag{34}$$

Apparently, the series (32), (34) and (33) are built up from functions

$$\begin{aligned}
W^{(n,r)}(g^J) &= y^{k/2} e(n\tau + rz + m(\kappa + i\lambda^2 y)) e^{ik\vartheta}, \\
\tilde{W}^{(n,r)}(q) &= (yy')^{k/2} e(n\tau + rz + m\tau') e^{ik\vartheta}, \\
W^{(n,r,m)}(g) &= (yy')^{k/2} e(n\tau + rz + m\tau') e^{ik(\vartheta+\vartheta')}
\end{aligned}$$

All this here has a "Whittaker-type" background. One may ask to realize models of $\pi^{J+}_{m,k}$, $\tau^+_k = \sigma^+_k|_Q$ and σ^+_k on spaces of functions W on G^J, Q resp. G of the transformation type

$$W(ng_0) = \psi^{n,r,m}(n) W(g_0) \qquad \text{for all } g_0 = g^J, q \text{ resp. } g,$$

and $n \in N_1 := \{n(x, 0, \mu, \kappa)\}$, where $\psi^{n,r,m}$ is the character of N_1 given by

$$\psi^{n,r,m}(n(x, 0, \mu, \kappa)) = e(nx + r\mu + m\kappa).$$

Equivalently, one looks for realizations of $\pi^{J+}_{m,k}$, τ^+_k and σ^+_k as subrepresentations of

$$\mathrm{Ind}_{N_1}^{G^J}\big(\psi^{(n,r,m)}\big), \qquad \mathrm{Ind}_{N_1}^{Q}\big(\psi^{(n,r,m)}\big) \qquad \text{resp.} \qquad \mathrm{Ind}_{N_1}^{G}\big(\psi^{(n,r,m)}\big).$$

Using differential operators as before, one can easily verify that the functions $W^{(n,r)}$, $\tilde{W}^{(n,r)}$ and $W^{(n,r,m)}$ are vacuum vectors for $\psi^{(n,r,m)}$-models of $\pi^{J+}_{m,k}$, τ^+_k resp. σ^+_k.

3.7 Final remarks

We have associated to the vacuum vector Ψ^0 of σ_k^+ the system of vacuum vectors

$$\Phi_m^0 = i_m \Psi^0, \qquad m \in \mathbb{R}^*.$$

These functions belong to equivalent representations of Q, and are connected by the index shift formula

$$(i_m \Psi^0)(q) = |m|^{-1} (i_1 \Psi^0)\big(\mathrm{diag}(1, 1, m^{-1}, m^{-1}) q\big),$$

see (15). This goes well along with the fact that a *special* Siegel form F_f, coming as a Maaß lift from a Jacobi form $f \in J_{k,1}$, is characterized (see [EZ] Theorem 6.2) by the condition that the Fourier-Jacobi coefficients f_m of F grow out of $f_1 = f$ by the application of a Hecke-type index shift operator $V_m : J_{k,1} \to J_{k,m}$. There is a p-adic explanation for this Hecke operator which will be discussed in a subsequent paper.

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