The Saito-Kurokawa Lifting and Functoriality

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Abstract. Certain non-tempered liftings from $\mathrm{PGL}(2) \times \mathrm{PGL}(2)$ to $\mathrm{PGSp}(4)$ are constructed using the theory of (local and global) theta lifts. The resulting representations on $\mathrm{PGSp}(4)$ are the Saito–Kurokawa representations. The lifting is shown to be functorial under certain reasonable assumptions on the local Langlands correspondence for $\mathrm{PGSp}(4)$.

Introduction

The classical Saito–Kurokawa lifting associates to each eigenform $f \in S_{2k-2}(\mathrm{SL}(2,\mathbb{Z}))$ with even k a cuspidal Siegel eigenform F of degree 2 and weight k such that the (finite parts of the) L–functions of f and F are related by the formula

$$L(s,F) = \zeta(s-k+1)\zeta(s-k+2)L(s,f)$$

(see [5], §6). Within the framework of functoriality of automorphic representations, the Saito–Kurokawa lifting can be explained as follows (see [17], §3). Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$. Let π_1 be the automorphic representation of $\mathrm{PGL}(2,\mathbb{A})$ corresponding to the eigenform f. Let π_2 be the *anomalous* automorphic representation of $\mathrm{PGL}(2,\mathbb{A})$ whose archimedean component is the lowest discrete series representation, and each of whose non-archimedean components is the trivial representation. We consider the (conjectural) lifting of $\mathrm{PGL}(2,\mathbb{A}) \times \mathrm{PGL}(2,\mathbb{A})$ to $\mathrm{PGSp}(4,\mathbb{A})$ coming from the standard embedding of L–groups

$$\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{Sp}(4,\mathbb{C}). \tag{1}$$

The image of the automorphic representation $\pi_1 \otimes \pi_2$ under this lifting turns out to be a (holomorphic) cusp form Π on $\mathrm{PGSp}(4,\mathbb{A})$ that corresponds to the Saito–Kurokawa lift F of f.

The main purpose of this paper is to prove the following generalization of the Saito–Kurokawa lifting. Let F be any number field and $\mathbb{A}$ its ring of adeles. Let $\pi = \otimes\pi_v$ be a cuspidal automorphic representation of $\mathrm{PGL}(2,\mathbb{A})$ and Σ the set of places v of F such that π_v is square integrable. In generalization of the above representation π_2 we shall define a global representation π_S of $\mathrm{PGL}(2,\mathbb{A})$ for any finite set of places S. Our basic lifting theorem (Theorem 3.1) states that *if $S \subset \Sigma$ and the parity of $\#S$ is such that $(-1)^{\#S} = \varepsilon(1/2,\pi)$, then the representation $\pi \otimes \pi_S$ of $\mathrm{PGL}(2,\mathbb{A}) \times \mathrm{PGL}(2,\mathbb{A})$ has a cuspidal lifting to $\mathrm{PGSp}(4,\mathbb{A})$*, except when $L(1/2,\pi) \neq 0$ and $S = \emptyset$, where the lifting exists but is not cuspidal. The class of representations so obtained coincides with the Saito–Kurokawa representations defined in [4] in terms of packets. The main point here is however to show that $\Pi(\pi \otimes \pi_S)$ is a functorial lifting of $\pi \otimes \pi_S$ with respect to the L–morphism (1). To prove this we have to make some reasonable assumptions on the conjectural local Langlands correspondence for $\mathrm{GSp}(4)$.

To prove the lifting theorem, we shall use the theory of local and global theta liftings as developed in [39], [40]. First we shall define local representations $\Pi(\pi_v \otimes \pi_{S,v})$ as theta liftings from the metaplectic group, and then piece them together to obtain the global lifting. To show that the global representation of $\mathrm{PGSp}(4,\mathbb{A})$ thus obtained is automorphic, we use Waldspurger's results, together with the description of the residual spectrum of $\mathrm{GSp}(4,\mathbb{A}_F)$ in [11]. The sign condition comes in since we argue with global "Waldspurger packets" on $\widetilde{\mathrm{SL}}(2,\mathbb{A})$.

There is a conjectural description of local L–packets on $\mathrm{GSp}(4, F_v)$ in terms of theta lifts from orthogonal groups, see [30], [38]. By definition, our local liftings are another type of theta lift coming from $\widetilde{\mathrm{SL}}(2, F_v)$. What we will prove is an identity between local theta lifts on $\mathrm{GSp}(4, F_v)$ coming from $\mathrm{GO}(X, F_v)$, where X is an anisotropic four-dimensional quadratic space with discriminant 1, and others coming from $\widetilde{\mathrm{SL}}(2, F_v)$ (Proposition 5.8). Assuming the above mentioned description of L–packets for $\mathrm{GSp}(4, F_v)$, this will show that our lifting $\pi \otimes \pi_S \mapsto \Pi(\pi \otimes \pi_S)$ is functorial at *every* place.

To prove this local theta identity, we use a global method and a result of PIATETSKI-SHAPIRO [22] that characterizes CAP representations on $\mathrm{GSp}(4, \mathbb{A})$ in terms of theta lifts from $\widetilde{\mathrm{SL}}(2, \mathbb{A})$. The argument only works if certain global theta lifts from $\mathrm{GO}(X, \mathbb{A})$ to $\mathrm{GSp}(4, \mathbb{A})$ do not vanish, where X is a four-dimensional quadratic space. To assure this, we will modify the non-vanishing theorems of ROBERTS [29], [30] to make them work in our (non-tempered) situation, see Theorem 5.4.

In section 1 we shall introduce the anomalous automorphic representations π_S. Section 2 introduces various groups and lifting maps that will be used in the following. In section 3 we shall prove the main lifting theorem but without establishing functoriality. Section 4 is devoted to the archimedean case, where the theta liftings can be computed explicitly. Since the archimedean local Langlands correspondence is known, functoriality is easily established in this case. In section 5 we shall prove the above mentioned local theta identity in the p–adic case. This implies our lifting is functorial also at the finite places, assuming what is currently conjectured about the local Langlands correspondence for $\mathrm{GSp}(4)$. In section 6 we shall also discuss a refinement of the base change theory for Saito–Kurokawa representations in [4]. In the final section we shall give more explicit information on our liftings in the p–adic case.

We mention that our results can be applied to holomorphic cusp forms $f \in S_{2k-2}(\Gamma_0(N))$, hence generalizing the classical Saito–Kurokawa lifting. Since the parity condition is essentially all that has to be observed in the choice of the set of places S above, a single modular form f can potentially have many such Saito–Kurokawa lifts F. The main difficulty is to control the level of F. This application to classical modular forms will be the subject of a separate paper.

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Notations and preliminaries

We shall set up notation and recall some basic facts about theta liftings that will be needed in this paper.

Groups

Let J_n denote the $n \times n$–matrix

$$J_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}. \tag{2}$$

We shall realize the orthogonal group $\mathrm{SO}(n)$ using this matrix, i.e., $\mathrm{SO}(n) = \{g \in \mathrm{SL}(n) : \, {}^tgJ_ng = J_n\}$. The symplectic group $\mathrm{Sp}(2n)$ and the similitude group $\mathrm{GSp}(2n)$ shall be realized using the matrix

$\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}$. In particular, we let

$$\mathrm{GSp}(4) = \{g \in \mathrm{GL}(4) : \, {}^t g J g = \lambda(g) J \text{ for some } \lambda(g) \in \mathrm{GL}(1)\}, \qquad J = \begin{pmatrix} & J_2 \\ -J_2 & \end{pmatrix}.$$

If not stated differently, the symbol G will abbreviate the group $\mathrm{GSp}(4)$ throughout the paper. As a Borel subgroup B of G we choose upper triangular matrices. The two conjugacy classes of proper maximal parabolic subgroups are represented by the *Siegel parabolic subgroup* P, whose Levi factor is

$$M_P = \left\{ \begin{pmatrix} A & \\ & uA' \end{pmatrix} : \, u \in \mathrm{GL}(1), \, A \in \mathrm{GL}(2) \right\} \simeq \mathrm{GL}(1) \times \mathrm{GL}(2),$$

where $A' := \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} {}^t A^{-1} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, and the *Klingen parabolic subgroup* Q, whose Levi factor is

$$M_Q = \left\{ \begin{pmatrix} u & & \\ & A & \\ & & u^{-1}\det(A) \end{pmatrix} : \, u \in \mathrm{GL}(1), \, A \in \mathrm{GL}(2) \right\} \simeq \mathrm{GL}(1) \times \mathrm{GL}(2).$$

Note that the Levi of the Siegel parabolic subgroup of $\mathrm{PGSp}(4)$ is isomorphic to $\mathrm{GL}(1) \times \mathrm{PGL}(2)$ via

$$M_P \longrightarrow \mathrm{GL}(1) \times \mathrm{PGL}(2), \qquad \begin{pmatrix} A & \\ & uA' \end{pmatrix} \longmapsto \big(u^{-1}\det(A), \, [A]\big), \tag{3}$$

and the Levi of the Klingen parabolic subgroup of $\mathrm{PGSp}(4)$ is isomorphic to $\mathrm{GL}(2)$ via

$$M_Q \longrightarrow \mathrm{GL}(2), \qquad \begin{pmatrix} u & & \\ & A & \\ & & u^{-1}\det(A) \end{pmatrix} \longmapsto u^{-1}A. \tag{4}$$

The kernel of either map (3) or (4) is the center of $\mathrm{GSp}(4)$ consisting of scalar matrices.

Representations of GSp(4)

Let F be a local field. We shall employ the notations of [34] and [32] for induced representations of the group $\mathrm{GSp}(4,F)$. For characters χ_1, χ_2 and σ of F^* let $\chi_1 \times \chi_2 \rtimes \sigma$ be the representation of $G(F) = \mathrm{GSp}(4,F)$ induced from the character

$$\begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & ub^{-1} & * \\ & & & ua^{-1} \end{pmatrix} \longmapsto \chi_1(a)\chi_2(b)\sigma(u)$$

of the Borel subgroup. The induction is always normalized. Provided that $e(\chi_1) \geq e(\chi_2) > 0$, where $e(\chi_i)$ denotes the real number with $|\chi_i(x)| = |x|^{e(\chi_i)}$ (the *exponent*), let $L((\chi_1,\chi_2,\sigma))$ be the unique irreducible quotient (the *Langlands quotient*) of $\chi_1 \times \chi_2 \rtimes \sigma$ (see [34], section 6). If π is a representation of $\mathrm{GL}(2,F)$ and σ a character of F^* let $\pi \rtimes \sigma$ be the representation of $G(F)$ induced from the representation

$$\begin{pmatrix} A & * \\ & uA' \end{pmatrix} \longmapsto \sigma(u)\pi(A)$$

of $P(F)$. The exponent $e(\pi)$ is the unique real number such that $|\,|^{-e(\pi)}\pi$ is unitarizable. Provided that π is essentially square integrable and $e(\pi) > 0$, the induced representation $\pi \rtimes \sigma$ has a unique Langlands quotient, denoted by $L((\pi,\sigma))$. Finally, assume that χ is a character of F^* and σ a representation of $\mathrm{GL}(2,F)$. Then $\chi \rtimes \sigma$ denotes the representation of $G(F)$ induced from the representation

$$\begin{pmatrix} u & * & * \\ & A & * \\ & & u^{-1}\det(A) \end{pmatrix} \longmapsto \chi(u)\sigma(A)$$

of $Q(F)$. If $e(\chi) > 0$ and σ is essentially tempered,
there is a unique Langlands quotient $L((\chi,\sigma))$. For parabolically induced representations of $\mathrm{GL}(2,F)$ we shall write either the common symbol $\pi(\chi_1,\chi_2)$, as in [9], or $\chi_1 \times \chi_2$, to fit into the systematic notational context of [34].

As in [32] we shall write $\nu(x) = |x|$ for the normalized absolute value on the local field F.

Occasionally symbols like $\chi_1 \times \chi_2 \rtimes \sigma$ and $\pi \rtimes \sigma$ will also denote the elements of the Grothendieck group of the category of all smooth representations of $G(F)$ of finite length defined by the corresponding induced representation (see the introduction to [32]). This should cause no confusion.

