Some remarks on local newforms for GL(2)

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Abstract. Local newforms for representations of GL(2) over a non-archimedean local field are computed in various models. Several formulas relating newforms and ε–factors are obtained.

Contents

Introduction

Let F be a non-archimedean local field and (π,V) an irreducible, infinite-dimensional representation of $\mathrm{GL}(2,F)$. By a well-known theorem of Casselman ([Cas], see also [De]) there exists a distinguished vector $v\in V$, unique up to multiples, of "best possible level". This level is an ideal $\mathfrak{p}^{c(\pi)}$ in $\mathfrak{o}$, the ring of integers of F, and is called the conductor of π. The number $c(\pi)$ appears also in the exponent of the ε–factor $\varepsilon(s,\pi,\psi)$ attached to π (and an additive character ψ of F).

In this note we shall collect several formulas for the local newform for different types of representations π and various models. We shall compute the action of intertwining operators and of Whittaker functionals on local newforms. Along the way we shall obtain a list of conductors for supercuspidal, special and principal series representations π.

If the representation π has trivial central character, there is the so-called *Atkin–Lehner involution* acting on the one-dimensional space of newvectors. This defines a sign ±1 which is canonically attached to π. We shall give a proof that this sign coincides with the sign defined by the ε–factor. For example, if π is a local component of a global automorphic representation that corresponds to a classical newform $f\in S_k(\Gamma_0(N))$, these signs enter in the functional equation of the L–functions $L(s,f)$. We remark that

Atkin–Lehner involutions are really a *symplectic* phenomenon. They can be defined for representations of $\mathrm{PGSp}(2n)$, but for $n > 1$ the relation to ε–factors is unclear.

The first two sections of this paper are preliminary, recalling properties of representations of $\mathrm{GL}(2, F)$ and their ε–factors. In sections 2.1 and 2.2 we shall treat principal series and special representations. We shall compute conductors and the local newform in various models and compute the action of the intertwining operator in the standard induced model. Section 2.3 deals with conductors and newforms for dihedral supercuspidal representations. More on supercuspidal representations and their ε–factors can be found in [GK]. In section 2.4 we shall compute the newform in the Kirillov model. Finally, sections 3.1 and 3.2 are concerned with the case of representations of $\mathrm{PGL}(2, F)$ that was mentioned above.

Very little in this note is really new, with most of the results being well-known to experts. But since the results are scattered throughout the literature and sometimes hard to locate, it seemed useful to have them written down coherently in one place. I would like to thank the referee for various suggestions to improve the exposition.

Notations

Throughout F denotes a non-archimedean local field. The ring of integers of F will be denoted by $\mathfrak{o}$, and $\mathfrak{p}$ is its maximal ideal. Let $q = \#\mathfrak{o}/\mathfrak{p}$ be the cardinality of the residue field. We let ϖ denote a fixed generator of $\mathfrak{p}$. The normalized valuation v on F has the property that $v(\varpi) = 1$. For the normalized absolute value we have $|\varpi| = q^{-1}$.

Let G be the algebraic group $\mathrm{GL}(2)$. As subgroups we have B, the standard Borel subgroup consisting of upper triangular matrices, and N, the unipotent upper triangular matrices. We shall also use $\bar{N}$, the subgroup of lower triangular unipotent matrices.

The group $K_r = \mathrm{GL}(r, \mathfrak{o})$ is a maximal compact-open subgroup of $\mathrm{GL}(r, F)$. For $n \geq 1$, let $K_r(n)$ be the subgroup of $\mathrm{GL}(r, \mathfrak{o})$ consisting of matrices of the form

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix}, \qquad A \in \mathrm{GL}(r-1, \mathfrak{o}), \quad c \equiv 0 \bmod \mathfrak{p}^n, \quad d \in 1 + \mathfrak{p}^n. \tag{1}$$

For $r = 1$ this is just $1 + \mathfrak{p}^n$. We let $K_r(0) := \mathrm{GL}(r, \mathfrak{o})$.

1 Preliminaries

1.1 ε–factors

A factor $\varepsilon(s, \pi, \psi)$ is attached to every irreducible, admissible representation π of $\mathrm{GL}(r, F)$ and choice of additive character ψ of F. We shall summarize a few properties of ε–factors, beginning with the *abelian* case.

ε–factors for $\mathrm{GL}(1)$

Let ψ be an additive character of F. For a locally constant function Φ on F^* with compact support we define the Fourier transform as

$$\hat{\Phi}(x) = \int\limits_F \Phi(y)\psi(xy)\,dy,$$

where the Haar measure is normalized such that $\hat{\hat{\Phi}}(x) = \Phi(-x)$. Under this normalization the volume of $\mathfrak{o}$ is $q^{c(\psi)/2}$, where $c(\psi)$ is the smallest integer such that ψ is trivial on $\mathfrak{p}^{c(\psi)}$. By [Ta1], there exists, for any character χ of F^*, an entire function $\varepsilon(s,\chi,\psi)$ with the property that

$$\frac{\int \hat{\Phi}(x)|x|^{1-s}\chi(x)^{-1}\,d^*x}{L(1-s,\chi^{-1})} = \varepsilon(s,\chi,\psi)\frac{\int \Phi(x)|x|^s\chi(x)\,d^*x}{L(s,\chi)} \tag{2}$$

for any test function Φ. The integrals in the numerators are only convergent in some half planes, but have meromorphic continuation to all of $\mathbb{C}$. The Haar measure on F^* is $d^*x = \frac{dx}{|x|}$, where dx is an arbitrary Haar measure on F. The L–factors are defined as usual,

$$L(s,\chi) = \begin{cases} \left(1-\chi(\varpi)q^{-s}\right)^{-1} & \text{if } \chi \text{ is unramified,} \\ 1 & \text{if } \chi \text{ is ramified.} \end{cases}$$

We cite from [Ta2] the following properties of ε–factors.

i) For any $a \in F^*$,

$$\varepsilon(s,\,\chi,\,\psi^a) = \chi(a)|a|^{s-1/2}\varepsilon(s,\,\chi,\,\psi), \qquad \text{where } \psi^a(x) = \psi(ax) \tag{3}$$

(to obtain this from [Ta2] (3.2.3), note that in our definition of $\varepsilon(s,\chi,\psi)$ we always use the self-dual measure giving $\mathfrak{o}$ volume $q^{c(\psi)/2}$).

ii) If χ is unramified, then

$$\varepsilon(s,\,\chi,\,\psi) = \chi(\varpi)^{-c(\psi)}q^{c(\psi)(s-1/2)}. \tag{4}$$

In particular, if χ is unramified and $c(\psi) = 0$, then $\varepsilon(s,\,\chi,\,\psi) = 1$.

iii) If χ is ramified, let $c(\chi)$ be the smallest positive integer such that χ is trivial on $1+\mathfrak{p}^{c(\chi)}$. Then

$$\varepsilon(s,\,\chi,\,\psi) = \int\limits_{\varpi^l\mathfrak{o}^*} |x|^{-s}\chi^{-1}(x)\psi(x)\,dx, \qquad \text{where } \; l = c(\psi)-c(\chi). \tag{5}$$

Here dx is the self-dual Haar measure giving $\mathfrak{o}$ volume $q^{c(\psi)/2}$. Since the function under the integral is invariant under multiplication with elements of $1+\mathfrak{p}^m$ for large enough m, the integral reduces to a local Gaussian sum.

iv) If ν is unramified, then

$$\varepsilon(s,\,\chi\nu,\,\psi) = \nu(\varpi)^{c(\chi)-c(\psi)}\varepsilon(s,\,\chi,\,\psi). \tag{6}$$

v) For every χ and ψ,

$$\varepsilon(s,\,\chi,\,\psi)\,\varepsilon(1-s,\,\chi^{-1},\,\psi)=\chi(-1). \tag{7}$$

In the following lemma a certain $\mathfrak{p}$–adic integral is evaluated that typically shows up in computations involving Whittaker functionals. We will use the result in the proof of Lemma 2.2.1 below.

1.1.1 Lemma. *Let ψ be a character of F and χ a character of F^*. Let $c(\psi)$ be the smallest integer such that ψ is trivial on $\mathfrak{p}^{c(\psi)}$, and let $c(\chi)$ be the smallest non-negative integer such that χ is trivial on $(1+\mathfrak{p}^{c(\chi)})\cap\mathfrak{o}^*$. For $m\in\mathbb{Z}$ consider the integral*

$$I(m)=\int\limits_{\varpi^l\mathfrak{o}^*}\chi^{-1}(x)\psi(x)\,dx,\qquad \text{where } l=c(\psi)-c(\chi)+m,$$

and where dx is the self-dual Haar measure with respect to ψ. If χ is unramified (i.e., $c(\chi)=0$), then

$$I(m)=\begin{cases}0, & \text{if } m\le -2,\\ -\chi\big(\varpi^{-c(\psi)+1}\big)q^{-c(\psi)/2}, & \text{if } m=-1,\\ \chi\big(\varpi^{-c(\psi)-m}\big)q^{-c(\psi)/2-m}(1-q^{-1}), & \text{if } m\ge 0,\end{cases}$$

while if χ is ramified, then

$$I(m)=\begin{cases}\varepsilon(0,\chi,\psi), & \text{if } m=0,\\ 0, & \text{if } m\ne 0.\end{cases}$$

Proof: In any case we have

$$I(m)=\chi^{-1}(\varpi^l)\,q^{-l}\int\limits_{\mathfrak{o}^*}\chi^{-1}(x)\,\psi(x\varpi^l)\,dx.$$

The easy computation in the unramified case is left to the reader. Assuming that χ is ramified, our assertion holds for $m=0$ by (5). If $m\ne 0$ we write the integral as

$$\int\limits_{\mathfrak{o}^*/(1+\mathfrak{p}^r)}\chi^{-1}(x)\Bigg(\int\limits_{(1+\mathfrak{p}^r)\cap\mathfrak{o}^*}\chi^{-1}(y)\,\psi(xy\varpi^l)\,dy\Bigg)\,dx,\qquad r\ge 0. \tag{8}$$

If $m<0$, we put $r=c(\chi)$. The inner integral reduces to

$$\int\limits_{1+\mathfrak{p}^{c(\chi)}}\psi\big(xy\varpi^l\big)\,dy=\psi(x\varpi^l)\int\limits_{\mathfrak{p}^{c(\chi)}}\psi\big(xz\varpi^{c(\psi)-c(\chi)+m}\big)\,dz.$$

Since x is a unit and $m<0$, the latter integral is zero. If $m>0$, we put $r=c(\chi)-1$ in (8). Then

$$xy\varpi^l\in x\varpi^l+\mathfrak{p}^{c(\psi)+m-1},$$

and therefore the inner integral in (8) equals

$$\int\limits_{(1+\mathfrak{p}^r)\cap\mathfrak{o}^*}\chi^{-1}(y)\,\psi(x\varpi^l)\,dy=\psi(x\varpi^l)\int\limits_{(1+\mathfrak{p}^r)\cap\mathfrak{o}^*}\chi^{-1}(y)\,dy.$$

Since χ is not trivial on $(1+\mathfrak{p}^r)\cap\mathfrak{o}^*$, this integral is 0. ∎

ε–factors for $\mathrm{GL}(r)$

We now turn to ε–factors for representations of $\mathrm{GL}(r,F)$. In (1) we have defined the compact subgroup $K_r(n)$ of $\mathrm{GL}(r,F)$. For the following theorem see [JPSS].

1.1.2 Theorem. *Let* (π,V) *be an irreducible admissible generic representation of* $\mathrm{GL}(r,F)$.

i) There exists a least non-negative integer $c(\pi)$ *such that* $V^{K_r(c(\pi))}\neq 0$.

ii) $\dim_{\mathbb{C}}\left(V^{K_r(c(\pi))}\right)=1$.

The ideal $\mathfrak{p}^{c(\pi)}$ with $c(\pi)$ as in this theorem is called the *conductor* of π.

Generalizing the abelian factors $\varepsilon(s,\chi,\psi)$, a factor $\varepsilon(s,\pi,\psi)$ is attached to every irreducible, admissible representation π of $\mathrm{GL}(r,F)$ (for the precise definition, see [Ja]). It incorporates a number $c(\pi)$ which for generic representations coincides with the constant defined in Theorem 1.1.2, see [JPSS]. We now summarize some properties of ε–factors for representations π of $\mathrm{GL}(r,F)$. Let ψ be a non-trivial additive character of F with conductor $\mathfrak{p}^{c(\psi)}$.

i) The ε–factor of π is given by

$$\varepsilon(s,\pi,\psi)=cq^{(rc(\psi)-c(\pi))s} \tag{9}$$

with some constant $c\in\mathbb{C}$ independent of s. If π is unramified, i.e., $c(\pi)=0$, and if ψ has conductor $\mathfrak{o}$, then $\varepsilon(s,\pi,\psi)=1$.

ii) For any $a\in F^*$,

$$\varepsilon(s,\,\pi,\,\psi^a)=\omega_\pi(a)|a|^{r(s-1/2)}\varepsilon(s,\,\pi,\,\psi),\qquad\text{where }\psi^a(x)=\psi(ax). \tag{10}$$

Here ω_π denotes the central character of π.

iii) For any $t\in\mathbb{C}$,

$$\varepsilon(s,\,|\,|^t\pi,\,\psi)=\varepsilon(s+t,\,\pi,\,\psi)=q^{(rc(\psi)-c(\pi))t}\varepsilon(s,\,\pi,\,\psi). \tag{11}$$

iv) If $\hat{\pi}$ denotes the representation contragredient to π, then

$$\varepsilon(s,\,\pi,\,\psi)\,\varepsilon(1-s,\,\hat{\pi},\,\psi)=\omega_\pi(-1). \tag{12}$$

1.2 Representations of $\mathrm{GL}(2,F)$

We briefly recall the various types of irreducible, admissible representations of $\mathrm{GL}(2,F)$.

Induced representations

Let χ_1 and χ_2 be characters of F^*. Let $V(\chi_1,\chi_2)$ be the space of the standard induced representation $\pi = \pi(\chi_1,\chi_2)$ of $G(F)$. It consists of all locally constant functions $f : G(F) \to \mathbb{C}$ with the transformation property

$$f\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} g\right) = \chi_1(a)\chi_2(d)|ad^{-1}|^{1/2} f(g) \qquad \text{for all } g \in G(F),\ a,d \in F^*,\ b \in F.$$

The action of $G(F)$ on $V(\chi_1,\chi_2)$ is by right translation. In view of the Iwasawa decomposition $G(F) = B(F)K$, another model for $\pi(\chi_1,\chi_2)$ is obtained by restricting functions in $V(\chi_1,\chi_2)$ to K. Let us denote this space of functions on K by $V(\chi_1,\chi_2)_K$. A third model is obtained by restricting functions in $V(\chi_1,\chi_2)$ to $N_-(F)$. Since $B(F)N_-(F)$ is dense in $G(F)$, this is an injective operation. We denote the resulting space of functions on $F \simeq N_-(F)$ by $V(\chi_1,\chi_2)_-$.

It is a well-known theorem that $\pi(\chi_1,\chi_2)$ is irreducible if and only if $\chi_1\chi_2^{-1} \neq |\,|^{\pm 1}$. In this case $\pi(\chi_1,\chi_2)$ is called a *principal series representation*.

The induced representation $\pi(|\,|^{1/2},|\,|^{-1/2})$ is not irreducible and has two constituents. The unique irreducible quotient is the trivial representation. The unique irreducible subrepresentation is the *Steinberg representation* which we denote by St. More generally, if χ is a character of F^*, then $\pi(\chi|\,|^{1/2},\chi|\,|^{-1/2})$ contains a unique irreducible subrepresentation which is just the twist χSt of the Steinberg representation. The resulting quotient is the one-dimensional representation $\chi \circ \det$. The representations χSt are also called *special representations*.

The only equivalences between the representations listed so far are

$$\pi(\chi_1,\chi_2) \simeq \pi(\chi_2,\chi_1)$$

for principal series representations. Every irreducible representation that is not a subquotient of some $\pi(\chi_1,\chi_2)$ is called *supercuspidal*.

Supercuspidal representations

We recall the definition of the so-called *dihedral* supercuspidal representations ω_ξ that are associated with a quadratic field extension E/F and a character ξ of E^* that is not trivial on the kernel of the norm map $N_{E/F}$ from E^* to F^*. If the residue characteristic of F is not 2, then every supercuspidal representation of $\mathrm{GL}(2,F)$ is isomorphic to some ω_ξ. As a reference for the construction, see [JL], section 1.1 or [Bu], section 4.8.

The Weil representation of $\mathrm{SL}(2,F)$ associated to E/F, call it ω_E, depends on the choice of an additive character ψ_F of F. It acts on the space $C_c^\infty(E)$ of locally constant, compactly supported functions on E by the following well-known formulas.

$$\left(\omega_E\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} f\right)(v) = \psi_F\big(xN_{E/F}(v)\big)\, f(v), \tag{13}$$

$$\left(\omega_E\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} f\right)(v) = |a|_F\, \chi_{E/F}(a)\, f(av), \tag{14}$$

$$\big(\omega_E(w_1)f\big)(v)=\gamma\int\limits_E f(u)\psi_E(u\bar v)\,du,\qquad w_1=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}. \tag{15}$$

Here $\psi_E=\psi_F\circ\mathrm{tr}_{E/F}$, and $\chi_{E/F}$ is the quadratic character of F^* that is trivial on $N_{E/F}E^*$. The Haar measure on E is normalized such that

$$\mathrm{vol}(\mathfrak{O})=(\#\mathfrak{O}/\mathfrak{P})^{c(\psi_E)/2}=q^{c(\psi)-d(E/F)/2}. \tag{16}$$

Here $\mathfrak{O}/\mathfrak{P}$ is the residue class field of E and $q=\#\mathfrak{o}/\mathfrak{p}$ is the cardinality of the residue class field of F. The number $d(E/F)$ denotes the valuation of the discriminant of the extension E/F. The second equality in (16) follows from Lemma 2.3.1 which we shall prove later.

The constant γ in (15) is a fourth root of unity with the property

$$\gamma^2=\chi_{E/F}(-1). \tag{17}$$

This follows by applying $\mathrm{id}=\omega_E(w_1)\omega_E(w_1)\omega_E(-\mathbf{1})$ to the characteristic function of $\mathfrak{O}$. Now let ξ be a regular character of E^*, meaning that $\xi\neq\xi^\sigma$, where σ is the non-trivial Galois automorphism of E/F (equivalently, ξ is non-trivial on the kernel of the norm map $N_{E/F}:E^*\to F^*$). Let U_ξ be the subspace of $C_c^\infty(E)$ consisting of functions f that satisfy

$$f(yv)=\xi(y)^{-1}f(v)\qquad\text{for all } v\in E \text{ and all } y\in\ker(N_{E/F}).$$

The Weil representation induces a representation of SL$(2,F)$ on U_ξ. It can be extended to a representation, call it $\omega_{\xi,\psi}$, of the subgroup GL$(2,F)^+$ of elements of GL$(2,F)$ whose determinant is a norm from E by requiring that

$$\left(\omega_{\xi,\psi}\begin{pmatrix} a & \\ & 1\end{pmatrix}f\right)(v)=|a|_F^{1/2}\xi(b)f(bv),\qquad a=N_{E/F}(b). \tag{18}$$

We let ω_ξ be the representation of GL$(2,F)$ that is obtained by induction from the representation $\omega_{\xi,\psi}$ of GL$(2,F)^+$. Then ω_ξ is an irreducible admissible supercuspidal representation of GL$(2,F)$. It is easy to see that the representations ω_ξ have the following properties.

- The isomorphism class of ω_ξ is independent of the choice of ψ.
- The central character of ω_ξ is $\chi_{E/F}\xi\big|_{F^*}$.
- For any character μ of F^* there is an isomorphism $\omega_\xi\otimes\mu\simeq\omega_{\xi(\mu\circ N_{E/F})}$.

Conductors

Every irreducible admissible representation of GL$(2,F)$, if not one-dimensional, is generic. Theorem 1.1.2 and formula (9) therefore imply the first part of the following result. For the second part see [Cas], Theorem 1, or [De], Théorème 2.2.6.

1.2.1 Theorem. *Let (π,V) be an infinite-dimensional irreducible admissible representation of* GL$(2,F)$*. Let $V^{(m)}$ denote the space of $K_2(m)$–fixed vectors in V. Then:*

i) There exists an $m \geq 0$ such that $V^{(m)} \neq 0$. Let $c(\pi)$ denote the least non-negative integer with this property. Then the ε–factor of π has the form

$$\varepsilon(s,\pi,\psi) = cq^{(2c(\psi)-c(\pi))s} \tag{19}$$

for a constant $c \in \mathbb{C}^*$ which is independent of s.

ii) For all $m \geq c(\pi)$ we have $\dim(V^{(m)}) = m - c(\pi) + 1$.

We now present the table of the conductors $\mathfrak{p}^{c(\pi)}$ in the sense of Theorem 1.1.2 for the various classes of infinite-dimensional irreducible representations.

representation π	$c(\pi)$
$\pi(\chi_1,\chi_2),\ \chi_1\chi_2^{-1} \neq \mid\,\mid^{\pm 1}$	$c(\chi_1)+c(\chi_2)$
$\chi\mathrm{St},\ \chi$ unramified	1
$\chi\mathrm{St},\ \chi$ ramified	$2c(\chi)$
$\omega_\xi,\ \xi$ regular character of E^*	$f(E/F)c(\xi)+d(E/F)$

In the last row, $f(E/F)$ is the degree of the residue class field extension of the quadratic extension E/F. These statements about the conductors will be proved in the following. In Proposition 2.1.2 we determine the local newform in induced representations. In Theorem 2.3.2 we shall compute the conductor of the dihedral supercuspidal representations.