The theta correspondence

Let V be a finite-dimensional non-degenerate symmetric bilinear space and W a finite-dimensional non-degenerate symplectic space, defined over a number field F. We view the orthogonal group $H = \mathrm{O}(V)$ and the symplectic group $G = \mathrm{Sp}(W)$ as algebraic F–groups. The well-known *theta correspondence* is a correspondence between subsets of the set of irreducible, admissible representations of H and of the metaplectic cover of G. If $\dim(V)$ is even, the metaplectic cover can be replaced by G itself. There is a local and a global version of the theta correspondence, and the two are compatible. See [21] for an introduction to the p–adic theta correspondence.

The theta correspondence was extended to a correspondence for similitude groups (instead of isometry groups) in [26]. We refer to that paper and the references therein for general background on the subject. In this paper we shall be dealing with theta correspondences between the following groups.

i) $\widetilde{\mathrm{SL}}(2)$ (metaplectic cover) and $\mathrm{PGL}(2) \simeq \mathrm{SO}(3)$ (split orthogonal group).

ii) $\widetilde{\mathrm{SL}}(2)$ and PD^*, where D is a quaternion algebra over F.

iii) $\widetilde{\mathrm{SL}}(2)$ and $\mathrm{PGSp}(4) \simeq \mathrm{SO}(5)$ (split orthogonal group).

iv) $\mathrm{GSp}(4)$ and $\mathrm{GO}(4)$. In the local case we assume that the 4–dimensional orthogonal space has discriminant 1.

There is extensive local and global information on the first two types of correspondences in [39] and [40]. For the third type of theta correspondence, see [4], [22], [23] and [40]. Finally, the correspondence iv) was closely investigated in [27].

Let F be a local field and consider the local theta correspondence from $H = \mathrm{GO}(4,F)$ to $G = \mathrm{GSp}(4,F)$. It is easy to prove that if $\pi \in \mathrm{Irr}(H)$ has trivial central character, then its theta lift $\theta(\pi) \in \mathrm{Irr}(G)$ has also. This fact will be used later without comment.

The relation between theta liftings and Langlands' functoriality is not yet fully understood. In [25], the theta correspondence for *unramified* representations was shown to be functorial with certain morphisms on the L–group. However, global functoriality usually fails. The present work hopes to give some insight into the relation between the theta correspondence and functoriality in a low–rank situation.

1 Global induced representations on $\mathrm{PGL}(2)$

Let F be a number field and $\mathbb{A}$ its ring of adeles. Let B be the standard Borel subgroup of $G = \mathrm{GL}(2)$, and consider the global induced representation

$$\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\left(|\,|^{1/2}, |\,|^{-1/2}\right) = \bigotimes_v \mathrm{Ind}_{B_v}^{G_v}\left(|\,|_v^{1/2}, |\,|_v^{-1/2}\right). \tag{5}$$

The constituents of this global representation are all automorphic ([16], Proposition 2), and are obtained by taking an irreducible constituent of the local induced representation at each place v, with the Langlands quotient for almost every v ([16], Lemma 1). The Langlands quotient is the trivial representation $\mathbf{1}_v$. There is exactly one other constituent (a subrepresentation), which we denote by St_v because for finite v it is the Steinberg representation. We shall now describe these local representations in more detail, in particular giving their L– and ε–factors.

<u>v real</u>: In this case $\mathrm{St}_v = \mathcal{D}(1)$ is the lowest discrete series representation of $\mathrm{PGL}(2)$; it has a lowest weight vector of weight 2 and a highest weight vector of weight -2. The corresponding local parameters (representations of the Weil group) are as follows (see [13]). The Weil group is $W_{\mathbb{R}} = \mathbb{C}^* \sqcup j\mathbb{C}^*$ with $j^2 = -1$ and $jzj^{-1} = \bar{z}$. The parameter for the trivial representation is given by

$$z = re^{i\vartheta} \longmapsto \begin{pmatrix} r^{1/2} & \\ & r^{-1/2} \end{pmatrix}, \qquad j \longmapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}. \tag{6}$$

The parameter for $\mathcal{D}(1)$ is

$$z = re^{i\vartheta} \longmapsto \begin{pmatrix} e^{i\vartheta} & \\ & e^{-i\vartheta} \end{pmatrix}, \qquad j \longmapsto \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

The L– and ε–factors for these representations can be taken from [37] or [13]; they are given by

$$L_v(s, \mathbf{1}_v) = 2(2\pi)^{-s+1/2}\,\Gamma\Big(s - \frac{1}{2}\Big), \qquad \varepsilon_v(s, \mathbf{1}_v, \psi_v) = 1,$$

resp.

$$L_v(s, \mathrm{St}_v) = (2\pi)^{-s-1/2}\,\Gamma\Big(s + \frac{1}{2}\Big), \qquad \varepsilon_v(s, \mathrm{St}_v, \psi_v) = -1.$$

Here we have chosen the standard character $\psi_v(x) = e^{2\pi i x}$ of $\mathbb{R}$.

<u>v complex</u>: In this case we shall not allow to take St_v, for reasons that will become clear later. Thus we do not care about this representation, and only give the local factors for the trivial representation:

$$L_v(s, \mathbf{1}_v) = 2\sqrt{2}\,(4\pi)^{1/2-2s}\,\Gamma\Big(2s - \frac{1}{2}\Big), \qquad \varepsilon_v(s, \mathbf{1}_v, \psi_v) = 1.$$

Here the character is $\psi_v(z) = e^{2\pi i(z+\bar{z})}$ for $z \in \mathbb{C}$.

<u>v p–adic</u>: In the p–adic case St_v is really the Steinberg or special representation. The local parameters are representations $\rho = (\tilde{\rho}, N)$ of the Weil-Deligne group, with $\tilde{\rho}$ a representation of the Weil group W_v, and N a nilpotent endomorphism of the representation space such that $\tilde{\rho}(w)N\tilde{\rho}(w)^{-1} = |w|N$ for any $w \in W_F$ (see [37] (4.1.2)). Here $|\,|$ is the character of W_F coming from the absolute value on F^* via the isomorphism $W_F^{\mathrm{ab}} \simeq F^*$ which the Weil group comes equipped with. The parameter ρ_{triv} for the trivial representation is given by

$$\tilde{\rho}(w) = \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}, \qquad N = 0. \tag{7}$$

The parameter ρ_{St} for the Steinberg representation is given by

$$\tilde{\rho}(w) = \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{8}$$

The local factors are

$$L_v(s, \mathbf{1}_v) = \big((1 - q_v^{-s-1/2})(1 - q_v^{-s+1/2})\big)^{-1}, \qquad \varepsilon_v(s, \mathbf{1}_v, \psi_v) = 1,$$

resp.

$$L_v(s, \mathrm{St}_v) = \big(1 - q_v^{-s-1/2}\big)^{-1}, \qquad \varepsilon_v(s, \mathrm{St}_v, \psi_v) = -q_v^{1/2-s}.$$

Here ψ_v must have conductor $\mathfrak{o}_v$.

This concludes the description of the relevant local data. Let now S denote a finite set of places, not including any complex ones, and let $\pi_S = \otimes \pi_{S,v}$ be the constituent of the global induced representation (5) such that

$$\pi_{S,v} = \begin{cases} \mathbf{1}_v & \text{for } v \notin S, \\ \mathrm{St}_v & \text{for } v \in S. \end{cases} \tag{9}$$

The global L–function of π_S is given by

$$\begin{aligned} L(s, \pi_S) &= \Big(\prod_{v \notin S} L_v(s, \mathbf{1}_v)\Big)\Big(\prod_{v \in S} L_v(s, \mathrm{St}_v)\Big) = \Big(\prod_{v} L_v(s, \mathbf{1}_v)\Big)\Big(\prod_{v \in S} \frac{L_v(s, \mathrm{St}_v)}{L_v(s, \mathbf{1}_v)}\Big) \\ &= Z\Big(s + \frac{1}{2}\Big) Z\Big(s - \frac{1}{2}\Big)\Big(\prod_{v \in S} \frac{L_v(s, \mathrm{St}_v)}{L_v(s, \mathbf{1}_v)}\Big). \end{aligned} \tag{10}$$

Here $Z(s)$ denotes the global L–function of the trivial character (if $F = \mathbb{Q}$ this is just the completed Riemann zeta function). From the above description of local factors we get

$$\frac{L_v(s, \mathrm{St}_v)}{L_v(s, \mathbf{1}_v)} = \begin{cases} \frac{1}{4\pi}(s - \frac{1}{2}) & v \text{ real}, \\ 1 - q_v^{-s+1/2} & v \text{ } p\text{-adic}. \end{cases} \tag{11}$$

Thus we see that each place in S increases the order of the L–function at $s = 1/2$ by one.

1.1 Proposition. *Let S be a finite set of places, not including any complex ones, and let π_S be the automorphic representation of $\mathrm{PGL}(2, \mathbb{A})$ with local components (9).*

- *i) The global L–function $L(s, \pi_S)$ has simple poles at $s = -1/2$ and $s = 3/2$, and no other poles except possibly at $s = 1/2$.*
- *ii) The order of $L(s, \pi_S)$ at $s = 1/2$ is $\#S - 2$.*
- *iii) We have the functional equation $L(s, \pi_S) = \varepsilon(s, \pi_S) L(1 - s, \pi_S)$ with*

$$\varepsilon(s, \pi_S) = (-1)^{\#S} \prod_{v \in S,\, v \nmid \infty} q_v^{1/2-s}.$$

Proof: It is known that $Z(s)$ is holomorphic except for simple poles at $s = 0$ and $s = 1$. The Euler product for $Z(s)$ is convergent for $\mathrm{Re}(s) > 1$, so there are no zeros for $\mathrm{Re}(s) > 1$ or $\mathrm{Re}(s) < 0$. Thus $Z(s + 1/2)Z(s - 1/2)$ has simple poles at $s = -1/2$ and $s = 3/2$, and a double pole at $s = 1/2$. By (10) and (11), every place in S adds another zero at $s = 1/2$, and nowhere else on the real axis. This proves i) and ii). The last assertion is immediate from the above description of local factors. ∎

2 Various theta liftings

If $(V, (\ ,\ ))$ is a symmetric bilinear space over some field F, let $\mathrm{GO}(V)$ denote the group of linear automorphisms g of V such that there exists a scalar $\lambda(g)$ such that

$$(gx, gy) = \lambda(g)(x, y) \qquad \text{for all } x, y \in V.$$

The homomorphism $\lambda : \mathrm{GO}(V) \to F^*$ is called the *multiplier*. The relation with the determinant is $\det(g)^2 = \lambda(g)^m$, where $m = \dim(V)$. If this dimension is *even*, consider the homomorphism

$$\mathrm{sgn} : \mathrm{GO}(V) \longrightarrow \{\pm 1\}, \qquad \mathrm{sgn}(g) = \frac{\det(g)}{\lambda(g)^{m/2}}.$$

Its kernel is denoted by $\mathrm{GSO}(V)$.

Now suppose that F is a non-archimedean local field of characteristic 0. To be able to apply results involving the local theta correspondence, we shall make the (usual) assumption in this section that *F has odd residue characteristic*. There are exactly two isomorphism classes of quadratic spaces of dimension 4 and discriminant 1 over F, the split space V^s, and the anisotropic space V^a. Explicitly, V^s is a sum of two hyperbolic planes and may be realized as $V^s = M(2, F)$ with the quadratic form $q(A) = -\det(A)$. The anisotropic space V^a can be realized as the unique quaternion division algebra over F endowed with the reduced norm.