2 Local newforms in various models

2.1 The local newform in induced representations

Let χ_1 and χ_2 be characters of F^*. As before, we let $\pi(\chi_1,\chi_2)$ be the associated induced representation on its standard model $V(\chi_1,\chi_2)$. For $i=1,2$ let n_i be the exponent of the conductor of χ_i. We shall prove that the conductor of π is $\mathfrak{p}^n$ with $n=n_1+n_2$, meaning n is the smallest integer such that there exists a non-zero vector invariant under

$$K_2(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K :\ c \in \mathfrak{p}^n,\ d \in 1+\mathfrak{p}^n \right\}.$$

The space of such vectors is one-dimensional, and we shall determine a canonical element.

2.1.1 Lemma. *We have the double coset decomposition*

$$K = \bigsqcup_{i=0}^{n} B(\mathfrak{o})\,\gamma_i\,K_2(n),$$

where $\gamma_i = \begin{pmatrix} 1 & \\ \varpi^i & 1 \end{pmatrix}$ for $0 \leq i \leq n-1$ and $\gamma_n = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$. For $0 \leq i < n$ the coset $B(\mathfrak{o})\,\gamma_i\,K_2(n)$ consists of those matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$ with $v(c)=i$.

Proof: If $i := v(c) > 0$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ac^{-1}d\varpi^i - b\varpi^i & b \\ & d \end{pmatrix} \begin{pmatrix} 1 & \\ \varpi^i & 1 \end{pmatrix} \begin{pmatrix} \varpi^{-i}d^{-1}c & \\ & 1 \end{pmatrix},$$

while if c is a unit, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ac^{-1}d - b & a + (b - ac^{-1}d)(1+\varpi^n) \\ & c \end{pmatrix} \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1+\varpi^n & (1+\varpi^n)c^{-1}d - 1 \\ -\varpi^n & 1 - \varpi^n c^{-1}d \end{pmatrix}.$$

■

We fix some $i \in \{0, \ldots, n\}$ and try to define a function $f \in V(\chi_1, \chi_2)_K$ by

$$f(g) := \begin{cases} \chi_1(a)\chi_2(d) & \text{if } g \in \begin{pmatrix} a & * \\ & d \end{pmatrix} \gamma_i\, K_2(n), \\ 0 & \text{if } g \notin B(\mathfrak{o})\gamma_i\, K_2(n). \end{cases}$$

This is well-defined if and only if

$$\chi_1(a)\chi_2(d) = 1 \qquad \text{for all} \qquad \begin{pmatrix} a & b \\ & d \end{pmatrix} \in B(\mathfrak{o}) \cap \gamma_i K_2(n)\gamma_i^{-1}.$$

Let $a \in (1+\mathfrak{p}^{n-i}) \cap \mathfrak{o}^*$ and $d \in (1+\mathfrak{p}^i) \cap \mathfrak{o}^*$ be arbitrary. Then

$$\begin{pmatrix} a & (d-1)\varpi^{-i} \\ & d \end{pmatrix} \in B(\mathfrak{o}) \cap \gamma_i K_2(n)\gamma_i^{-1}.$$

For f to be well-defined, we must therefore have

$$\chi_1 \text{ trivial on } (1+\mathfrak{p}^{n-i}) \cap \mathfrak{o}^* \qquad \text{and} \qquad \chi_2 \text{ trivial on } (1+\mathfrak{p}^i) \cap \mathfrak{o}^*.$$

This is equivalent to $n - i \geq n_1$ and $i \geq n_2$, a condition that can be satisfied if and only if $n \geq n_1 + n_2$. If $n = n_1 + n_2$, it is fulfilled precisely for $i = n_2$. This proves the first statement in the following proposition, while the other statements follow by an easy calculation (observing the second statement of Lemma 2.1.1).

2.1.2 Proposition. *The space of $K_2(n)$–invariant vectors, $n = n_1 + n_2$, in the induced representation $\pi(\chi_1, \chi_2)$ is one-dimensional. In the model $V(\chi_1, \chi_2)$ a non-trivial $K_2(n)$–invariant vector is given by*

$$f(g) = \begin{cases} \chi_1(\varpi^{-n_2})\chi_1(a)\chi_2(d)|ad^{-1}|^{1/2} & \text{if } g \in \begin{pmatrix} a & * \\ & d \end{pmatrix} \gamma_{n_2}\, K_2(n), \\ 0 & \text{if } g \notin B(F)\gamma_{n_2}\, K_2(n), \end{cases} \tag{20}$$

where $\gamma_{n_2} = \begin{pmatrix} 1 & \\ \varpi^{n_2} & 1 \end{pmatrix}$. In the model $V(\chi_1, \chi_2)_-$, this function becomes a function on F of the following shape:

- *If χ_1 and χ_2 are ramified, then*

$$f\begin{pmatrix}1 & \\ x & 1\end{pmatrix} = \begin{cases} \chi_1(x)^{-1}, & \text{if } v(x) = n_2, \\ 0, & \text{if } v(x) \neq n_2. \end{cases} \tag{21}$$

- *If χ_1 is unramified and χ_2 is ramified, then*

$$f\begin{pmatrix}1 & \\ x & 1\end{pmatrix} = \begin{cases} \chi_1(\varpi)^{-n_2}, & \text{if } v(x) \geq n_2, \\ 0, & \text{if } v(x) < n_2. \end{cases} \tag{22}$$

- *If χ_1 is ramified and χ_2 is unramified, then*

$$f\begin{pmatrix}1 & \\ x & 1\end{pmatrix} = \begin{cases} \chi_1(x)^{-1}\chi_2(x)|x|^{-1}, & \text{if } v(x) \leq 0 \\ 0, & \text{if } v(x) > 0. \end{cases} \tag{23}$$

- *If χ_1 and χ_2 are unramified, then*

$$f\begin{pmatrix}1 & \\ x & 1\end{pmatrix} = \begin{cases} \chi_1(x)^{-1}\chi_2(x)|x|^{-1}, & \text{if } v(x) \leq 0 \\ 1, & \text{if } v(x) > 0. \end{cases} \tag{24}$$

Remark: The factor $\chi_1(\varpi^{-n_2})$ in (20) makes the function f independent of the choice of ϖ.

Special representations

If $\pi(\chi_1, \chi_2)$ is irreducible, Lemma 2.1.2 tells us the shape of a local newform in the model $V(\chi_1, \chi_2)$. We now consider the case that $\pi(\chi_1, \chi_2)$ is reducible, which, as is well known, happens if and only if $\chi_1\chi_2^{-1} = |\ |$ or $\chi_1\chi_2^{-1} = |\ |^{-1}$. Let us therefore assume that

$$\chi_1 = \chi|\ |^{1/2}, \qquad \chi_2 = \chi|\ |^{-1/2}$$

for some character χ. In this case the unique invariant subspace $V_0(\chi_1, \chi_2)$ of $V(\chi_1, \chi_2)$ is infinite-dimensional, the quotient is one-dimensional. The representation of $G(F)$ on $V_0(\chi_1, \chi_2)$ is the special representation $\chi\mathrm{St}$, a twist of the Steinberg representation. $V_0(\chi_1, \chi_2)$ can be characterized as the kernel of the intertwining operator $M : V(\chi_1, \chi_2) \to V(\chi_2, \chi_1)$, given by

$$(Mf)(g) = \int\limits_F f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix} g\right)\, dx.$$

Here we are normalizing the measure such that $\int_{\mathfrak{o}} dx = 1$. By [Bu], Proposition 4.5.6, the integral is absolutely convergent in the case under consideration.

First we consider the case where the character χ is ramified. We shall prove that the function f in (20) is an element of $V_0(\chi_1, \chi_2)$. Indeed, for all $y \in F$ we have

$$(Mf)\begin{pmatrix}1 & \\ y & 1\end{pmatrix} = \int\limits_F f\begin{pmatrix} -y & -1 \\ 1+xy & x\end{pmatrix} dx = \int\limits_{F^*} f\left(\begin{pmatrix} x^{-1} & -1 \\ & x\end{pmatrix}\begin{pmatrix} 1 & \\ x^{-1}+y & 1\end{pmatrix}\right) dx$$

$$= \int\limits_{F^*} |x|^{-1} f\begin{pmatrix} 1 & \\ x^{-1}+y & 1 \end{pmatrix} \frac{dx}{|x|} = \int\limits_{F^*} |x|\, f\begin{pmatrix} 1 & \\ x+y & 1 \end{pmatrix} \frac{dx}{|x|}$$
$$= \int\limits_{F} f\begin{pmatrix} 1 & \\ x+y & 1 \end{pmatrix} dx = \int\limits_{F} f\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} dx.$$

In view of (21) and the fact that χ is ramified, the latter integral is zero. Thus $Mf = 0$, proving that $f \in V_0(\chi_1, \chi_2)$.

If χ is unramified, the function (20) is just the spherical vector in $V(\chi_1, \chi_2)$, and the above computation shows that it is not an element of $V_0(\chi_1, \chi_2)$. Indeed, the conductor of $\chi\mathrm{St}$ is known to be $\mathfrak{p}$ if χ is unramified. To prove this, we note that in the unramified case a function $f \in V(\chi_1, \chi_2)$ is $K_2(1)$–invariant if and only if it is I–invariant. Here I is the Iwahori subgroup consisting of all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$ with $c \in \mathfrak{p}$. Thus, considering the model $V(\chi_1, \chi_2)_K$, we are looking for functions on

$$B(\mathfrak{o})\backslash K/I \simeq B(k)\backslash G(k)/B(k),$$

where $k = \mathfrak{o}/\mathfrak{p}$ is the residue field. By the Bruhat decomposition, this double coset space is represented by the elements $\mathbf{1} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ and $w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. Therefore any $K_2(1)$–invariant function f in $V(\chi_1, \chi_2)$ is determined by the values

$$\alpha := f(\mathbf{1}) \qquad \text{and} \qquad \beta := f(w), \tag{25}$$

and these values can be chosen arbitrarily. Consequently, the space of such functions is two-dimensional. Let us apply the intertwining operator M to the $K_2(1)$–invariant function f as in (25). With the measure being normalized such that $\int_{\mathfrak{o}} dx = 1$ we compute

$$(Mf)(\mathbf{1}) = \int\limits_{\mathfrak{o}} f(w)\, dx + \int\limits_{F\backslash\mathfrak{o}} f\left(\begin{pmatrix} x^{-1} & \\ & x \end{pmatrix}\begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix}\right) dx$$
$$= \beta + \int\limits_{F\backslash\mathfrak{o}} |x|^{-1} f(1) \frac{dx}{|x|} = \beta + \alpha \sum_{n=0}^{\infty} \int\limits_{\varpi^{-n-1}\mathfrak{o}^*} |x|^{-1} \frac{dx}{|x|}$$
$$= \beta + \alpha \sum_{n=0}^{\infty} q^{-n-1} \int\limits_{\mathfrak{o}^*} dx = \beta + \alpha q^{-1}.$$

A similar computation shows $(Mf)(w) = \beta + \alpha q^{-1}$. Thus we see that the space of $K_2(1)$–invariant vectors of $V_0(\chi_1, \chi_2)$ is one-dimensional, a non-zero element f being determined by the values

$$(f(\mathbf{1}),\, f(w)) = (q,\, -1). \tag{26}$$

2.2 Local newforms and Whittaker models

Let ψ be an additive character of F with conductor $\mathfrak{p}^{c(\psi)}$. For $i = 1, 2$ we shall write the characters χ_i as $\chi_i = \xi_i|\,|^{s_i}$ with a unitary character ξ_i and $s_i \in \mathbb{C}$.

Let Λ_ψ be the ψ–Whittaker functional on $V(\chi_1,\chi_2)$ given by

$$\Lambda_\psi f = \lim_{N\to\infty} \int\limits_{\mathfrak{p}^{-N}} f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right)\psi^{-1}(x)\,dx. \tag{27}$$

(limit exists for all χ_1,χ_2). Here we normalize the measure such that $\int_{\mathfrak{o}} dx = 1$.

2.2.1 Lemma. *Let f be the newform given by (20).*

- *If χ_1 and χ_2 are ramified, then*

$$\Lambda_\psi f = \begin{cases} \varepsilon(0,\,\chi_2^{-1}|\,|,\,\psi^{-1}) = \varepsilon(1-s_2,\,\xi_2^{-1},\,\psi^{-1}), & \text{if } c(\psi)=0, \\ 0, & \text{if } c(\psi)\neq 0. \end{cases} \tag{28}$$

- *If χ_1 is unramified and χ_2 is ramified, then*

$$\Lambda_\psi f = \begin{cases} \chi_1(\varpi)^{-c(\psi)}q^{-c(\psi)/2}\,\varepsilon(1-s_2,\,\xi_2^{-1},\,\psi^{-1}), & \text{if } c(\psi)\le 0, \\ 0, & \text{if } c(\psi)>0. \end{cases} \tag{29}$$

- *If χ_1 is ramified and χ_2 is unramified, then*

$$\Lambda_\psi f = \begin{cases} 1, & \text{if } c(\psi)\le 0, \\ 0, & \text{if } c(\psi)>0. \end{cases} \tag{30}$$

- *If χ_1 and χ_2 are unramified, let $\alpha=\chi_1(\varpi)$, $\beta=\chi_2(\varpi)$. If $\alpha\neq\beta$, then*

$$\Lambda_\psi f = \begin{cases} (1-\alpha\beta^{-1}q^{-1})\frac{1-(\alpha\beta^{-1})^{-c(\psi)+1}}{1-\alpha\beta^{-1}}, & \text{if } c(\psi)\le 0, \\ 0, & \text{if } c(\psi)>0, \end{cases} \tag{31}$$

  while if $\alpha=\beta$, then

$$\Lambda_\psi f = \begin{cases} (1-q^{-1})(1-c(\psi)) & \text{if } c(\psi)\le 0 \\ 0, & \text{if } c(\psi)>0. \end{cases} \tag{32}$$

  If we interpret the quotient in (31) as $1+\alpha\beta^{-1}+\ldots+(\alpha\beta^{-1})^{-c(\psi)}$, then (32) becomes a special case of (31).

Proof: Let us first assume that χ_1 and χ_2 are ramified, i.e., $n_1>0$ and $n_2>0$. We compute

$$\begin{aligned} \Lambda_\psi f &= \lim_{N\to\infty} \int\limits_{\mathfrak{p}^{-N}} f\begin{pmatrix} & -1 \\ 1 & x \end{pmatrix}\psi^{-1}(x)\,dx \\ &= \lim_{N\to\infty} \int\limits_{\mathfrak{p}^{-N}\setminus\mathfrak{o}} f\left(\begin{pmatrix} x^{-1} & -1 \\ & x \end{pmatrix}\begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix}\right)\psi^{-1}(x)\,dx \end{aligned}$$

$$= \lim_{N\to\infty} \int_{\mathfrak{p}^{-N}\setminus\mathfrak{o}} \chi_1(x)^{-1}\chi_2(x)|x|^{-1} f\begin{pmatrix} 1 & \\ x^{-1} & 1\end{pmatrix}\psi^{-1}(x)\,dx$$

$$\overset{(21)}{=} \int_{\varpi^{-n_2}\mathfrak{o}^*} \chi_2(x)|x|^{-1}\psi^{-1}(x)\,dx. \tag{33}$$

By Lemma 1.1.1, we get

$$\Lambda_\psi f = \begin{cases} \varepsilon(0,\,\chi_2^{-1}|\,|,\,\psi^{-1}), & \text{if } c(\psi)=0,\\ 0, & \text{if } c(\psi)\neq 0,\end{cases} \tag{34}$$

hence (28). Next assume that $n_1 = 0$ and $n_2 > 0$. Then, using (22), we arrive similarly at

$$\Lambda_\psi f = \int_{v(x)\le -n_2} \chi_1(x)^{-1}\chi_2(x)|x|^{-1}\chi_1(\varpi)^{-n_2}\psi^{-1}(x)\,dx$$

$$= \chi_1(\varpi)^{-n_2}\sum_{l=n_2}^{\infty}\int_{\varpi^{-l}\mathfrak{o}^*}(\chi_1^{-1}\chi_2)(x)|x|^{-1}\psi^{-1}(x)\,dx.$$

By Lemma 1.1.1, the last integral is non-zero if and only if $-l = c(\psi) - n_2$, in which case it takes the value $q^{-c(\psi)/2}\,\varepsilon(0,\,|\,|\chi_1\chi_2^{-1},\,\psi^{-1})$ (the factor $q^{-c(\psi)/2}$ comes from the different normalizations of Haar measures). Using (6), we get (29).

Now assume that $n_1 > 0$ and $n_2 = 0$. In this case we have $f\begin{pmatrix} & -1\\ 1 & x\end{pmatrix} = 0$ unless $x \in \mathfrak{o}$, so that

$$\Lambda_\psi f = \int_{\mathfrak{o}} f\begin{pmatrix} & -1\\ 1 & x\end{pmatrix}\psi^{-1}(x)\,dx$$

$$= \int_{\mathfrak{o}} f\left(\begin{pmatrix} 1 & -(1+\varpi^n)\\ & 1\end{pmatrix}\begin{pmatrix} 1 & \\ 1 & 1\end{pmatrix}\begin{pmatrix} 1+\varpi^n & (1+\varpi^n)x-1\\ -\varpi^n & 1-\varpi^n x\end{pmatrix}\right)\psi^{-1}(x)\,dx$$

$$= \int_{\mathfrak{o}} f\begin{pmatrix} 1 & \\ 1 & 1\end{pmatrix}\psi^{-1}(x)\,dx = \int_{\mathfrak{o}}\psi^{-1}(x)\,dx. \tag{35}$$

From this we obtain (30). If χ_1 and χ_2 are both unramified, then the computation of $\Lambda_\psi f$ is a standard computation of p–adic integrals which is left to the reader. ∎

Remark: The lemma shows that if $c(\psi) = 0$, then the newform f is a *test vector* for the functional Λ_ψ in the sense that $\Lambda_\psi f \neq 0$.

Special representations

Consider a special representation $\chi\mathrm{St}$, realized as a subspace of $V(\chi|\,|^{1/2},\,\chi|\,|^{-1/2})$. A Whittaker functional on this space is still given by formula (27). Assuming that χ is ramified, we proved in the previous section that the function f given by (20) lies in this subspace. The computation of $\Lambda_\psi f$ is therefore the same as for (28), and we get

$$\Lambda_\psi f = \begin{cases} \varepsilon(0,\, \chi_2^{-1}|\,|,\, \psi) = \varepsilon(0,\, \chi^{-1}|\,|^{3/2},\, \psi), & \text{if } c(\psi) = 0, \\ 0, & \text{if } c(\psi) \neq 0. \end{cases} \tag{36}$$

Now assume that χ is unramified. In this case the newform in the subspace of $V(\chi|\,|^{1/2},\, \chi|\,|^{-1/2})$ realizing $\chi\mathrm{St}$ is the unique Iwahori–invariant function f taking the values $f(\mathbf{1}) = q$ and $f(w) = -1$, see (26). Let us decompose $\Lambda_\psi f = A + B$ with

$$A = \int\limits_{\mathfrak{o}} f\left(w\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \psi^{-1}(x)\, dx, \qquad B = \lim_{N\to\infty} \int\limits_{\mathfrak{p}^N\setminus\mathfrak{o}} f\left(w\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \psi^{-1}(x)\, dx.$$

Since f is right invariant under matrices $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ with $x \in \mathfrak{o}$, we get

$$A = -\int\limits_{\mathfrak{o}} \psi^{-1}(x)\, dx = \begin{cases} -1, & \text{if } c(\psi) \leq 0, \\ 0, & \text{if } c(\psi) > 0. \end{cases}$$

As for B, we have

$$\begin{aligned} B &= \lim_{N\to\infty} \int\limits_{\mathfrak{p}^N\setminus\mathfrak{o}} f\left(\begin{pmatrix} x^{-1} & -1 \\ & x \end{pmatrix}\begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix}\right) \psi^{-1}(x)\, dx \\ &= \sum_{l=1}^{\infty} \int\limits_{\varpi^{-l}\mathfrak{o}^*} |x^{-2}|\, f(\mathbf{1})\, \psi^{-1}(x)\, dx \\ &= \sum_{l=1}^{\infty} q^{1-2l} \int\limits_{\varpi^{-l}\mathfrak{o}^*} \psi^{-1}(x)\, dx. \end{aligned}$$

An easy computation shows

$$\int\limits_{\varpi^{-l}\mathfrak{o}^*} \psi^{-1}(x)\, dx = \begin{cases} q^l - q^{l-1}, & \text{if } l < 1 - c(\psi), \\ -q^{-c(\psi)}, & \text{if } l = 1 - c(\psi), \\ 0, & \text{if } l > 1 - c(\psi) \end{cases}$$

(one can also apply Lemma 1.1.1). From this it is straightforward to compute that

$$B = \begin{cases} 1 - q^{c(\psi)}(1 + q^{-1}), & \text{if } c(\psi) \leq 0, \\ 0, & \text{if } c(\psi) > 0. \end{cases}$$

Taking everything together, we see that

$$\Lambda_\psi f = \begin{cases} -q^{c(\psi)}(1 + q^{-1}), & \text{if } c(\psi) \leq 0, \\ 0, & \text{if } c(\psi) > 0. \end{cases} \tag{37}$$

Intertwining operators

Consider the standard intertwining operator $M : V(\chi_1, \chi_2) \to V(\chi_2, \chi_1)$, given in its region of convergence by

$$(Mf)(g) = \int_F f\left(w\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g\right)\,dx, \qquad w = \begin{pmatrix} & -1\\ 1 & \end{pmatrix}. \tag{38}$$

This operator maps newform to newform, therefore, if f denotes the element (20) of $V(\chi_1, \chi_2)$, and $\tilde{f}$ denotes the analogously defined element of $V(\chi_2, \chi_1)$, then

$$Mf = c_1\tilde{f} \qquad \text{for some } c_1 \in \mathbb{C}^*.$$

We are trying to determine this constant c_1. Apparently it is important to fix the measure in (38), and we shall do so by requiring that $\mathrm{vol}(\mathfrak{o}) = 1$.