The groups $\mathrm{GSO}(V^s)$ and $\mathrm{GSO}(V^a)$ can be explicitly described as follows. Each element $(g, h) \in \mathrm{GL}(2, F) \times \mathrm{GL}(2, F)$ defines an automorphism $\rho(g, h)$ of $V^s = M(2, F)$ by $\rho(g, h)(x) = gxh^{-1}$. It is easy to see that $\rho(g, h) \in \mathrm{GSO}(V^s)$, and it is known that the sequence

$$1 \longrightarrow F^* \xrightarrow{\Delta} \mathrm{GL}(2, F) \times \mathrm{GL}(2, F) \xrightarrow{\rho} \mathrm{GSO}(V^s) \longrightarrow 1$$

is exact, where Δ is the diagonal embedding. Similarly, if D is the division quaternion algebra over F, there is an exact sequence

$$1 \longrightarrow F^* \xrightarrow{\Delta} D^* \times D^* \xrightarrow{\rho} \mathrm{GSO}(V^a) \longrightarrow 1.$$

Therefore we have isomorphisms

$$\mathrm{GSO}(V^s) \simeq (\mathrm{GL}(2, F) \times \mathrm{GL}(2, F))/\Delta F^* \qquad \text{and} \qquad \mathrm{GSO}(V^a) \simeq (D^* \times D^*)/\Delta F^*.$$

As a consequence, the irreducible representations of $\mathrm{GSO}(V^s)$ (resp. $\mathrm{GSO}(V^a)$) correspond bijectively to the pairs of irreducible representations of $\mathrm{GL}(2, F)$ (resp. D^*) with the same central character. If (π_1, π_2) is such a pair, we denote the corresponding representation of $\mathrm{GSO}(V^s)$ (resp. $\mathrm{GSO}(V^a)$) with the symbol $\pi_1 \otimes \pi_2^\vee$. Its space is that of the tensor product representation $\pi_1 \otimes \pi_2^\vee$, where $\pi_2^\vee$ denotes the contragredient.

An irreducible representation τ of $\mathrm{GSO}(V^s)$ (resp. $\mathrm{GSO}(V^a)$) is called *regular* if the induced representation of τ to $\mathrm{GO}(V^s)$ (resp. $\mathrm{GO}(V^a)$) is irreducible. In this case we denote the induced representation by τ^+. The regular representations can easily be described (see [27], Proposition 3.1).

2.1 Lemma. *An irreducible representation $\pi_1 \otimes \pi_2$ of $\mathrm{GSO}(V^s)$ (resp. $\mathrm{GSO}(V^a)$) is regular if and only if $\pi_1 \not\simeq \pi_2^\vee$.*

The representations of $\mathrm{GSO}(V^s)$ (resp. $\mathrm{GSO}(V^a)$) of the form $\tau = \pi \otimes \pi^\vee$ have exactly two extensions to a representation of $\mathrm{GO}(V^s)$ (resp. $\mathrm{GO}(V^a)$). Precisely one of these, denoted τ^+, participates in the theta correspondence with $\mathrm{GSp}(4,F)$ ([27], Theorem 6.8). Such representations do already participate in the theta correspondence between $\mathrm{GO}(V^s)$ (resp. $\mathrm{GO}(V^a)$) and $\mathrm{GSp}(2,F)=\mathrm{GL}(2,F)$ ([27], Theorem 7.4). For example, if π is a square integrable representation of $\mathrm{GL}(2,F)$, and if $\pi^{\mathrm{JL}}\in\mathrm{Irr}(D^*)$ corresponds to π under the Jacquet–Langlands correspondence, then the representation $(\pi^{\mathrm{JL}}\otimes\pi^{\mathrm{JL}\vee})^+$ of $\mathrm{GO}(V^a)$ lifts to π on $\mathrm{GL}(2,F)$.

Now consider the following diagram of "lifting maps", in which we have omitted the ground field F as well as the symbol for admissible representations of the respective groups.

$$\begin{tikzcd}
 & & \mathrm{GSp}(4) & & \\
\mathrm{GO}(V^a) \arrow[urr, "\theta"] & & & & \mathrm{GO}(V^s) \arrow[ull, "\theta"'] \\
\mathrm{GSO}(V^a) \arrow[u, "+"] & & \widetilde{\mathrm{SL}(2)} \arrow[uu, "\theta'_\psi"] & & \mathrm{GSO}(V^s) \arrow[u, "+"'] \\
PD^* \arrow[u, "_\otimes\mathbf{1}"] \arrow[urr, "\theta_\psi"] \arrow[rrrr, leftrightarrow, "\mathrm{JL}"] & & & & \mathrm{PGL}(2) \arrow[u, "_\otimes\mathbf{1}"'] \arrow[ull, "\theta_\psi"']
\end{tikzcd} \tag{12}$$

On the bottom we have the Jacquet–Langlands correspondence between irreducible representations of $PD^*=D^*/F^*$ and discrete series representations (supercuspidal and special representations) of $\mathrm{PGL}(2,F)$. The map $\_\otimes\mathbf{1}$ associates to a representation π of D^* (resp. $\mathrm{GL}(2,F)$) with trivial central character the representation $\pi\otimes\mathbf{1}$ of $\mathrm{GSO}(V^a)$ (resp. $\mathrm{GSO}(V^s)$), where $\mathbf{1}$ denotes the trivial representation. The arrows labeled "+" denote essentially induction, except for representations of the type $\pi\otimes\pi^\vee$, where they mean the unique extension of $\pi\otimes\pi^\vee$ to $\mathrm{GO}(V^a)$ (resp. $\mathrm{GO}(V^s)$) that has non-zero theta–lift to $\mathrm{GSp}(4,F)$. The upper θ's denote the theta–correspondence between $\mathrm{GO}(V^*)$ and $\mathrm{GSp}(4,F)$, see [27]. The arrows in the middle are also theta–correspondences, but, in contrast to the upper θ's, depend on the choice of an additive character ψ. Those local correspondences were studied by Waldspurger in [40]. They are given by explicit integral transformations and can be defined also in even residue characteristic and in the archimedean case. Note that the bottom triangle is definitely not commutative. The target group on top is really $\mathrm{PGSp}(4,F)$.

For π an irreducible representation of PD^* or of $\mathrm{PGL}(2,F)$, we define the *local Saito–Kurokawa lift* of π as

$$\mathrm{SK}(\pi):=\theta'_\psi(\theta_\psi(\pi)). \tag{13}$$

This is an irreducible representation of $\mathrm{PGSp}(4,F)$. By [4], Corollary to Proposition 2.1, $\mathrm{SK}(\pi)$ is independent of
the choice of ψ; whence our notation. This result also holds in even residue characteristic.

Note that for a discrete series representation π of $\mathrm{PGL}(2,F)$ there are four ways in the above diagram to reach the top. Our main local result will be that (exactly) two of the resulting representations of $\mathrm{GSp}(4)$ coincide. More precisely, we will prove in Proposition 5.8 that the left half of diagram (12) is commutative. The three different representations we can obtain in this way from a discrete series representation of $\mathrm{PGL}(2,F)$ are the ones that appear in diagram (21) below.

2.2 Lemma. *Suppose F is a local field of characteristic 0. For any irreducible, unitary representation π of $\mathrm{PGL}(2,F)$ the Saito–Kurokawa lift of π can be described as*

$$\mathrm{SK}(\pi) = \text{ unique irreducible quotient of } \nu^{1/2}\pi \rtimes \nu^{-1/2}.$$

This representation is unitary and not generic.

Proof: Translated into our notation, Lemme 49 in [40] says that $\mathrm{SK}(\pi)$ is a subrepresentation of $\nu^{-1/2}\pi \rtimes \nu^{1/2}$. Weyl group action gives an equality $\nu^{-1/2}\pi \rtimes \nu^{1/2} = \nu^{1/2}\pi \rtimes \nu^{-1/2}$ in the Grothendieck group, but the subrepresentation and the quotient become interchanged. Thus $\mathrm{SK}(\pi)$ is a quotient of $\nu^{1/2}\pi \rtimes \nu^{-1/2}$.

In the p–adic case it follows from [32] Lemmas 3.3, 3.7, 3.8, 3.6 and Proposition 4.6 that $\nu^{1/2}\pi \rtimes \nu^{-1/2}$ has length 2. Since the inducing representation is generic, it follows from [3] that the quotient is not generic. It follows further from [32] Theorem 4.4 and [33] Theorem 5.1 that this quotient is unitarizable.

If $F = \mathbb{R}$ and π is a discrete series representation, see section 4 where $\mathrm{SK}(\pi)$ is explicitly determined. Suppose that F is archimedean and $\pi = \pi(\chi, \chi^{-1}) = \chi \times \chi^{-1}$ is a unitary principal series representation. We have formally

$$\begin{aligned} \nu^{1/2}\pi \rtimes \nu^{-1/2} &= \nu^{1/2}\chi \times \nu^{1/2}\chi^{-1} \rtimes \nu^{-1/2} = \nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \chi^{-1} \\ &= \chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \chi^{-1} + \chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \chi^{-1} \end{aligned}$$

in the Grothendieck group. Since $\mathrm{SK}(\pi)$ is a quotient of the reducible representation $\nu^{1/2}\pi \rtimes \nu^{-1/2}$, it is not generic. It is easily seen that $\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \chi^{-1}$ is a subrepresentation and $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \chi^{-1}$ is a quotient of $\nu^{1/2}\pi \rtimes \nu^{-1/2}$. Hence $\mathrm{SK}(\pi)$ is a constituent of $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \chi^{-1}$. But this representation is irreducible and unitary by [18]. ∎

3 The main lifting theorem

If F is a local field of characteristic zero, possibly archimedean, and if π is an irreducible, admissible, infinite-dimensional representation of $\mathrm{PGL}(2,F)$, we define a representation of $\mathrm{PGSp}(4,F)$ as

$$\Pi(\pi \otimes \mathbf{1}) := \mathrm{SK}(\pi). \tag{14}$$

By Lemma 2.2, $\Pi(\pi \otimes \mathbf{1})$ can also be described as the unique irreducible quotient of $\nu^{1/2}\pi \rtimes \nu^{-1/2}$. Provided π is square-integrable, we define another representation of $\mathrm{PGSp}(4,F)$ as

$$\Pi(\pi \otimes \mathrm{St}) := \mathrm{SK}(\pi^{\mathrm{JL}}) \qquad (\pi \text{ square-integrable}). \tag{15}$$

Assuming certain facts on the local Langlands correspondence, we will later recognize $\Pi(\pi \otimes \mathbf{1})$ and $\Pi(\pi \otimes \mathrm{St})$ as functorial liftings from $\mathrm{PGL}(2) \times \mathrm{PGL}(2)$ to $\mathrm{PGSp}(4)$, which will explain our notations.

Now assume F is a global number field and π is a cuspidal automorphic representation of $\mathrm{PGL}(2,\mathbb{A}_F)$. Let S be a set of places of F contained in the set of places v where π_v is a discrete series representation. Let π_S be the corresponding automorphic representation of $\mathrm{PGL}(2,\mathbb{A}_F)$ considered in section 1. We define a global representation of $\mathrm{PGSp}(4,\mathbb{A}_F)$ by

$$\Pi(\pi \otimes \pi_S) := \bigotimes \Pi(\pi_v \otimes \pi_{S,v}). \tag{16}$$

The local liftings on the right hand side have been defined by (14) and (15). Our main result about the representations $\Pi(\pi \otimes \pi_S)$ is the following theorem. In the statement the number $\varepsilon(1/2,\pi)$ for a global cusp form π on $\mathrm{PGL}(2,\mathbb{A}_F)$ occurs. This number is just a sign. See section 3 of [31] for more information on the signs defined by ε–factors.

3.1 Theorem. *Let $\pi = \otimes\pi_v$ be a cusp form on $\mathrm{PGL}(2,\mathbb{A})$. Let S be a set of places of F such that π_v is a discrete series representation for each place $v \in S$, and let π_S be the corresponding constituent of the global induced representation (5). Assume that*

i) The order at $s = 1/2$ of the L–function $L(s,\pi)L(s,\pi_S)$ is even.

An equivalent condition is

ii) $(-1)^{\#S} = \varepsilon(1/2,\pi)$.

Then:

a) The global lifting $\Pi(\pi\otimes\pi_S)$ is an automorphic representation of $\mathrm{PGSp}(4,\mathbb{A})$ which appears discretely in the space of automorphic forms.

b) If $L(1/2,\pi) = 0$ or if $S \neq \emptyset$, then $\Pi(\pi\otimes\pi_S)$ is a cuspidal automorphic representation.