2.2.2 Proposition. *Let χ_1 and χ_2 be characters of F^* with $\chi_1\chi_2^{-1} \neq |\,|^{\pm 1}$. Let $f \in V(\chi_1, \chi_2)$ (resp. $\tilde{f} \in V(\chi_2, \chi_1)$) be the normalized newform given in Lemma 2.1.2. Then the intertwining operator $M : V(\chi_1, \chi_2) \to V(\chi_2, \chi_1)$ maps f to $c_1\tilde{f}$, where c_1 is given as follows.*

- *If χ_1 or χ_2 is ramified, then*

$$c_1 = q^{-c(\psi)/2}\frac{\varepsilon(1-s_1+s_2,\,\xi_1^{-1}\xi_2,\,\psi)\,\varepsilon(1-s_2,\,\xi_2^{-1},\,\psi)}{\varepsilon(1-s_1,\,\xi_1^{-1},\,\psi)}\cdot\frac{L(s_1-s_2,\,\xi_1\xi_2^{-1})}{L(1-s_1+s_2,\,\xi_1^{-1}\xi_2)}. \tag{39}$$

Here ψ is an arbitrary non-trivial character of F.

- *If χ_1 and χ_2 are nonramified, then*

$$c_1 = \frac{1-\alpha\beta^{-1}q^{-1}}{1-\alpha\beta^{-1}}, \tag{40}$$

where $\alpha = \chi_1(\varpi)$, $\beta = \chi_2(\varpi)$.

Proof: First assume that χ_1 or χ_2 is ramified. It follows from (3) that the value of the right side of (39) does not change if we replace ψ by ψ^a, $a \in F^*$ (note $c(\psi^a) = c(\psi) - v(a)$). We may therefore assume that $c(\psi) = 0$. Then it follows from Lemma 2.2.1 and (4) that

$$\Lambda_\psi f = \varepsilon(1-s_2,\,\xi_2^{-1},\,\psi^{-1}). \tag{41}$$

Consider the diagram

$$\begin{tikzcd} V(\chi_1,\chi_2) \arrow[rr, "M"] \arrow[dr, "\Lambda_\psi"'] & & V(\chi_2,\chi_1) \arrow[dl, "\tilde{\Lambda}_\psi"] \\ & \mathbb{C} & \end{tikzcd} \tag{42}$$

Here $\tilde{\Lambda}_\psi$ is the ψ–Whittaker functional defined on $V(\chi_2,\chi_1)$ by the same formula as in (27). The diagram (42) is not commutative, but since Whittaker models are unique, there exists a constant c_2 such that $\tilde{\Lambda}_\psi \circ M = c_2\Lambda_\psi$. This constant is computed in [Bu], Proposition 4.5.9. It is given by

$$c_2 = \gamma(1 - s_1 + s_2,\, \xi_1^{-1}\xi_2,\, \psi^{-1}), \tag{43}$$

where

$$\gamma(s,\,\chi,\,\psi) = \frac{\varepsilon(s,\,\chi,\,\psi)L(1-s,\,\chi^{-1})}{L(s,\,\chi)}.$$

Our assertion now follows from (41), the analogous formula $\tilde{\Lambda}_\psi f = \varepsilon(1-s_1,\,\xi_1^{-1},\,\psi^{-1})$, and $\tilde{\Lambda}_\psi(Mf) = c_2\Lambda_\psi f$. Note that by (3) we can replace ψ^{-1} by ψ in the final formula.

The argument in the unramified case is the same, but instead of (41) one uses $\Lambda_\psi f = 1 - \alpha\beta^{-1}q^{-1}$, see (31) and (32). ∎

Remarks: a) For unramified χ_1 and χ_2, formula (39) does *not* reduce to (40).

b) Had we not used the measure with $\mathrm{vol}(\mathfrak{o}) = 1$ but the self-dual measure with respect to ψ in the definition (38) of the intertwining operator, then the factor $q^{-c(\psi)/2}$ in (39) would not appear.

c) Formula (40) is given in various places, e.g. in [Bu], Proposition 4.6.7.

2.3 Dihedral supercuspidal representations

In section 1.2 we have defined the supercuspidal representations ω_ξ associated with regular characters of E^*, where E/F is a quadratic extension. In the following we shall compute the conductor of ω_ξ and the local newform in the standard model. For further results, see [GK] and [Tu]. The formula for the conductor (see Theorem 2.3.2) was also obtained in [Sh], Proposition 1.

Notations. We shall be dealing with quadratic extensions E/F of p–adic fields, for which we employ the following notations.

	E	F
ring of integers	$\mathfrak{O}$	$\mathfrak{o}$
maximal ideal	$\mathfrak{P}$	$\mathfrak{p}$
uniformizer	Ω	ϖ
normalized valuation	v_E	v_F
cardinality of residue field	q^f	q

Here $f = f(E/F)$ is the degree of the residue field extension. We further let $e = e(E/F)$ be the ramification index, so that $ef = 2$.

Let ψ be a non-trivial character of the non-archimedean local field F. As before, we let $c(\psi)$ be the exponent of the conductor of ψ. This means that ψ is trivial on $\mathfrak{p}^{c(\psi)}$ but not on $\mathfrak{p}^{c(\psi)-1}$. We define a character ψ_E on the quadratic extension E by

$$\psi_E := \psi \circ \mathrm{tr}_{E/F}.$$

Let $d(E/F)$ be the valuation of the discriminant of the field extension E/F. We have $d(E/F) = 0$ if E/F is unramified, and $d(E/F) = 1$ if E/F is ramified and q is odd. The following lemma gives the conductor of ψ_E in terms of the conductor of ψ. It holds more generally for *any* extension of local fields.

2.3.1 Lemma. *With the above notations we have*

$$c(\psi_E) = e(E/F)\, c(\psi) - f(E/F)^{-1} d(E/F). \tag{44}$$

In particular, if E/F is unramified, then $c(\psi_E) = c(\psi)$.

Proof: Let $\mathfrak{D}$ be the different of E/F. This is an integral ideal of $\mathfrak{O}$, its inverse being defined by

$$\mathfrak{D}^{-1} = \{x \in E : \operatorname{tr}(x\mathfrak{O}) \subset \mathfrak{o}\}.$$

Let $m \geq 0$ such that $\mathfrak{D} = \Omega^m \mathfrak{O}$. We have

$$\begin{aligned} \psi_E(\Omega^l \mathfrak{O}) = 1 \quad &\Longleftrightarrow \quad \psi(\operatorname{tr}(\Omega^l \mathfrak{O})) = 1 \\ &\Longleftrightarrow \quad \operatorname{tr}(\Omega^l \mathfrak{O}) \subset \varpi^{c(\psi)} \mathfrak{o} \\ &\Longleftrightarrow \quad \operatorname{tr}(\Omega^{l-ec(\psi)} \mathfrak{O}) \subset \mathfrak{o} \\ &\Longleftrightarrow \quad \Omega^{l-ec(\psi)} \mathfrak{O} \subset \mathfrak{D}^{-1} \\ &\Longleftrightarrow \quad m \geq e\, c(\psi) - l. \end{aligned}$$

The norm of the different is the discriminant, i.e., $d(E/F) = mf(E/F)$. The assertion follows. ∎

2.3.2 Theorem. *Let E/F be a quadratic extension of p–adic fields and ξ a character of E^* that does not factor through $N_{E/F}$. The conductor of the supercuspidal representation ω_ξ is*

$$c(\omega_\xi) = f(E/F) c(\xi) + d(E/F), \tag{45}$$

where $f(E/F)$ is the degree of the residue field extension. In particular:

i) If E/F is unramified, then $c(\omega_\xi) = 2c(\xi)$.

ii) If the residue characteristic of F is odd and E/F is ramified, then $c(\omega_\xi) = c(\xi) + 1$.

Proof: By [JL], Theorem 4.7, there exists a constant γ, independent of s, such that

$$\varepsilon(s, \omega_\xi, \psi_F) = \gamma \cdot \varepsilon(s, \xi, \psi_E), \qquad \psi_E = \psi_F \circ \operatorname{tr}_{E/F}.$$

By (9) we find

$$q^{(2c(\psi_F) - c(\omega_\xi))s} = q^{f(c(\psi_E) - c(\xi))s},$$

thus $2c(\psi_F) - c(\omega_\xi) = f(c(\psi_E) - c(\xi))$. Therefore the assertion follows by Lemma 2.3.1. ∎

We note that formula (45) coincides with the formula for the Artin conductor of a two-dimensional induced representation of a local Galois group, see [Se], §4.3, Proposition 4 b). Knowing the conductor, we would now like to compute the local newform in the standard model of ω_ξ. First we compute a newvector in $\omega_{\xi,\psi}$.

2.3.3 Lemma. *Let ω_E be the Weil representation of* $\mathrm{GL}(2,F)^+$ *on the Schwartz space $C_c^\infty(E)$, defined using a character ψ of F with conductor $\mathfrak{o}$. Then the characteristic function $f = \mathbf{1}_{v_0+\mathfrak{P}^m} \in C_c^\infty(E)$, where $v_0 \in E$ and $m \geq 0$ are arbitrary, is invariant under all operators $\omega_E\begin{pmatrix}1&0\\c&1\end{pmatrix}$ with $c \in \mathfrak{p}^{f(E/F)m+d(E/F)}$.*

Proof: Since $\begin{pmatrix}1&0\\c&1\end{pmatrix} = -w_1\begin{pmatrix}1&-c\\0&1\end{pmatrix}w_1$, we compute

$$\begin{aligned}
\left(\omega_E\begin{pmatrix}1&0\\c&1\end{pmatrix}f\right)(v) &= \chi(-1)\left(\omega_E(w_1)\omega_E\begin{pmatrix}1&-c\\0&1\end{pmatrix}\omega_E(w_1)f\right)(-v)\\
&= \chi(-1)\gamma\int\limits_E\left(\omega_E\begin{pmatrix}1&-c\\0&1\end{pmatrix}\omega_E(w_1)f\right)(u)\,\psi_E(-u\bar v)\,du\\
&= \chi(-1)\gamma\int\limits_E\psi_E(-u\bar v)\,\psi_F(-cN(u))\big(\omega_E(w_1)f\big)(u)\,du\\
&= \chi(-1)\gamma^2\int\limits_E\int\limits_E\psi_E(u'\bar u)\,\psi_E(-u\bar v)\,\psi_F(-cN(u))f(u')\,du'\,du\\
&\overset{(17)}{=}\int\limits_E\int\limits_{\mathfrak{P}^m}\psi_E((u'+v_0)\bar u)\,\psi_E(-u\bar v)\,\psi_F(-cN(u))\,du'\,du.
\end{aligned}$$

By Lemma 2.3.1 and our hypothesis on ψ_F, the conductor of ψ_E is $\mathfrak{P}^{-d(E/F)/f(E/F)}$. Therefore

$$\int\limits_{\mathfrak{P}^m}\psi_E(u'\bar u)\,du' = \begin{cases}\mathrm{vol}(\mathfrak{P}^m)\overset{(16)}{=}q^{-f(E/F)m-d(E/F)/2} & \text{if } u\in\mathfrak{P}^{-m-d(E/F)/f(E/F)},\\ 0 & \text{if } u\notin\mathfrak{P}^{-m-d(E/F)/f(E/F)}\end{cases}$$

(see (16); note that $\#\mathfrak{O}/\mathfrak{P} = q^{f(E/F)}$). Hence our integral equals

$$\left(\omega_E\begin{pmatrix}1&0\\c&1\end{pmatrix}f\right)(v) = q^{-f(E/F)m-d(E/F)/2}\int\limits_{\mathfrak{P}^{-m-d(E/F)/f(E/F)}}\psi_E(u(\bar v_0-\bar v))\,\psi_F(-cN(u))\,du.$$

Now, if $v_F(c) \geq f(E/F)m + d(E/F)$, then

$$\begin{aligned}
v_F(-cN(u)) &= v_F(c) + e^{-1}v_E(u\bar u) = v_F(c) + 2e^{-1}v_E(u)\\
&\geq v_F(c) + 2e^{-1}(-m - d(E/F)/f(E/F))\\
&= v_F(c) - f(E/F)m - d(E/F) \geq 0,
\end{aligned}$$

so that the term $\psi_F(-cN(u))$ in the integral can be omitted. Our assertion therefore follows from

$$\begin{aligned}
&\int\limits_{\mathfrak{P}^{-m-d(E/F)/f(E/F)}}\psi_E(u(\bar v_0-\bar v))\,du\\
&\qquad = \begin{cases}\mathrm{vol}(\mathfrak{P}^{-m-d(E/F)/f(E/F)}) = q^{f(E/F)m+d(E/F)/2} & \text{if } v_0-v\in\mathfrak{P}^m,\\ 0 & \text{if } v_0-v\notin\mathfrak{P}^m.\end{cases}
\end{aligned}$$

■

2.3.4 Proposition. *Let (ω_E, U_ξ) be the representation of* $\mathrm{GL}(2,F)^+$ *constructed as above from a character ψ of F with conductor $\mathfrak{o}$ and a character ξ of E^*. Then the element*

$$\varphi_0(v) = \begin{cases} \xi^{-1}(v) & \text{if } v \in \mathfrak{O}^*, \\ 0 & \text{if } v \notin \mathfrak{O}^*, \end{cases}$$

of U_ξ is invariant under $K_2\big(f(E/F)c(\xi) + d(E/F)\big) \cap \mathrm{GL}(2,F)^+$.

Proof: Let $H = K_2\big(fc(\xi) + d(E/F)\big) \cap \mathrm{GL}(2,F)^+$. By the Iwahori decomposition,

$$H = (H \cap N_-(F))(H \cap T(F))(H \cap N(F)),$$

where N (resp. N_-, resp. T) denotes unipotent upper triangular (resp. unipotent lower triangular, resp. diagonal) matrices.

The function φ_0 is obviously an element of U_ξ. It has the property that

$$\varphi_0(v + x\varpi^{c(\xi)}) = \varphi_0(v(1 + v^{-1}x\varpi^{c(\xi)})) = \xi(1 + v^{-1}x\varpi^{c(\xi)})\varphi_0(v) = \varphi_0(v) \qquad \text{for all } x \in \mathfrak{O}.$$

It follows that φ_0 is a linear combination of functions of the form $\mathbf{1}_{v_0 + \mathfrak{P}^{c(\xi)}}$. By Lemma 2.3.3, φ_0 is fixed by all elements $\omega_E\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ with $v_F(c) \geq f(E/F)c(\xi) + d(E/F)$, i.e., by all elements of $H \cap N_-(F)$. Since φ_0 is supported in $\mathfrak{O}$, it is further invariant under all $\omega_E\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ with $b \in \mathfrak{o}$. Since

$$1 + \mathfrak{p}^{f(E/F)c(\xi)+d(E/F)} \subset N_{E/F}E^*,$$

it is clear that φ_0 is fixed by elements of the form $\omega_E\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ with $a \in 1 + \mathfrak{p}^{f(E/F)c(\xi)+d(E/F)}$. By (18), φ_0 is also fixed by $\omega_E\begin{pmatrix} a & \\ & 1 \end{pmatrix}$, where $a \in \mathfrak{o}^* \cap N_{E/F}E^*$. Thus it is fixed by $H \cap T(F)$. ∎

The standard model V_ξ of the representation ω_ξ consists of functions $f : \mathrm{GL}(2,F) \to U_\xi$ with the property

$$f(hg) = \omega_{\xi,\psi}(h)f(g) \qquad \text{for all } h \in \mathrm{GL}(2,F)^+. \tag{46}$$

Let us fix an element $x \in F^*$ that is not a norm from E. If E/F is ramified, we choose x to be a unit. The matrix

$$t = \begin{pmatrix} x & \\ & 1 \end{pmatrix} \in \mathrm{GL}(2,F)$$

is not an element of $\mathrm{GL}(2,F)^+$. Every element $f \in V_\xi$ is determined by its values $f(\mathbf{1})$ and $f(t)$. Conversely, given arbitrary vectors $\varphi_1, \varphi_2 \in U_\xi$, there exists a unique $f \in V_\xi$ such that $f(\mathbf{1}) = \varphi_1$ and $f(t) = \varphi_2$. Explicitly, it is given by

$$f(g) = \begin{cases} \omega_{\xi,\psi}(g)\varphi_1, & \text{if } g \in \mathrm{GL}(2,F)^+, \\ \omega_{\xi,\psi}(gt^{-1})\varphi_2, & \text{if } g \notin \mathrm{GL}(2,F)^+. \end{cases} \tag{47}$$

2.3.5 Theorem. *With the above notations, we define a function $f \in V_\xi$ as follows. Let $\varphi_0 \in U_\xi$ be the function defined in Proposition 2.3.4 (the character ψ is assumed to have conductor $\mathfrak{o}$).*

- *If E/F is unramified, then let*

$$\varphi_1 = \varphi_0, \qquad \varphi_2 = 0.$$

- *If E/F is ramified, then let*

$$\varphi_1 = \varphi_2 = \varphi_0.$$

Then the function f as in (47) is a newvector in the standard model V_ξ of ω_ξ.

Proof: We have to check that f is invariant under $H = K_2(f(E/F)c(\xi) + d(E/F))$, see Theorem 2.3.2. It is easy to see that f is invariant under $H^+ = H \cap \mathrm{GL}(2,F)^+$. In the unramified case we have $H = H^+$ since every unit is a norm, so we are done. In the ramified case we have to check in addition that f is invariant under $\omega_\xi(t)$. This is a very easy computation. ∎

2.4 The newform in the Kirillov model

Again choose a non-trivial additive character ψ of F. We know from [JL] that for every irreducible, admissible representation π of $\mathrm{GL}(2,F)$ there is a unique space $\mathcal{K}(\pi,\psi)$ of locally constant functions $\phi:\ F^* \to \mathbb{C}$ with the following property: $\mathrm{GL}(2,F)$ acts on $\mathcal{K}(\pi,\psi)$ in a way such that

$$\left(\begin{pmatrix} a & b \\ & d \end{pmatrix}\phi\right)(x) = \omega_\pi(d)\psi(bx/d)\phi(ax/d), \tag{48}$$

and the resulting representation of $\mathrm{GL}(2,F)$ is equivalent to π (here ω_π is the central character of π). This is the *Kirillov model* of π (with respect to ψ). If V is the space of π, and if Λ_ψ is a ψ–Whittaker functional on V, then $\mathcal{K}(\pi,\psi)$ consists of all functions ϕ_v, $v \in V$, where

$$\phi_v(a) = \Lambda_\psi\left(\pi\begin{pmatrix} a & \\ & 1 \end{pmatrix} v\right), \qquad a \in F^*.$$

It is known that $\mathcal{K}(\pi,\psi)$ contains the space $\mathcal{S}(F^*)$ of locally constant, compactly supported functions in F^*, and that there is equality if and only if π is supercuspidal. We shall now determine the local newform in the Kirillov model.

Induced representations

Assume that $\chi_1\chi_2^{-1} \neq |\,|^{\pm 1}$, so that $\pi = \pi(\chi_1,\chi_2)$ is irreducible. For every $f \in V(\chi_1,\chi_2)$ consider the function ϕ_f on F^* given by

$$\phi_f(a) = \Lambda_\psi\left(\pi\begin{pmatrix} a & \\ & 1 \end{pmatrix} f\right),$$

where Λ_ψ is as in (27). The space of all such ϕ_f is the Kirillov model $\mathcal{K}(\pi,\psi)$ of π with respect to ψ. A straightforward computation shows that

$$\phi_f(a) = \chi_2(a)|a|^{1/2}\Lambda_{\psi^a} f, \qquad \text{where } \psi^a(x) = \psi(ax).$$

Since $c(\psi^a) = c(\psi) - v(a)$, Lemma 2.2.1 is equivalent to the computation of the local newform in the Kirillov model. For simplicity, choose ψ to have conductor $\mathfrak{o}$. Let f be the newform given by (20).

- If χ_1 and χ_2 are ramified, then

$$\phi_f(a) = \begin{cases} \varepsilon(1 - s_2, \xi_2^{-1}, \psi^{-1}), & \text{if } v(a) = 0, \\ 0, & \text{if } v(a) \neq 0. \end{cases} \tag{49}$$

- If χ_1 is unramified and χ_2 is ramified, then

$$\phi_f(a) = \begin{cases} \chi_1(a)|a|^{1/2}\,\varepsilon(1 - s_2, \xi_2^{-1}, \psi^{-1}), & \text{if } v(a) \geq 0, \\ 0, & \text{if } v(a) < 0. \end{cases} \tag{50}$$

- If χ_1 is ramified and χ_2 is unramified, then

$$\phi_f(a) = \begin{cases} \chi_2(a)|a|^{1/2}, & \text{if } v(a) \geq 0, \\ 0, & \text{if } v(a) < 0. \end{cases} \tag{51}$$

- If χ_1 and χ_2 are unramified, let $\alpha = \chi_1(\varpi)$, $\beta = \chi_2(\varpi)$. If $\alpha \neq \beta$, then

$$\phi_f(a) = \begin{cases} (1 - \alpha\beta^{-1}q^{-1})|a|^{1/2}\frac{\alpha^{v(a)+1} - \beta^{v(a)+1}}{\alpha - \beta}, & \text{if } v(a) \geq 0, \\ 0, & \text{if } v(a) < 0, \end{cases} \tag{52}$$

  while if $\alpha = \beta$, then

$$\phi_f(a) = \begin{cases} (1 - q^{-1})|a|^{1/2}\alpha^{v(a)}(1 + v(a)) & \text{if } v(a) \geq 0 \\ 0, & \text{if } v(a) < 0. \end{cases} \tag{53}$$

  If we interpret the quotient in (52) as $\alpha^{v(a)} + \alpha^{v(a)-1}\beta + \ldots + \beta^{v(a)}$, then (53) is actually a special case of (52).