Proof: Let us first assume that the cusp condition in b) is *not* fulfilled. This means that $L(1/2,\pi) \neq 0$ and $S = \emptyset$. In this case, by Lemma 2.2, the representation $\Pi(\pi\otimes\pi_S)$ is the unique irreducible quotient of the global induced representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GSp}(4,\mathbb{A})}\left(|\,|_{\mathbb{A}}^{1/2}\pi\otimes|\,|_{\mathbb{A}}^{-1/2}\right) \qquad \text{(induction from the Siegel parabolic)}$$

and is therefore automorphic. But such representations for $L(1/2,\pi)\neq 0$ are known to appear in the residual spectrum of $\mathrm{PGSp}(4)$, see [11], Theorem 7.1.

From now on we may assume that the cusp condition ii) is fulfilled. Let Σ be the set of places v where π_v is a discrete series representation. We fix an additive character $\psi = \otimes\psi_v$ of F. Denote by Wald_ψ the global Waldspurger lifting from cuspidal automorphic representations of $\widetilde{\mathrm{SL}}(2,\mathbb{A})$ to cuspidal automorphic representations on $\mathrm{PGL}(2,\mathbb{A})$ defined in [39]. Similarly, denote by Wald_{ψ_v} the local Waldspurger lifting from $\mathrm{Irr}(\widetilde{\mathrm{SL}}(2,F))$ to $\mathrm{Irr}(\mathrm{PGL}(2,F))$ defined in [40] VI. The global and local liftings are compatible. We have

$$\#\mathrm{Wald}_{\psi_v}^{-1}(\pi_v) = \begin{cases} 1, & \text{if } \pi_v \text{ is a principal series representation (i.e., if } \pi_v\notin\Sigma), \\ 2, & \text{otherwise.}\end{cases} \tag{17}$$

In each case these "local L–packets" for the metaplectic group contain the theta lift $\theta(\pi_v,\psi_v)$, which is a ψ_v–generic representation. We denote it by $\tilde\pi_{v,\mathrm{gen}}$. If π_v is square integrable, then $\mathrm{Wald}_{\psi_v}^{-1}(\pi_v)$ contains moreover the ψ_v–non-generic representation $\tilde\pi_{v,\mathrm{ng}} := \theta(\pi_v^{\mathrm{JL}},\psi_v)$, where π_v^{JL} is the representation of PD^* corresponding to π_v under the Jacquet–Langlands correspondence.

If $\pi = \otimes\pi_v$ is a cusp form on $\mathrm{PGL}(2,\mathbb{A})$, let $C := \mathrm{Wald}_\psi^{-1}(\pi)$ be the corresponding fiber of the global Waldspurger lifting. Let $\tilde\pi = \otimes\tilde\pi_v$ be any element of C. Decompose

$$\Sigma = \Sigma_{\mathrm{gen}}\cup\Sigma_{\mathrm{ng}},$$

where Σ_{gen} (resp. Σ_{ng}) is the set of places $v\in\Sigma$ such that $\tilde\pi_v = \tilde\pi_{v,\mathrm{gen}}$ (resp. $\tilde\pi_v = \tilde\pi_{v,\mathrm{ng}}$). We claim that

$$(-1)^{\#\Sigma_{ng}} = \varepsilon(1/2,\pi), \tag{18}$$

where $\varepsilon(s,\pi) = \prod \varepsilon(s,\pi_v)$ is the global ε–factor of π. To see this, let $\varepsilon(\tilde\pi_v,\psi_v) \in \{\pm 1\}$ be the sign attached to $\tilde\pi_v$ (and ψ_v), see [40] I.4 b). Since $\tilde\pi$ is automorphic, we have $\prod_v \varepsilon(\tilde\pi_v,\psi_v) = 1$ (equation (1) in [40] VI). By [40], Lemme 6 and Théorème 2, 3), we have

$$\varepsilon(\tilde\pi_{v,\mathrm{gen}},\psi_v) = \varepsilon(1/2,\pi_v) \qquad \text{and} \qquad \varepsilon(\tilde\pi_{v,\mathrm{ng}},\psi_v) = -\varepsilon(1/2,\pi_v).$$

It follows that

$$\begin{aligned} 1 = \prod_v \varepsilon(\tilde\pi_v,\psi_v) &= \Big(\prod_{v\in\Sigma_{\mathrm{ng}}} \varepsilon(\tilde\pi_v,\psi_v)\Big)\Big(\prod_{v\notin\Sigma_{\mathrm{ng}}} \varepsilon(\tilde\pi_v,\psi_v)\Big) \\ &= (-1)^{\#\Sigma_{\mathrm{ng}}}\Big(\prod_{v\in\Sigma_{\mathrm{ng}}} \varepsilon(1/2,\pi_v)\Big)\Big(\prod_{v\notin\Sigma_{\mathrm{ng}}} \varepsilon(1/2,\pi_v)\Big), \end{aligned}$$

hence our claim. These arguments can be reversed, and we see that if $\tilde\pi = \otimes\tilde\pi_v$ is *any* global representation with $\tilde\pi_v \in \mathrm{Wald}_{\psi_v}^{-1}$ for all v, then $\tilde\pi \in C$ as soon as (18) holds.

Now assume the representation π_S is as in the theorem. In view of Proposition 1.1 ii) we have the equivalences

$$\begin{aligned} \varepsilon(1/2,\pi) = 1 \quad &\Longleftrightarrow \quad \mathrm{ord}_{1/2}(L(s,\pi)) \text{ is even} \\ &\Longleftrightarrow \quad \mathrm{ord}_{1/2}(L(s,\pi_S)) \text{ is even} \quad \Longleftrightarrow \quad \#S \text{ is even.} \end{aligned}$$

Therefore, if we define $\tilde\pi = \otimes\tilde\pi_v$ by $\tilde\pi_v = \tilde\pi_{v,\mathrm{ng}}$ for $v \in S$ and $\tilde\pi_v = \tilde\pi_{v,\mathrm{gen}}$ for $v \notin S$, the condition (18) is fulfilled and it follows that $\tilde\pi \in C$.

We have assumed that the cusp condition in b) holds. If $L(1/2,\pi) = 0$, then, by [39], no $\tau \in C$ is globally ψ–generic. On the other hand, if $L(1/2,\pi) \neq 0$, then there is exactly one ψ–generic member in C, namely $\tau = \otimes\tau_v$ with $\tau_v = \tilde\pi_{v,\mathrm{gen}}$ for all v. (It is this representation which participates in the global theta correspondence with $\mathrm{PGL}(2,\mathbb{A})$.) If $S \neq \emptyset$, then our $\tilde\pi$ constructed above is different from τ. We see that the cusp condition in b) implies that $\tilde\pi$ is not ψ–generic.

Now let $\theta'(\cdot,\psi)$ denote the theta lifting from $\widetilde{\mathrm{SL}}(2)$ to $\mathrm{PGSp}(4)$. Since $\tilde\pi$ is not ψ–generic, it follows from [22], Theorem 2.3, that $\theta'(\tilde\pi,\psi)$ is a non-vanishing irreducible cuspidal automorphic representation of $\mathrm{PGSp}(4,\mathbb{A})$. By our definitions (14) and (15) we have

$$\theta'(\tilde\pi_v,\psi_v) = \begin{cases} \Pi(\pi_v \otimes \mathbf{1}_v) & \text{for } v \notin S, \\ \Pi(\pi_v \otimes \mathrm{St}_v) & \text{for } v \in S. \end{cases} \tag{19}$$

Since $\theta'(\tilde\pi,\psi) \simeq \otimes\theta'(\tilde\pi_v,\psi_v)$, it follows from (19) that our global lift $\Pi(\pi\otimes\pi_S)$ is a theta lift from $\widetilde{\mathrm{SL}}(2,\mathbb{A})$: $\theta'(\tilde\pi,\psi) \simeq \Pi(\pi\otimes\pi_S)$. This shows that $\Pi(\pi\otimes\pi_S)$ is a cuspidal automorphic representation of $\mathrm{PGSp}(4,\mathbb{A})$. ■

3.2 Remarks. a) As mentioned above, under some reasonable assumptions on the local Langlands correspondence for $\mathrm{GSp}(4)$ (Conjecture 6.1 below), the representation $\Pi(\pi\otimes\pi_S)$ is a functorial lifting of the representation $\pi\otimes\pi_S$ of $\mathrm{PGL}(2,\mathbb{A}) \times \mathrm{PGL}(2,\mathbb{A})$ under the natural embedding of L–groups $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Sp}(4,\mathbb{C})$. Assuming this, the L–function of the lift $\Pi = \Pi(\pi\otimes\pi_S)$, defined via local Langlands correspondence and using the standard representation of the L–group, is given by

$$L(s,\Pi) = L(s,\pi)L(s,\pi_S).$$

Using Proposition 1.1, we see that $L(s,\Pi)$ has simple poles at $s = -1/2$ and $s = 3/2$. The only other possible pole is at $s = 1/2$. If the cuspidality condition in the theorem holds, then $L(s,\Pi)$ is holomorphic at $s = 1/2$.

b) Since at almost all places our local liftings are quotients of induced representations $\nu^{1/2}\pi_v \rtimes \nu^{-1/2}$, the global liftings $\Pi(\pi\otimes\pi_S)$ are examples for CAP representations (cuspidal associated to parabolics).

c) If $\Pi = \Pi(\pi \otimes \pi_S)$ is one of the cuspidal representations constructed in the theorem, and if χ is an idele class character, we can consider the twist $\chi\otimes\Pi$, where we put χ on the multiplier. This is another cusp form on $\mathrm{GSp}(4,\mathbb{A})$, and if χ is quadratic, the central character remains trivial. Obviously $\chi\otimes\Pi$ is a lifting of $(\chi\pi)\otimes(\chi\pi_S)$, and therefore is still a CAP representation. But, as soon as χ is non-trivial, its L–function does not have poles, since, over $\mathbb{Q}$ say, the Riemann zeta functions are replaced by Dirichlet L–functions. These twists must be taken into consideration in any proper formulation of the Ramanujan conjecture for $\mathrm{GSp}(4)$.

d) By construction, the liftings that can be obtained through Theorem 3.1 are essentially the same as the Saito–Kurokawa liftings defined in [4] in terms of packets. In section 6 we will comment on the consequences of functoriality for the base change theory of Saito–Kurokawa representations.

3.3 Example. Assume that the ground field is $\mathbb{Q}$ and that π is the automorphic representation corresponding to a classical eigenform $f \in S_{2k-2}(\Gamma)$, where $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. For the global ε–factor we have $\varepsilon(1/2,\pi) = (-1)^{k-1}$. Thus, if k is even, we can put $S = \{\infty\}$ in Theorem 3.1 and get a cuspidal lifting Π to $\mathrm{PGSp}(4,\mathbb{A})$. As we will see in the next section, the archimedean component Π_∞ of the lift is the holomorphic discrete series representation σ_k^+ with scalar minimal K–type (k,k). At every finite place we get an unramified representation. Therefore Π corresponds to a classical Siegel modular form F of weight k and degree 2 for the full modular group $\mathrm{Sp}(4,\mathbb{Z})$ (see [2]). This F is the classical Saito–Kurokawa lift of f.

3.4 Example. Assume that $f \in S_{2k-2}(\Gamma_0(N))^{\mathrm{new}}$ is an eigenform for some level $N > 1$ and that the corresponding automorphic representation π is square integrable at some finite place p (this is the case, for example, if p divides N to an odd order). Then we can find a suitable S that does not contain the archimedean place. The archimedean component of the resulting lift Π is then the cohomological representation σ_k^- described more precisely in section 4. Thus we are able to produce many non-holomorphic Saito–Kurokawa lifts.

4 The lifting at real places

We shall now examine more closely the liftings defined in (14) and (15) over the real field. Since the real theta liftings can be computed explicitly, and since the archimedean local Langlands correspondence is known, we will be able to check that the lifting constructed in Theorem 3.1 is functorial as described in Remark 3.2 a) at least over archimedean places. The situation over $\mathbb{R}$ gives a good picture of what is happening at finite places, where proofs are much harder.