We can make similar computations for the special representations, using (36) and (37). The result is as follows.

- If χ is unramified, then a local newform ϕ in the Kirillov model of $\chi\mathrm{St}$ is given by

$$\phi(a) = \begin{cases} \chi(a)|a|, & \text{if } v(a) \geq 0, \\ 0, & \text{if } v(a) < 0. \end{cases} \tag{54}$$

- If χ is ramified, then a local newform in $\mathcal{K}(\chi\mathrm{St}, \psi)$ is given as the characteristic function of $\mathfrak{o}^*$.

Supercuspidal representations

Let E/F be a quadratic extension and ξ a character of E^* that is not trivial on the kernel of $N_{E/F}$. As in the previous section, let V_ξ be standard model of the supercuspidal representation ω_ξ. It consists of functions $f:\ \mathrm{GL}(2,F)\to U_\xi$ with the transformation property (46). From the definition of ω_ξ and $\omega_{\xi,\psi}$, in particular from (13), it is immediate that

$$\Lambda_\psi f = \big(f(\mathbf{1})\big)(1_E), \qquad\qquad f\in V_\xi,$$

is a ψ–Whittaker functional on V_ξ. Here ψ is the character of F that was used to define the Weil representation.

Let f be the newform in V_ξ as given by Theorem 2.3.5. Then it is an easy computation that

$$\Lambda_\psi\left(\omega_\xi\begin{pmatrix} a & \\ & 1\end{pmatrix}f\right)=\begin{cases}1, & \text{if } v(a)=0,\\ 0, & \text{if } v(a)\neq 0\end{cases}\qquad\qquad (a\in F^*).$$

Therefore the newform in the Kirillov model is given as the characteristic function of $\mathfrak{o}^*$.

This result was also obtained in [Sh]. It is in fact true for *every* supercuspidal representation that $\mathfrak{o}^*$ is the newform in the Kirillov model. This was proved in [GP], section 4, but we shall sketch an elementary proof that is taken from [Bo]. Thus let π be a supercuspidal representation of $\mathrm{GL}(2,F)$ and $\mathcal{K}$ its Kirillov model with respect to a character ψ of F with conductor $\mathfrak{o}$. Since π is supercuspidal, we have $\mathcal{K}=\mathcal{S}(F^*)$. Denote by $\mathcal{K}^{(m)}$ the subspace of vectors fixed under $K_2(m)$ (see (1)). Let n be the level of π and f a generator for $\mathcal{K}^{(n)}$, i.e., f is the local newform that we are trying to determine. It follows from (48) that f is supported on $\mathfrak{o}$ and that $f(x)$ depends only on the valuation of x. It follows that we can write

$$f=\sum_{j=0}^{m} d_j\mathbf{1}_{\varpi^j\mathfrak{o}^*}=\sum_{j=0}^{r} d_j\,\pi\begin{pmatrix}\varpi^{-j} & \\ & 1\end{pmatrix}\mathbf{1}_{\mathfrak{o}^*},\qquad\qquad m\geq 0,\ d_j=f(\varpi^j)\in\mathbb{C}. \tag{55}$$

Now $\mathbf{1}_{\mathfrak{o}^*}$, being a smooth vector and being fixed by $B(\mathfrak{o})=\begin{pmatrix}\mathfrak{o}^* & \mathfrak{o}\\ & \mathfrak{o}^*\end{pmatrix}$, lies in $\mathcal{K}^{(n+m)}$ for some $m\geq 0$. A basis of this space is easily seen to be

$$f,\ \pi\begin{pmatrix}\varpi^{-1} & \\ & 1\end{pmatrix}f,\ \ldots,\ \pi\begin{pmatrix}\varpi^{-m} & \\ & 1\end{pmatrix}f,$$

see Theorem 1.2.1 ii). Accordingly we can write

$$\mathbf{1}_{\mathfrak{o}^*}=\sum_{j=0}^{m} c_j\,\pi\begin{pmatrix}\varpi^{-j} & \\ & 1\end{pmatrix}f,\qquad\qquad m\geq 0,\ c_j\in\mathbb{C}. \tag{56}$$

We may assume that the m's in (55) and (56) are the same. We normalize f such that $f(1)=c_0=1$. Then also $d_0=1$. Subtracting $\mathbf{1}_{\mathfrak{o}^*}$ from (55) and f from (56) and adding the two resulting equations, we get

$$\sum_{j=1}^{m}\Big(c_jf(\varpi^{-j}x)+d_j\mathbf{1}_{\mathfrak{o}^*}(\varpi^{-j}x)\Big)=0\qquad\qquad \text{for all } x\in F^*.$$

Substituting ϖ^i for x for $i \in \{1,\ldots,m\}$ yields $d_i + \sum_{j=1}^m c_j d_{i-j} = 0$, or

$$\sum_{j=0}^{i} c_j d_{i-j} = 0 \qquad \text{for all } i = 1,\ldots,m. \tag{57}$$

Similarly, substituting ϖ^i for x for $i \in \{m+1,\ldots,2m\}$ yields

$$\sum_{j=i-m}^{m} c_j d_{i-j} = 0 \qquad \text{for all } i = m+1,\ldots,2m. \tag{58}$$

One can now prove by induction on m that if complex numbers c_j, d_j fulfill the equations (57) and (58), and if $c_0 = d_0 = 1$, then $c_j = d_j = 0$ for $j \geq 1$. This shows that $f = \mathbf{1}_{\mathfrak{o}^*}$.

Summary

The following table summarizes the shape of the newform in the Kirillov model (we have made convenient normalizations). The character ψ used in the construction of $\mathcal{K}(\pi,\psi)$ is always assumed to have conductor $\mathfrak{o}$. See also section 3 of [Ro], where a very similar table is given.

representation	local newform
$\pi(\chi_1,\chi_2)$, $\chi_1\chi_2^{-1} \neq \lvert\,\rvert^{\pm 1}$, χ_i unramified	$\lvert x\rvert^{1/2}\Big(\sum\limits_{k+l=v(x)} \chi_1(\varpi^k)\chi_2(\varpi^l)\Big)\mathbf{1}_{\mathfrak{o}}(x)$
$\pi(\chi_1,\chi_2)$, $\chi_1\chi_2^{-1} \neq \lvert\,\rvert^{\pm 1}$, χ_1 ram., χ_2 unram.	$\lvert x\rvert^{1/2}\chi_2(x)\mathbf{1}_{\mathfrak{o}}(x)$
$\pi(\chi_1,\chi_2)$, $\chi_1\chi_2^{-1} \neq \lvert\,\rvert^{\pm 1}$, χ_1 unram., χ_2 ram.	$\lvert x\rvert^{1/2}\chi_1(x)\mathbf{1}_{\mathfrak{o}}(x)$
$\pi(\chi_1,\chi_2)$, $\chi_1\chi_2^{-1} \neq \lvert\,\rvert^{\pm 1}$, χ_i ramified	$\mathbf{1}_{\mathfrak{o}^*}(x)$
$\chi\mathrm{St}$, χ unramified	$\lvert x\rvert\chi(x)\mathbf{1}_{\mathfrak{o}}(x)$
$\chi\mathrm{St}$, χ ramified	$\mathbf{1}_{\mathfrak{o}^*}(x)$
supercuspidal	$\mathbf{1}_{\mathfrak{o}^*}(x)$

We note that in any case, if f is the newform in the Kirillov model, then $f(x)$ depends only on the valuation $v(x)$ of $x \in F^*$.

3 Representations of $\mathrm{PGL}(2,F)$

Since representations π of $\mathrm{PGL}(2,F)$ are self-dual, the value of the ε–factor at $1/2$ is just a sign ± 1. This sign $\varepsilon(1/2,\pi,\psi)$ is independent of ψ and therefore canonically attached to π. On the other hand, there is the so-called Atkin–Lehner involution acting on the one-dimensional space spanned by the newform. This also defines a sign, and we shall show that these two signs attached to π coincide. This was proved in a more classical context in [Fl], but we shall give a completely local proof.

3.1 Atkin–Lehner eigenvalues

Throughout F is a non-archimedean local field with $\mathfrak{o}$ its ring of integers and $\mathfrak{p} \subset \mathfrak{o}$ the maximal ideal. As before, let ϖ be a generator for $\mathfrak{o}$, and $q = \#\mathfrak{o}/\mathfrak{p}$.

Atkin–Lehner involutions

We recall the definition of the local Atkin–Lehner involutions. Let (π, V) be an irreducible, admissible representation of $\mathrm{PGL}(2,F)$, considered as a representation of $\mathrm{GL}(2,F)$ with trivial central character. Consider the congruence subgroup

$$K_0(\mathfrak{p}^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathfrak{o}) : \ c \in \mathfrak{p}^n \right\}.$$

This group is bigger than the group $K_2(n)$ defined in (1), but since the center acts trivially, a vector in V is $K_0(\mathfrak{p}^n)$–invariant if and only if it is $K_2(n)$–invariant.

Let $A_n = \begin{pmatrix} & 1 \\ \varpi^n & \end{pmatrix}$, where ϖ is a prime element. Then we have:

i) $K_0(\mathfrak{p}^n)$ is normalized by A_n.

ii) A_n^2 lies in the center of $\mathrm{GL}(2,F)$.

We see from i) that $\pi(A_n)$ induces an endomorphism B_n of $V^{(n)}$, the space of $K_0(\mathfrak{p}^n)$–invariant vectors in V. It follows from ii) that B_n^2 is the identity on $V^{(n)}$. We call B_n the *local Atkin-Lehner involution* of level n on π.

Now, if $\mathfrak{p}^{n_0}$ is the conductor of π, then B_{n_0} has eigenvalue 1 or -1 on the one-dimensional space $V^{(n_0)}$. This sign is canonically attached to π. We call it the *Atkin-Lehner eigenvalue* of π.

Principal series representations

We shall now determine the Atkin–Lehner eigenvalues of the principal series representations.

3.1.1 Proposition. *Let χ be a character of F^* with $\chi^2 \neq |\,|^{\pm 1}$. For the infinite-dimensional principal series representation $\pi = \pi(\chi, \chi^{-1})$ of $\mathrm{PGL}(2,F)$ the Atkin–Lehner eigenvalue is given by $\chi(-1)$.*

Proof: Since this is certainly true for unramified representations we may assume that the conductor of χ is $\mathfrak{p}^n$ with $n \geq 1$. Let f be the local newform in the standard model $V(\chi, \chi^{-1})$ of π. We compute

$$\begin{aligned} (\pi(A_{2n})f)\begin{pmatrix} 1 & \\ \varpi^n & 1 \end{pmatrix} &= f\left(\begin{pmatrix} 1 & \\ \varpi^n & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ \varpi^{2n} & \end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} -\varpi^n & \\ & \varpi^n \end{pmatrix}\begin{pmatrix} 1 & -\varpi^{-n} \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ \varpi^n & 1 \end{pmatrix}\right) \\ &= \chi(-1) f\begin{pmatrix} 1 & \\ \varpi^n & 1 \end{pmatrix}. \end{aligned}$$

By Proposition 2.1.2, f is nonzero on the element $\begin{pmatrix} 1 & \\ \varpi^n & 1 \end{pmatrix}$. We can therefore conclude that the Atkin–Lehner eigenvalue is $\chi(-1)$. ∎

Special representations

The Steinberg representation St of $\mathrm{GL}(2,F)$ is the unique non-trivial irreducible subrepresentation of the standard induced representation $V(|\,|^{1/2},|\,|^{-1/2})$ (the quotient is the trivial representation). We shall now determine the Atkin–Lehner eigenvalue of St and its quadratic twists $\chi\mathrm{St}$. Note that $\chi\mathrm{St}$ appears as the unique non-trivial subrepresentation of $V(\chi|\,|^{1/2},\chi|\,|^{-1/2})$.

3.1.2 Proposition. *Let χ be a quadratic character of F^*.*

i) If χ is unramified, then the Atkin–Lehner eigenvalue of $\chi\mathrm{St}$ is $-\chi(\varpi)$, where ϖ is a uniformizer. In particular, the Atkin–Lehner eigenvalue of St is -1.

ii) If χ is ramified, then the Atkin–Lehner eigenvalue of $\chi\mathrm{St}$ is $\chi(-1)$.

Proof: i) In section 2.1 we have already computed the local newform in the model of $\chi\mathrm{St}$ that is given as a subspace of $V(\chi|\,|^{1/2},\chi|\,|^{-1/2})$. It is the unique function $f\in V(\chi|\,|^{1/2},\chi|\,|^{-1/2})$ that is right $K_0(\mathfrak{p})$–invariant and has $f(\mathbf{1})=q$, $f(w)=-1$. If ρ denotes right translation, it is now very easily computed that $\rho\begin{pmatrix} & 1\\ \varpi & \end{pmatrix}f=-\chi(\varpi)f$.

ii) We have proved in section 2.1 that the function exhibited in (20) lies in the subspace that defines $\chi\mathrm{St}$. The calculation is therefore the same as in the case of the principal series representations. ■

3.2 Equality of the signs

The local ε–factor also attaches a sign to an irreducible, admissible representation of $\mathrm{PGL}(2,F)$, and we shall prove that it coincides with the Atkin–Lehner eigenvalue. Recall the following basic result.

3.2.1 Lemma. *Let π be an irreducible, admissible representation of $\mathrm{PGL}(2,F)$ with conductor $\mathfrak{p}^n$, $n\geq 0$. Let ψ be a character of F with conductor $\mathfrak{p}^{c(\psi)}$. Then there exists an $\varepsilon\in\{\pm1\}$, independent of ψ, such that the ε-factor of π is given by*

$$\varepsilon(s,\pi,\psi)=\varepsilon q^{(2c(\psi)-n)(s-1/2)}.$$

In particular, if ψ has conductor $\mathfrak{o}$, then $\varepsilon(s,\pi,\psi)=\varepsilon q^{n(1/2-s)}$.

Proof: In general it is known that

$$\varepsilon(s,\pi,\psi)=cq^{(2c(\psi)-n)s}\qquad\text{for some } c\in\mathbb{C}\tag{59}$$

(see (9), or [Cas], p. 307). There is also the general formula

$$\varepsilon(s,\pi,\psi)\varepsilon(1-s,\hat{\pi},\psi)=\omega_\pi(-1)\tag{60}$$

(see (12), or [JL], p. 77). But π is self-dual, and $\omega_\pi=1$, thus

$$\varepsilon\Big(\frac{1}{2},\pi,\psi\Big)=\varepsilon\in\{\pm1\}.$$

It is further known that this sign is independent of ψ (follows from (10), see also [Ta2] (3.4.4)). Putting $s = \frac{1}{2}$ in (59), it follows that

$$c = \varepsilon q^{n/2 - c(\psi)}.$$

Substituting this into (59) gives the result. ∎

In particular, for every π there is a sign

$$\varepsilon\Big(\frac{1}{2}, \pi\Big) := \varepsilon\Big(\frac{1}{2}, \pi, \psi\Big) \in \{\pm 1\}$$

depending only on π and not on ψ.

3.2.2 Theorem. *For each infinite-dimensional, irreducible, admissible representation π of* PGL$(2, F)$, *the Atkin-Lehner eigenvalue equals $\varepsilon(1/2, \pi)$.*

Proof: The assertion is true if π is unramified (spherical), since then $\varepsilon(s, \pi, \psi) = 1$ for ψ with conductor $\mathfrak{o}$ and the Atkin-Lehner involution is trivial. Let us therefore assume that $n \geq 1$, where $\mathfrak{p}^n$ is the conductor of π. We shall compute the action of the Atkin-Lehner involution in the Kirillov model. According to the definition we have to compute the action of

$$A_n = aw, \qquad \text{where} \qquad a = \begin{pmatrix} 1 & \\ & -\varpi^n \end{pmatrix}, \qquad w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

We shall use the following description of the action of w, to be found in [JL], Proposition 2.10. For a function $f \in \mathcal{K}(\pi, \psi)$ and a character ν of $\mathfrak{o}^*$ define

$$\hat{f}_m(\nu) = \int\limits_{\mathfrak{o}^*} f(u\varpi^m)\nu(u)\,du, \qquad\qquad m \in \mathbb{Z},$$

$$\hat{f}(\nu, t) = \sum_{m \in \mathbb{Z}} \hat{f}_m(\nu) t^m.$$

The function f is determined by the power series $\hat{f}(\nu, t)$ for all ν. The action of w is characterized by

$$\widehat{(wf)}(\nu, t) = C(\nu, t)\hat{f}(\nu^{-1}, t^{-1}), \tag{61}$$

where $C(\nu, t)$ is a formal series such that

$$C(\mathbf{1}, q^{s-1/2}) = \frac{L(1-s, \pi)\varepsilon(s, \pi, \psi)}{L(s, \pi)}, \qquad\qquad \mathbf{1} \text{ the trivial character of } \mathfrak{o}^*$$

($L(s, \pi)$ is the standard local Euler factor attached to π).

Case $n \geq 2$.

In this case we have $L(s, \pi) = 1$ and, using Lemma 3.2.1,

$$C(\mathbf{1}, q^{s-1/2}) = \varepsilon(1/2, \pi) q^{-n(s-1/2)}.$$

Consequently $C(\mathbf{1},t) = \varepsilon(1/2,\pi)t^{-n}$. By the table in section 2.4 the local newform is under our assumption $n \geq 2$ given as the characteristic function of $\mathfrak{o}^*$,

$$f = \mathbf{1}_{\mathfrak{o}^*}.$$

It is immediate that

$$\hat{f}_m(\nu) = \begin{cases} 1 & \text{if } m = 0 \text{ and } \nu = \mathbf{1}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\hat{f}(\nu,t) = 0$ unless $\nu = \mathbf{1}$, where we have $\hat{f}(\nu,t) = 1$. By (61), $\widehat{(wf)}(\nu,t)$ also vanishes except for trivial ν, where

$$\widehat{(wf)}(\mathbf{1},t) = C(\mathbf{1},t) = \varepsilon(1/2,\pi)t^{-n}.$$

Now a function g such that $\hat{g}(\nu,t) = 0$ for $\nu \neq \mathbf{1}$ and $\hat{g}(\mathbf{1},t) = t^{-n}$ is easily seen to be given by $g(x) = \mathbf{1}_{\mathfrak{o}^*}(\varpi^n x)$. It follows that

$$(wf)(x) = \varepsilon(1/2,\pi)\mathbf{1}_{\mathfrak{o}^*}(\varpi^n x).$$

By (48) it is then obvious that

$$(awf)(x) = \varepsilon(1/2,\pi)\mathbf{1}_{\mathfrak{o}^*}(x).$$

This is the local newform again, up to the factor $\varepsilon(1/2,\pi)$, completing the proof in the case $n \geq 2$.

Case $n = 1$.

This is the case when $\pi = \chi\mathrm{St}$ is a special representation with an unramified quadratic character χ. According to the table in section 2.4 the newform is given by

$$f(x) = |x|\,\chi(x)\mathbf{1}_{\mathfrak{o}}(x).$$

Since χ is unramified it is clear that $\hat{f}_m(\nu) = 0$ if ν is non-trivial. It is also immediate that $\hat{f}_m(\mathbf{1}) = \chi(\varpi)^m q^{-m}$ for $m \geq 0$, and zero for $m < 0$. Thus

$$\hat{f}(\mathbf{1},t) = \sum_{m \geq 0} \chi(\varpi)^m q^{-m} t^m = \frac{1}{1 - \chi(\varpi)q^{-1}t}.$$

The Euler factor is $L(s,\pi) = (1 - \chi(\varpi)q^{-s-1/2})^{-1}$ (see [JL] Proposition 3.6), so

$$C(\mathbf{1},q^{s-1/2}) = \varepsilon(1/2,\pi)q^{-(s-1/2)}\frac{1 - \chi(\varpi)q^{-(s-1/2)-1}}{1 - \chi(\varpi)q^{s-1/2-1}}.$$

It follows that $C(\mathbf{1},t) = \varepsilon(1/2,\pi)t^{-1}\frac{1-\chi(\varpi)q^{-1}t^{-1}}{1-\chi(\varpi)q^{-1}t}$. By (61),

$$\begin{aligned}\widehat{(wf)}(\mathbf{1},t) = C(\mathbf{1},t)\frac{1}{1 - \chi(\varpi)q^{-1}t^{-1}} &= \varepsilon(1/2,\pi)t^{-1}\frac{1}{1 - \chi(\varpi)q^{-1}t} \\ &= \varepsilon(1/2,\pi)\sum_{m \geq 0} \chi(\varpi)^m q^{-m} t^{(m-1)}.\end{aligned}$$

For non-trivial ν we have $\widehat{(wf)}(\nu,t)=0$. It is easily computed that the function having these Fourier transforms is

$$(wf)(x)=\varepsilon(1/2,\pi)|x\varpi|\,\chi(x\varpi)\mathbf{1}_{\mathfrak{o}}(x\varpi).$$

From (48) we then get $(awf)(x)=\varepsilon(1/2,\pi)f(x)$. This concludes the proof. ∎

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