The embedding $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Sp}(4,\mathbb{C})$ in (1) is explicitly given by

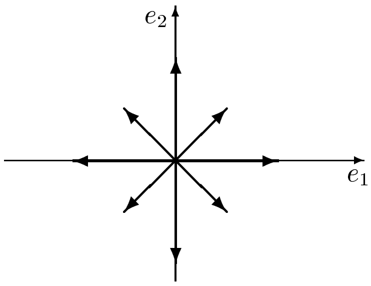
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \longmapsto \begin{pmatrix} a' & & & b' \\ & a & b & \\ & c & d & \\ c' & & & d' \end{pmatrix}. \tag{20}$$

If we have two local parameters $\rho_1, \rho_2 : W_\mathbb{R} \to \mathrm{SL}(2,\mathbb{C})$, we shall denote by $\rho_1 \oplus \rho_2$ the composition of (ρ_1,ρ_2) with the L–morphism (20). Thus we associate to every parameter for $\mathrm{PGL}(2) \times \mathrm{PGL}(2)$ a parameter for $\mathrm{PGSp}(4)$.

If π is any infinite-dimensional, unitary, irreducible, admissible representation of $\mathrm{PGL}(2,\mathbb{R})$ with local parameter ρ, consider the parameter $\rho \oplus \rho_{\mathrm{triv}}$, where ρ_{triv} is given in (6). Its image is contained in the Klingen parabolic subgroup of $\mathrm{Sp}(4,\mathbb{C})$, and the corresponding representation is therefore induced from the Siegel parabolic subgroup of $\mathrm{PGSp}(4,\mathbb{R})$. More precisely, pulling back to $\mathrm{GSp}(4,\mathbb{R})$, the parameter $\rho \oplus \rho_{\mathrm{triv}}$ corresponds to the unique irreducible quotient of $\nu^{1/2}\pi \rtimes \nu^{-1/2}$. By Lemma 2.2 and the definition in (14), $\Pi(\pi \otimes \mathbf{1})$ is a functorial lifting of $\pi \otimes \mathbf{1}_{\mathrm{GL}(2)}$ under the morphism (20) of L–groups.

We shall now give a more precise description of the real liftings. If $\pi = \pi(\chi, \chi^{-1})$ is a unitary principal series representation, then, as explained in the proof of Lemma 2.2, the lifting $\Pi(\pi \otimes \mathbf{1})$ is the unitary and non-generic degenerate principal series representation $\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \chi^{-1}$. Little more can be said in this case, hence we shall from now on concentrate on the liftings of discrete series representations.

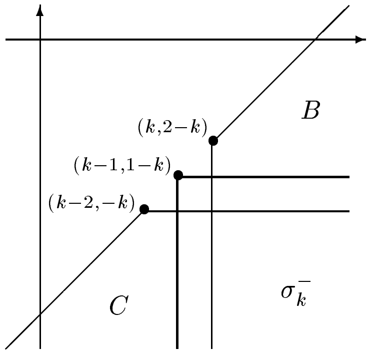
We are working with representations of $\mathrm{PGSp}(4,\mathbb{R})$, but it is convenient to consider the closely related group $\mathrm{Sp}(4,\mathbb{R})$. Let e_1, e_2 be a basis for the character lattice of $\mathrm{Sp}(4)$ such that $\pm e_1 \pm e_2$ and $\pm 2e_i$ are the roots of this group:



We will sometimes write $(a, b) := ae_1 + be_2$ for a point in this plane. Let the numbering be such that $\pm(e_1 - e_2)$ are the compact roots. The possible K–types correspond to integer points (l, l') with $l \geq l'$. Since we are interested in representations with trivial central character, only the K–types (l, l') with $l + l'$ even will be relevant.

Consider $\pi = \mathcal{D}(2k-3)$ with $k \geq 2$, the discrete series representation of $\mathrm{PGL}(2,\mathbb{R})$ with lowest weight $2k-2$. We shall first describe $\Pi(\pi \otimes \mathbf{1})$, which by the definition in (14) is a double theta lifting $\theta'(\theta(\pi))$. If the character used to define the theta correspondence is $x \mapsto e^{imx}$ with $m > 0$, then, by [40], Proposition 5, the inner lifting $\theta(\pi)$ is the discrete series representation of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ with a highest weight vector of weight $-k + 1/2$. One can then use the paper [19] to compute the lifting to $\mathrm{PGSp}(4,\mathbb{R}) \simeq \mathrm{SO}(3,2)$. Putting $p = 3$, $q = 2$, $r = 1$ and $s = 0$ in Example (I_4) of this paper shows that $\Pi(\pi \otimes \mathbf{1})$ is a certain cohomologically induced representation $A_{\mathfrak{q}}(\lambda)$ (notation of [14]), where $\lambda = (k-3, 3-k)$, and where $\mathfrak{q}$ is the θ–stable parabolic subalgebra of $\mathfrak{sp}(4,\mathbb{C})$ which has the short non-compact roots in the Levi. Let us denote this representation by σ_k^-.

This σ_k^- has infinitesimal character $(k-1, 2-k)$ and minimal K–type $(k-1, 1-k)$. It can be realized as the unique irreducible quotient of $\nu^{1/2}\pi \rtimes \nu^{-1/2}$. This induced representation has length 2 and decomposes as $\pi_W + \sigma_k^-$, where π_W is a generic representation (a *large* discrete series representation if $k \geq 3$). Upon restriction to $\mathrm{Sp}(4,\mathbb{R})$ the large representation π_W decomposes into two parts B and C, and we get a picture of the K–types as follows.



To give another description of σ_k^-, define a character χ of the Siegel parabolic subgroup $P = MN$ by

$$\begin{pmatrix} A & * \\ & uA' \end{pmatrix} \longmapsto \begin{cases} |u^{-1}\det(A)|^{k-3/2} & \text{if } k \text{ is odd,} \\ \mathrm{sgn}(u^{-1}\det(A))|u^{-1}\det(A)|^{k-3/2} & \text{if } k \text{ is even,} \end{cases}$$

and consider the degenerate principal series representation $\sigma_\chi := \mathrm{Ind}_{P(\mathbb{R})}^{\mathrm{GSp}(4,\mathbb{R})}(\chi)$. This is really a representation of $\mathrm{PGSp}(4,\mathbb{R})$, and σ_k^- appears as the unique non-trivial proper subrepresentation of σ_χ. This follows from the paper [18], where the reducibilities of the degenerate principal series representations have been determined. Our σ_k^- is L_{21} in LEE's Theorem 5.2. From [18], Lemma 5.1, we see that σ_k^- is unitarizable.

Next we consider the lift $\Pi(\pi\otimes\mathrm{St})$, where $\pi = \mathcal{D}(2k-3)$ with $k \geq 2$, and where $\mathrm{St} = \mathcal{D}(1)$, the lowest discrete series representation of $\mathrm{PGL}(2,\mathbb{R})$. By definition, $\Pi(\pi\otimes\mathrm{St}) = \theta'(\theta(\pi^{\mathrm{JL}}))$, see (15). By [40], Proposition 9, $\theta(\pi^{\mathrm{JL}})$ is a discrete series representation of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of lowest weight $k-1/2$ (provided the character used for the theta lifting is e^{imx} with $m > 0$). By example (I_4) in §6 of [19] (put $p = 3$, $q = 2$, $r = 0$, $s = 1$), the theta lift of this representation to $\mathrm{PGSp}(4,\mathbb{R})$ is σ_k^+, the holomorphic discrete series representation on $\mathrm{PGSp}(4,\mathbb{R})$ with scalar minimal K–type (k,k). Actually, if $k = 2$, this representation is only in the limit of the discrete series.

To see why this lifting is functorial, let $\rho_n : W_{\mathbb{R}} \to \mathrm{SL}(2,\mathbb{R})$ be the parameter of the discrete series representation $\mathcal{D}(n)$. Explicitly, it maps

$$z = re^{i\vartheta} \longmapsto \begin{pmatrix} e^{in\vartheta} & \\ & e^{-in\vartheta} \end{pmatrix}, \qquad j \longmapsto \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

(recall $W_{\mathbb{R}} = \mathbb{C}^* \sqcup j\mathbb{C}^*$). We note that $\rho_{\mathrm{St}} = \rho_1$ and consider the parameter $\rho_{2k-3}\oplus\rho_1$. If $k \geq 3$, this is recognized as the parameter for an L-packet on $\mathrm{PGSp}(4,\mathbb{R})$ containing discrete series representations with Harish-Chandra parameter $(k-1, k-2)$ (some care has to be taken with the duality between $\mathrm{PGSp}(4) \simeq \mathrm{SO}(5)$ and $\mathrm{Sp}(4)$). This L-packet in particular contains the discrete series representation σ_k^+ with one-dimensional minimal K-type of weight (k,k) (combining a holomorphic and an anti-holomorphic discrete series representation of $\mathrm{Sp}(4,\mathbb{R})$). This shows that $\Pi(\mathcal{D}(2k-3)\otimes\mathrm{St}) = \sigma_k^+$ is a functorial lifting of $\mathcal{D}(2k-3)\otimes\mathrm{St}_{\mathrm{GL}(2)}$ if $k \geq 3$.

Now consider the case $k = 2$, i.e., the lift $\Pi(\mathrm{St}\otimes\mathrm{St})$, with $\mathrm{St} = \mathcal{D}(1)$. In this case, the parameter $\rho_1\otimes\rho_1$ can be conjugated into the standard Siegel parabolic subgroup of $\mathrm{Sp}(4,\mathbb{C})$ by a suitable Cayley transformation. This shows that the functorial lifting of $\mathcal{D}(1)\otimes\mathcal{D}(1)$ is the unique irreducible quotient of $\mathbf{1}_{\mathbb{R}^*}\rtimes\mathcal{D}(1)$ (induction from the Klingen parabolic subgroup). It can be shown that this quotient equals the lowest weight representation σ_2^+ considered above. Hence our lifting is also functorial in this case.

Since the large representation π_W has a Whittaker model, σ_k^- has not. Neither has σ_k^+, since it is only the large discrete series representations that are generic. Thus neither of our lifts is generic. The quotient σ_k^- is not tempered, while the σ_k^+ are tempered (and even square integrable for $k \geq 3$). All our lifts are unitary.

With $\pi = \mathcal{D}(2k-3)$ we see that we are dealing with three representations, any two of which constitute some kind of "packet":

$$\begin{tikzcd}
& \pi_W \arrow[dl, "\text{ind. rep.}"', <->] \arrow[dr, "L\text{–packet}", <->] & \\
\Pi(\pi \otimes \mathbf{1}) \arrow[rr, <->, "A\text{–packet}"'] & & \Pi(\pi \otimes \mathrm{St})
\end{tikzcd} \tag{21}$$

Here the pair $(\Pi(\pi \otimes \mathbf{1}), \Pi(\pi \otimes \mathrm{St})) = (\sigma_k^-, \sigma_k^+)$ constitutes an *Arthur packet*, see Example 1.4.1 in [1]. The arrow "ind. rep." indicates that the two representations are the two constituents of the induced representation $\nu^{1/2}\pi \rtimes \nu^{-1/2}$. The pair (π_W, σ_k^+) is an L–packet of discrete series representations.

4.1 Lemma. *Let* $\pi = \mathcal{D}(2k-3)$.

i) The lifting $\Pi(\pi \otimes \mathrm{St})$ *can be obtained as a theta lifting from the anisotropic* $\mathrm{GO}(4, \mathbb{R})$*:*

$$\Pi(\pi \otimes \mathrm{St}) \stackrel{\text{def}}{=} \mathrm{SK}(\pi) = \sigma_k^+ = \theta((\pi^{\mathrm{JL}} \otimes \mathbf{1}_{D^*})^+).$$

ii) The large discrete series representation π_W *with minimal* K*–type* $(k, 2-k)$ *can be obtained as a theta lifting from the split* $\mathrm{GO}(2,2)$*, namely* $\pi_W = \theta((\pi \otimes \mathcal{D}(1))^+)$.

Proof: Both assertions can be deduced from example (I_0) in [19]. For i) put $n = 2$, $p = 4$, $q = 0$, for ii) put $n = p = q = 2$. For the relationship between theta liftings for isometry groups and similitude groups in the real case see section 1 of [30]. ∎

Section 5 is devoted to proving the p–adic analogue of part i) of this lemma. Since the (conjectural) L–packets on $\mathrm{GSp}(4)$ are defined in terms of theta liftings from $\mathrm{GO}(V)$, where V is a four-dimensional quadratic space, this is the key to proving that the lifting constructed in Theorem 3.1 is functorial.

We summarize the basic properties of the three representations in (21) for π a discrete series representation of $\mathrm{PGL}(2, \mathbb{R})$. We will encounter exactly the same situation in the p–adic case.

- $\Pi(\pi \otimes \mathbf{1}) \stackrel{\text{def}}{=} \mathrm{SK}(\pi)$, unique irreducible quotient of $\nu^{1/2}\pi \rtimes \nu^{-1/2}$, unitary, non-generic, non-tempered.
- $\Pi(\pi \otimes \mathrm{St}) \stackrel{\text{def}}{=} \mathrm{SK}(\pi^{\mathrm{JL}})$, unitary, non-generic, tempered, square-integrable if $\pi \neq \mathrm{St}$, obtained as a theta lifting $\theta((\pi^{\mathrm{JL}} \otimes \mathbf{1}_{D^*})^+)$ from the anisotropic $\mathrm{GO}(4)$.
- π_W, unique irreducible subrepresentation of $\nu^{1/2}\pi \rtimes \nu^{-1/2}$, unitary, generic, tempered, square-integrable if $\pi \neq \mathrm{St}$, obtained as a theta lifting $\theta((\pi \otimes \mathrm{St}_{\mathrm{GL}(2)})^+)$ from the split $\mathrm{GO}(4)$.

5 A local theta identity

As in section 2 let V^s be the split quadratic space of dimension 4 over a local field F. After choosing a suitable basis, we may realize $\mathrm{SO}(V^s)$ and $\mathrm{GSO}(V^s)$ as matrix groups using the form J_4 as in (2). Let $B \subset \mathrm{SO}(V^s)$ be the subgroup of upper triangular matrices. If μ_1 and μ_2 are characters of F^*, we denote by $\mu_1 \times \mu_2$ the representation of $\mathrm{SO}(V^s)$ unitarily induced from the character

$$\begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & b^{-1} & * \\ & & & a^{-1} \end{pmatrix} \longmapsto \mu_1(a)\mu_2(b)$$

of B. Assuming that μ_1 and μ_2 are unramified, $\mu_1 \times \mu_2$ has a unique spherical constituent $\sigma_0(\mu_1, \mu_2)$. Its L–factor is given by

$$L(s, \sigma_0(\mu_1, \mu_2)) = L(s, \mu_1)L(s, \mu_1^{-1})L(s, \mu_2)L(s, \mu_2^{-1}). \tag{22}$$

5.1 Lemma. *Let $\pi_1 = \pi(\chi_1, \chi_1')$ and $\pi_2 = \pi(\chi_2, \chi_2')$ be two standard induced representations of* $\mathrm{GL}(2, F)$ *with $\chi_1\chi_1' = \chi_2\chi_2'$. Let $\pi = \pi_1 \otimes \pi_2^\vee$ be the corresponding representation of* $\mathrm{GSO}(V^s)$*. Then $\pi\big|_{\mathrm{SO}(V^s)}$ is isomorphic to the induced representation $\mu_1 \times \mu_2$ with*

$$\mu_1 = \chi_1\chi_2^{-1}, \qquad \mu_2 = {\chi_1'}^{-1}\chi_2.$$

Proof: This can be checked in a straightforward way. Realizing π_1 and π_2 in their standard models as functions on $\mathrm{GL}(2, F)$, the space of $\pi_1 \otimes \pi_2^\vee$ becomes a space of functions on $(\mathrm{GL}(2, F) \times \mathrm{GL}(2, F))/\Delta F^* \simeq \mathrm{GSO}(V^s)$ that transform correctly under upper triangular matrices. Restriction of these functions to $\mathrm{SO}(V^s)$ is an injective map. ■

For the next lemma see also [30], Lemma 8.1.

5.2 Lemma. *Let π_1 and π_2 be unramified representations of* $\mathrm{GL}(2, F)$ *with the same central character. Let σ_0 be the spherical component of the restriction of the corresponding representation $\pi_1 \otimes \pi_2^\vee$ of* $\mathrm{GSO}(V^s)$ *to* $\mathrm{SO}(V^s)$*. Then*

$$L(s, \sigma_0) = L(s, \pi_1 \times \pi_2^\vee),$$

where on the right we have the Rankin–Selberg L–factor of π_1 and $\pi_2^\vee$.

Proof: We can realize π_1 and π_2 as constituents of induced representations with unramified characters. The assertion thus follows from Lemma 5.1 and (22). ■

The statement of the next lemma is analogous to the formula for Jacquet modules of induced representations of $\mathrm{GSp}(4, F)$ given in section 2 of [32]. It is a consequence of Theorem 5.3 of [35]. We shall use the notations of these papers.

5.3 Lemma. *Let π be an admissible representation of* $\mathrm{GL}(2, F)$ *with Jacquet module $m^*(\pi) = 1 \otimes \pi + \sum_i \pi_i^1 \otimes \pi_i^2 + \pi \otimes 1$. Then the Jacquet module of the representation $\pi \rtimes 1$ of* $\mathrm{Sp}(4, F)$ *(induction from the Siegel parabolic) is given by*

$$\mu^*(\pi \rtimes 1) = 1 \otimes \pi \rtimes 1 + \left[\sum_i \pi_i^1 \otimes \pi_i^2 \rtimes 1 + \sum_i \tilde{\pi}_i^2 \otimes \pi_i^1 \rtimes 1\right] + \left[\pi \otimes 1 + \tilde{\pi} \otimes 1 + \sum_i \pi_i^1 \times \tilde{\pi}_i^2 \otimes 1\right].$$

For the next theorem, which is a modification of Theorem 8.3 of the paper [30], let F be an algebraic number field. Let D be a global quaternion algebra over F. If π_1 and π_2 are automorphic representations of $D^*(\mathbb{A})$ with the same central character, we can consider the automorphic representation $\pi = \pi_1 \otimes \pi_2^\vee$ of $\mathrm{GSO}(D, \mathbb{A})$. This π may have more than one "extension" to an automorphic representation σ of $\mathrm{GO}(D, \mathbb{A})$; see [30], Theorem 7.1 (which in turn is taken from [7]) for a more precise statement.

5.4 Theorem. *Let D be a quaternion algebra over the totally real number field F which ramifies at all archimedean places. Let π be a cuspidal automorphic representation of $PD^*(\mathbb{A})$ and let σ be an automorphic representation of $\mathrm{GO}(D,\mathbb{A})$ lying over the representation $\pi \otimes \mathbf{1}$ of $\mathrm{GSO}(D,\mathbb{A})$. If $\pi_v = \mathbf{1}_{D^*(F_v)}$ at a place v, then we assume that $\sigma_v = \mathbf{1}_{\mathrm{GO}(D,F_v)}$. If*

$$L(1/2,\pi) \neq 0,$$

then the global theta lift $\theta(\sigma)$ from $\mathrm{GO}(D,\mathbb{A})$ to $\mathrm{GSp}(4,\mathbb{A})$ is non-zero.

Proof: We will argue as in the proof of Theorem 8.3 in [30] and refer to that paper for some more details. Let σ_1 be an irreducible constituent of $\sigma\big|_{\mathrm{O}(D,\mathbb{A})}$. If we can show that σ_1 has a non-zero theta lift to $\mathrm{Sp}(4,\mathbb{A})$, we will be done. By our local hypothesis, each σ_v occurs in the theta correspondence between $\mathrm{GO}(D,F_v)$ and $\mathrm{GSp}(4,F_v)$ (see [30], Theorem 3.4). It follows that each local component $\sigma_{1,v}$ occurs in the theta correspondence between $\mathrm{O}(D,F_v)$ and $\mathrm{Sp}(4,F_v)$.

Let S be a finite set of places including all the archimedean ones and all places where $\sigma_{1,v}$ is ramified. By Lemma 5.2,

$$L^S(s,\sigma_1) = L^S(s,\pi \times \mathbf{1}_{\mathrm{GL}(2,\mathbb{A})}) = L^S(s-1/2,\pi)L^S(s+1/2,\pi).$$

Thus, by our hypothesis, $L^S(s,\sigma_1)$ does not vanish at $s=1$.

We would now like to apply the nonvanishing theorem (Theorem 1.2) of [29]. However, we need to find a substitute for the temperedness condition for the local components of σ_1 made in that theorem. As explained in the introduction of [29], the temperedness condition can be replaced by the non-vanishing of $L^S(s,\sigma_1)$ at certain points $s_X(k)$, $k>3$, together with an assumption on the Langlands data of the local theta lifts of $\sigma_{1,v}$ to $\mathrm{Sp}(2(k+1),F_v)$. The non-vanishing certainly holds, since in our case we have $s_X(k) = k-2$.

Let $\theta_k(\sigma_{1,v})$ denote any representation of $\mathrm{Sp}(2k,F_v)$ corresponding to $\sigma_{1,v}$ in the theta correspondence between $\mathrm{O}(D,F_v)$ and $\mathrm{Sp}(2k,F_v)$. We need to see that the Langlands data of $\theta_3(\sigma_{1,v})$ and $\theta_4(\sigma_{1,v})$ looks as follows:

$$\theta_3(\sigma_{1,v}) = L(\nu,\ldots), \qquad \theta_4(\sigma_{1,v}) = L(\nu^2,\ldots). \tag{23}$$

If D is ramified at v, then $\sigma_{1,v}$ is tempered since $\mathrm{O}(D,F_v)$ is compact, and we can use the results of [28]. By our hypothesis, this applies to all archimedean places. Let us therefore assume that v is finite and D splits at v. Then π_v is an irreducible, admissible representation of $\mathrm{PGL}(2,F_v)$. To prove (23) in this case, one can essentially follow the arguments in the proof of Theorem 4.4 of [28]. The temperedness hypothesis made in this theorem is used only at one point in the proof (top of page 1117), namely to exclude certain possibilities for irreducible subquotients of a Jacquet module of $\tau := \theta_2(\sigma_{1,v})$. Thus everything comes down to computing these Jacquet modules. We will only prove the first equation in (23), the argument for the second one being very similar.

Let $R_0(\tau)$ be the (twisted) Jacquet module of τ along the Siegel parabolic subgroup of $\mathrm{Sp}(4)$, and let $R_1(\tau)$ be the Jacquet module along the Klingen parabolic. Thus $R_0(\tau)$ is a representation of $\mathrm{GL}(2,F_v)$ and $R_1(\tau)$ is a representation of $\mathrm{GL}(1,F_v)\times\mathrm{SL}(2,F_v)$. To argue as in [28], we have to see that

$$R_0(\tau) \text{ has no irreducible subquotient of the form } |\det|^{-5/2}, \tag{24}$$

and

$$R_1(\tau) \text{ has no irreducible subquotient of the form } |\,|^{-2}\otimes\ldots \tag{25}$$

(this excludes the cases $i=0$ and $i=1$ on page 1117 of [28]; it would be automatically true if τ were tempered).

It is easily computed that the Jacquet module of the restriction of our representation $\pi_v\otimes\mathbf{1}$ to $\mathrm{SO}(4,F_v)$ along the Siegel parabolic is given by the representation $\nu^{1/2}\pi_v$ of $\mathrm{GL}(2,F_v)$. It follows by Frobenius reciprocity that $(\pi_v\otimes\mathbf{1})\big|_{\mathrm{SO}(4)}$ is a constituent of $\nu^{1/2}\pi_v\rtimes 1$ (induction from the Siegel parabolic subgroup). The Bernstein–Zelevinski data of this representation is therefore given by

$$[(\pi_v\otimes\mathbf{1})\big|_{\mathrm{SO}(4)}]=\begin{cases}[\nu^{1/2}\pi_v], & \text{if } \pi_v \text{ is supercuspidal},\\ [\xi\nu,\xi], & \text{if } \pi_v=\xi\,\mathrm{St}_{\mathrm{GL}(2)},\\ [\nu^{1/2}\chi,\nu^{1/2}\chi^{-1}], & \text{if } \pi_v=\pi(\chi,\chi^{-1}).\end{cases}$$

It follows from Theorem 2.5 of [15] that the theta lift $\tau=\theta_2(\sigma_{1,v})$ on $\mathrm{Sp}(4,F_v)$ has the same Bernstein–Zelevinski data. In other words,

$$\tau \text{ is a constituent of } \begin{cases}\nu^{1/2}\pi_v\rtimes 1, & \text{if } \pi_v \text{ is supercuspidal},\\ \xi\nu\times\xi\rtimes 1, & \text{if } \pi_v=\xi\,\mathrm{St}_{\mathrm{GL}(2)},\\ \nu^{1/2}\chi\times\nu^{1/2}\chi^{-1}\rtimes 1, & \text{if } \pi_v=\pi(\chi,\chi^{-1}).\end{cases}$$

The Jacquet modules of the representations on the right can be computed using the formula given in Lemma 5.3. It is then easily seen that (24) and (25) are indeed true. ∎

5.5 Lemma. *Let v_0 be a finite place of the totally real number field F and let D_0 be the unique division quaternion algebra over F_{v_0}. For any irreducible, admissible representation τ of $PD_0^*(F_{v_0})$ there exist a global quaternion algebra D over F and an automorphic cuspidal representation $\pi=\otimes\pi_v$ of $PD^*(\mathbb{A})$ with the following properties:*

i) $D\times_F F_{v_0}=D_0$, *i.e., D is ramified at v_0.*

ii) D is ramified at the archimedean places.

iii) $\pi_{v_0}=\tau$.

iv) $L(1/2,\pi)\neq 0$.

Proof: If F has an even number of real places, choose any finite place $v_2\neq v_0$ and let D be the unique quaternion algebra that is ramified precisely at v_0, v_2 and the real places. If F has an odd number of real places, let D be the unique quaternion algebra that is ramified precisely at v_0 and the real places. To have a unified notation, let in this case v_2 be any finite place different from v_0. In either case we can find *some* cuspidal automorphic representation $\tilde\pi=\otimes\tilde\pi_v$ of $PD^*(\mathbb{A})$ (greater than one-dimensional) such that $\tilde\pi_{v_0}=\tau$, see, for example, the end of [8]. Let $\sigma=\otimes\sigma_v$ be the Jacquet–Langlands lift of $\tilde\pi$. This is a cuspidal automorphic representation of $\mathrm{PGL}(2,\mathbb{A})$. To achieve iv) we are going to apply suitable quadratic twists.

Let v_1 be one of the real places of F. Let $\xi=\otimes\xi_v$ be a quadratic Hecke character of $\mathbb{A}_F^*$ such that

$$\xi_{v_1}(-1)=-1,\qquad \xi_v=1 \quad\text{for all real } v\neq v_1,\qquad \xi_{v_0}=1,\qquad \xi_{v_2}=1,$$
$$\text{and } \xi_v=1 \text{ for all finite } v \text{ such that } \sigma_v \text{ is square integrable}$$

(one can find such a ξ by considering suitable quadratic extensions of F). Since σ_{v_1} is a discrete series representation of $\mathrm{PGL}(2,\mathbb{R})$, it is invariant under quadratic twisting, so that in particular

$$\varepsilon(1/2,\xi_{v_1}\otimes\sigma_{v_1})=\varepsilon(1/2,\sigma_{v_1}).$$

On the other hand, for principal series representations, the ε–factor changes as follows:

$$\varepsilon(1/2,\, \xi_v \otimes \sigma_v) = \xi_v(-1)\varepsilon(1/2, \sigma_v) \qquad \text{for all } v \neq v_1$$

(this trivially holds for real $v \neq v_1$, for $v = v_0$, $v = v_2$, and all finite v such that σ_v is square integrable). It follows for the global ε–factors that

$$\varepsilon(1/2,\, \xi \otimes \sigma) = \Big(\prod_{v \neq v_1} \xi_v(-1)\Big)\varepsilon(1/2, \sigma) = \xi_{v_1}(-1)\varepsilon(1/2, \sigma) = -\varepsilon(1/2, \sigma).$$

We may thus assume from the beginning that $\varepsilon(1/2, \sigma) = 1$. Theorem B of [6] then tells us that $L(1/2, \chi \otimes \sigma) \neq 0$ for some quadratic character $\chi = \otimes \chi_v$ such that $\chi_{v_0} = 1$. If we let π be the cuspidal automorphic representation of $D^*(\mathbb{A})$ that corresponds to $\chi \otimes \sigma$ under the Jacquet–Langlands lifting, then all the conditions stated in the lemma are satisfied. ∎

5.6 Lemma. *Let F be a p–adic field and let $\pi = \pi(\chi, \chi^{-1})$ be a spherical (unramified) principal series representation of* $\mathrm{PGL}(2, F)$. *Then for the theta lift from* $\mathrm{GO}(V^s)$ *to* $\mathrm{GSp}(4, F)$ *we have*

$$\begin{aligned}\theta((\pi \otimes \mathbf{1})^+) &= \mathrm{SK}(\pi) = L((\nu^{1/2}\chi, \nu^{1/2}\chi^{-1}, \nu^{-1/2})) \\ &= \textit{unique irreducible quotient of } \nu^{1/2}\pi \rtimes \nu^{-1/2}.\end{aligned}$$

Proof: Theta lifts for spherical representations are known, see [25], section 6. For the Langlands data see [32], Lemma 3.3. ∎

5.7 Lemma. *Any p–adic field can be realized as the completion of a totally real number field.*

Proof: The following short argument uses the approximation theorem and is due to D. Prasad. Let the given p–adic field be generated over $\mathbb{Q}_p$ by the element α, and let f be the minimal polynomial of α.
Since $\mathbb{Q}$ is dense in $\mathbb{Q}_p \times \mathbb{R}$, we can find a rational polynomial g that is p–adically arbitrarily close to f and has only real roots. By Krasner's Lemma, one of these roots will generate the same field as α, proving our result. ∎

5.8 Proposition. *Let F be a p–adic field or $F = \mathbb{R}$, and let D be the unique quaternion division algebra over F. Then, using the notations of section 2, we have*

$$\mathrm{SK}(\tau) = \theta((\tau \otimes \mathbf{1}_{D^*})^+) \tag{26}$$

for every irreducible, admissible representation τ of PD^*.

Proof: For $F = \mathbb{R}$ this is just Lemma 4.1 i). Let F be p–adic. We shall write F_{v_0} instead of F and assume that F_{v_0} is the completion of the totally real number field F at the place v_0 (Lemma 5.7). Let us write D_0 instead of D and choose a global quaternion algebra D and a representation π of $D^*(\mathbb{A})$ as in Lemma 5.5. Let σ be any cusp form on $\mathrm{GO}(D, \mathbb{A})$ lying above the representation $\pi \otimes \mathbf{1}$ of $\mathrm{GSO}(D, \mathbb{A})$. We may assume that $\sigma_v = \mathbf{1}_{\mathrm{GO}(D, F_v)}$ whenever $\pi_v = \mathbf{1}_{D^*(F_v)}$. According to Theorem 5.4, the theta lift $\Pi := \theta(\sigma)$ is non-zero. Knowing the description of the local theta correspondence between GO(4) and GL(2) from [27], Theorem 7.4, it is an easy exercise to show that Π is cuspidal. By Lemma 5.6, Π is clearly a CAP representation of $\mathrm{PGSp}(4, \mathbb{A})$. More precisely, in the language of [22], Π is strongly associated to $(P, \mathrm{JL}(\pi), |\,|^{1/2})$, where $\mathrm{JL}(\pi)$ is the Jacquet–Langlands lift of π. According to [22], Theorem 2.2, there exists a cusp form $\tilde{\pi}$ on $\widetilde{\mathrm{SL}}(2,\mathbb{A})$ such that $\theta'_\psi(\tilde{\pi}) = \Pi$. Here θ'_ψ is the theta lifting from $\widetilde{\mathrm{SL}}(2)$ to $\mathrm{PGSp}(4) \simeq \mathrm{SO}(5)$ constructed using the additive character ψ. In particular we have

$$\theta'_{\psi_0}(\tilde{\pi}_{v_0}) = \Pi_{v_0} = \theta((\tau \otimes \mathbf{1}_{v_0})^+).$$

To identify $\tilde{\pi}_{v_0}$, consider the Waldspurger lift $\mathrm{Wald}_\psi(\tilde{\pi})$, see [39], [40]. This is a cusp form on $\mathrm{PGL}(2,\mathbb{A})$, and for almost every place v we have

$$\mathrm{Wald}_\psi(\tilde{\pi})_v = \theta^{-1}_{\psi_v}(\tilde{\pi}_v) = {\theta'}^{-1}_{\psi_v}\big(\theta^{-1}_{\psi_v}(\Pi_v)\big) = \pi_v,$$

where θ_ψ denotes the lifting from $\mathrm{PGL}(2)$ to $\widetilde{\mathrm{SL}}(2)$. For the last equality we have used Lemma 5.6 again. By strong multiplicity one it follows that $\mathrm{Wald}_\psi(\tilde{\pi}) = \mathrm{JL}(\pi)$. Thus we have at least identified the L–packet of $\tilde{\pi}_{v_0}$:

$$\tilde{\pi}_{v_0} \in \mathrm{Wald}^{-1}_{\psi_v}(\mathrm{JL}(\tau)) = \big\{\theta_\psi(\tau),\, \theta_\psi(\mathrm{JL}(\tau))\big\}.$$

But $\theta((\tau \otimes \mathbf{1}_{D^*})^+)$ is tempered (see [28], Theorem 4.2), and $\theta'_{\psi_0}(\theta_{\psi_0}(\mathrm{JL}(\tau))) = \mathrm{SK}(\mathrm{JL}(\tau))$ is not (Lemma 2.2). Consequently $\theta((\tau \otimes \mathbf{1}_{D^*})^+) = \theta'_\psi(\theta_\psi(\tau)) = \mathrm{SK}(\tau)$. ∎

Remark: By [30], Theorem 1.8, the formulation of Proposition 5.8 makes sense even if F is an extension of $\mathbb{Q}_2$.

6 Functoriality

Let F be a local field of characteristic 0. If we have two local parameters $\rho_1, \rho_2 : W'_F \to \mathrm{SL}(2,\mathbb{C})$ for $\mathrm{PGL}(2,F)$, their direct sum $\rho_1 \oplus \rho_2$ will be considered a parameter for $\mathrm{PGSp}(4,F)$. In the p–adic case its semisimple part is obtained by composing $(\tilde{\rho}_1, \tilde{\rho}_2)$ with the L–morphism (20).

The local Langlands correspondence for $\mathrm{GSp}(4)$ is presently not known, but some parts of it are conjectured with a certain amount of evidence. For this paper we are interested in the following special cases.

6.1 Conjecture. *Let F be local field of characteristic zero, possibly archimedean. Let $\rho : W'_F \to \mathrm{SL}(2,\mathbb{C})$ be the local parameter of the infinite-dimensional, irreducible, admissible, unitary representation π of $\mathrm{PGL}(2,F)$. Then we have the following special cases of the local Langlands correspondence for $\mathrm{PGSp}(4,F)$.*

i) The L–packet attached to the local parameter $\rho \oplus \rho_{\mathrm{triv}}$ consists of a single representation, namely the unique irreducible quotient of the induced representation $\nu^{1/2}\pi \rtimes \nu^{-1/2}$.

ii) If π is square-integrable, then the L–packet attached to the local parameter $\rho \oplus \rho_{\mathrm{St}}$ consists of two elements, namely

$$\pi_{\mathrm{ng}} = \theta((\pi^{\mathrm{JL}} \otimes \mathbf{1}_{D^*})^+) \qquad \text{and} \qquad \pi_W = \theta((\pi \otimes \mathrm{St}_{\mathrm{GL}(2)})^+).$$

Here π_W is obtained as a theta lift from $\mathrm{GO}(V^s)$ and is a generic representation. π_{ng} is a theta lift from $\mathrm{GO}(V^a)$ and is non-generic. (For notations see section 2.)

Remark: The fact that π_W is generic and π_{ng} is not generic is known. See [38], section 6. Note that by [30], Theorem 1.2, the formulation in ii) makes sense even if F is an extension of $\mathbb{Q}_2$.

It is very reasonable to assume part i) of this conjecture. The local parameter in question has image in the Klingen parabolic subgroup of $\mathrm{Sp}(4,\mathbb{C})$ and should therefore correspond to a representation induced from the Siegel parabolic subgroup of $\mathrm{GSp}(4,\mathbb{C})$. Strong evidence for part ii) of the conjecture is given in [30], where more general L–packets have been defined. See also Theorem 7, 1, of [24], which in turn is based on [38]. The archimedean case of the conjecture is true, see Lemma 4.1.

6.2 Proposition. *Let F be a local field of characteristic zero. We assume that Conjecture 6.1 holds. Let π be an infinite-dimensional, irreducible, admissible, unitary representation of* $\mathrm{PGL}(2,F)$.

i) $\Pi(\pi\otimes\mathbf{1})$ *as defined in (14) is a local functorial lifting of the representation* $\pi\otimes\mathbf{1}_{\mathrm{GL}(2)}$ *of* $\mathrm{PGL}(2,F)\times\mathrm{PGL}(2,F)$ *with respect to the L–morphism (20).*

ii) If π is square-integrable, then $\Pi(\pi\otimes\mathrm{St})$ *as defined in (15) is a local functorial lifting of the representation* $\pi\otimes\mathrm{St}_{\mathrm{GL}(2)}$.

Consequently, if F is now a number field, the global representation $\Pi(\pi\otimes\pi_S)$ constructed in Theorem 3.1 is a functorial lifting of the representation $\pi\otimes\pi_S$ of $\mathrm{PGL}(2,\mathbb{A})\times\mathrm{PGL}(2,\mathbb{A})$ *(at every place v of F).*

Proof: i) follows from Lemma 2.2, and ii) follows from Proposition 5.8. ∎

We now make some comments on base change for the representations $\Pi(\pi\otimes\pi_S)$ constructed in Theorem 3.1. In [4] a theory of base change for these representations was developed on the level of *packets*. More precisely, let F be a number field and E a cyclic extension of F of prime degree. For π a cusp form on $\mathrm{PGL}(2,\mathbb{A}_F)$, let $\Sigma(\pi)$ be the set of places v of F such that π_v is square integrable. Let $\mathrm{SK}(\pi)$ be the set of equivalence classes of representations $\Pi(\pi\otimes\pi_S)$, where S runs through subsets of $\Sigma(\pi)$ such that $(-1)^{\#S}=\varepsilon(1/2,\pi)$. In [4], the base change of the packet $\mathrm{SK}(\pi)$ is defined as

$$BC_{E/F}(\mathrm{SK}(\pi)):=\mathrm{SK}(BC_{E/F}(\pi)),$$

provided the right side exists ($BC_{E/F}(\pi)$ might no longer be cuspidal). With our knowledge on the functorial behaviour of Saito–Kurokawa liftings we can define base change on the level of individual representations. The definition that is compatible with L–groups is obviously

$$BC_{E/F}\big(\Pi(\pi\otimes\pi_S)\big):=\Pi\big(BC_{E/F}(\pi)\otimes BC_{E/F}(\pi_S)\big),$$

provided the right side exists. Note that $BC_{E/F}(\pi_S)$ is the automorphic representation π_T of $\mathrm{GL}(2,\mathbb{A}_E)$, where T is the set of all places of E dividing a place in S. According to Theorem 3.1, there are the following obstructions for $BC_{E/F}\big(\Pi(\pi\otimes\pi_S)\big)$ to be a Saito–Kurokawa representation.

i) $BC_{E/F}(\pi)$ might no longer be a cusp form.

ii) For some $w\in T$ the local component $BC_{E/F}(\pi)_w$ might be a principal series representation. Equivalently, for some $v\in S$ the square integrable representation π_v lifts to a principal series representation under BC_{E_w/F_v} for some $w|v$.

iii) The sign condition $(-1)^{\#T}=\varepsilon(1/2,BC_{E/F}(\pi))$ might be violated.

It turns out that if the degree $[E:F]$ is an odd prime number, that none of i), ii) or iii) can occur. This follows from known properties of base change for $\mathrm{GL}(2)$ and easy computations similar to those in section 4 of [4]. It also turns out that if the cuspidality condition in part b) of Theorem 3.1 is fulfilled for $\Pi(\pi\otimes\pi_S)$, then it is also fulfilled for $BC_{E/F}\big(\Pi(\pi\otimes\pi_S)\big)$. Thus, in the odd degree case, the base change of a (cuspidal) Saito–Kurokawa representation is again a (cuspidal) Saito–Kurokawa representation. It may however happen that $BC_{E/F}\big(\Pi(\pi\otimes\pi_S)\big)$ is cuspidal even if $\Pi(\pi\otimes\pi_S)$ is not.

The situation is more complicated if $[E:F]=2$, since in this case each of the obstructions above can occur. We refrain from formulating the precise conditions under which the base change of a Saito–Kurokawa representation is again a Saito–Kurokawa representation. A precise count of the number of elements of the packet $BC_{E/F}(\mathrm{SK}(\pi))$ is given in Theorem 4.2 of [4].

7 Description of the p–adic liftings

In the following we shall describe in more detail the local lifts $\Pi(\pi\otimes\mathbf{1})$ and $\Pi(\pi\otimes\mathrm{St})$ for $\pi\in\mathrm{Irr}(\mathrm{PGL}(2,F))$, where F is a p–adic field We shall make use of the notation of [32] for induced representations of $\mathrm{GSp}(4,F)$.

Principal series representations

Assume that the local field F is p–adic. Let $\pi=\pi(\chi,\chi^{-1})=\chi\times\chi^{-1}$ be a principal series representation of $\mathrm{PGL}(2,F)$, with χ a character of F^*. We shall only be interested in unitary representations, hence we assume that $|\chi|=|\;|^e$ with $0\le e<1/2$. By the definition in (14) and Lemma 2.2, $\Pi(\pi\otimes\mathbf{1})$ is the unique irreducible quotient of the induced representation $\nu^{1/2}\pi\rtimes\nu^{-1/2}=\nu^{1/2}\chi\times\nu^{1/2}\chi^{-1}\rtimes\nu^{-1/2}$. In the notation of [32],

$$\Pi\big(\pi(\chi,\chi^{-1})\otimes\mathbf{1}\big)=L((\nu^{1/2}\chi,\,\nu^{1/2}\chi^{-1},\,\nu^{-1/2})). \tag{27}$$

This is a unitary representation by [32], Theorem 4.4 (iii) and (v). Note that by [32], Lemma 3.3 (for $\chi^2\neq\mathbf{1}$) resp. Lemma 3.7 (for $\chi^2=\mathbf{1}$) we have

$$\Pi\big(\pi(\chi,\chi^{-1})\otimes\mathbf{1}\big)\simeq\chi\,\mathbf{1}_{\mathrm{GL}(2)}\rtimes\chi^{-1}, \tag{28}$$

a degenerate principal series representation.

The Steinberg representation

Now consider the Steinberg representation $\mathrm{St}_{\mathrm{GL}(2)}$ which has the two liftings $\Pi(\mathrm{St}\otimes\mathbf{1})$ and $\Pi(\mathrm{St}\otimes\mathrm{St})$. By definition, $\Pi(\mathrm{St}\otimes\mathbf{1})$ equals the Langlands quotient $L((\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}))$. We shall now determine $\Pi(\mathrm{St}\otimes\mathrm{St})$ more explicitly. By the definition given in (15) we have

$$\Pi(\mathrm{St}\otimes\mathrm{St})=\theta'(\theta(\mathrm{St}^{\mathrm{JL}}))=\theta'(\theta(\mathbf{1}_{D^*})),$$

where the inner theta is the lifting from the quaternion unit group PD^* to the metaplectic group $\widetilde{\mathrm{SL}}(2,F)$. By [39], [40], the lifting $\theta(\mathbf{1}_{D^*})$ is a *special representation* of the metaplectic group, which is a consituent of a certain induced representation. Hence we know the Bernstein–Zelevinski data of $\theta(\mathbf{1}_{D^*})$. By the results of [15], we then know the Bernstein–Zelevinski data of $\theta'(\theta(\mathbf{1}_{D^*}))$, which is a representation of $\mathrm{SO}(5,F)\simeq\mathrm{PGSp}(4,F)$. Pulling back to $\mathrm{GSp}(4,F)$, the result is that $\Pi(\mathrm{St}\otimes\mathrm{St})$ is a constituent of the induced representation $\nu\times\mathbf{1}_{F^*}\rtimes\nu^{-1/2}$. We quote from [32], Lemma 3.8, how this representation decomposes:

$$\nu\times\mathbf{1}_{F^*}\rtimes\nu^{-1/2}=\underbrace{\nu^{1/2}\,\mathrm{St}\rtimes\nu^{-1/2}}_{\text{sub}}+\underbrace{\nu^{1/2}\,\mathbf{1}_{\mathrm{GL}(2)}\rtimes\nu^{-1/2}}_{\text{quot}}=\underbrace{\mathbf{1}_{F^*}\rtimes\mathrm{St}}_{\text{sub}}+\underbrace{\mathbf{1}_{F^*}\rtimes\mathbf{1}_{\mathrm{GL}(2)}}_{\text{quot}}, \tag{29}$$

and each of the four representations on the right side again decomposes into two irreducible constituents. These are summarized in the following table. The quotients are on the bottom resp. on the right.

	$\mathbf{1}_{F^*}\rtimes\mathrm{St}$	$\mathbf{1}_{F^*}\rtimes\mathbf{1}_{\mathrm{GL}(2)}$
$\nu^{1/2}\,\mathrm{St}\rtimes\nu^{-1/2}$	$\tau(S,\nu^{-1/2})$	$L((\nu^{1/2}\,\mathrm{St},\nu^{-1/2}))$
$\nu^{1/2}\,\mathbf{1}_{\mathrm{GL}(2)}\rtimes\nu^{-1/2}$	$\tau(T,\nu^{-1/2})$	$L((\nu,\mathbf{1}_{F^*}\rtimes\nu^{-1/2}))$

(30)

Here $\tau(S,\nu^{-1/2})$ and $\tau(T,\nu^{-1/2})$ are certain essentially tempered but not square integrable representations. By Proposition 5.8 we have $\Pi(\mathrm{St}\otimes\mathrm{St})=\theta((\mathbf{1}_{D^*}\otimes\mathbf{1}_{D^*})^+)$. It therefore follows from [28], Theorem 4.2, that $\Pi(\mathrm{St}\otimes\mathrm{St})$ is tempered. Since the constituents of $\mathbf{1}_{F^*}\rtimes\mathbf{1}_{\mathrm{GL}(2)}$ are not tempered, $\Pi(\mathrm{St}\otimes\mathrm{St})$ must be equal to either $\tau(S,\nu^{-1/2})$ or $\tau(T,\nu^{-1/2})$. But we know from [38], section 6, that $\theta((\mathbf{1}_{D^*}\otimes\mathbf{1}_{D^*})^+)$ is not generic, while $\tau(S,\nu^{-1/2})$ is generic. It follows that

$$\Pi(\mathrm{St}\otimes\mathrm{St})=\tau(T,\nu^{-1/2}). \tag{31}$$

We have stated in Proposition 6.2 that if Conjecture 6.1 ii) is true, then $\Pi(\mathrm{St}\otimes\mathrm{St})$ is a functorial lifting of the representation $\mathrm{St}_{\mathrm{GL}(2)}\otimes\mathrm{St}_{\mathrm{GL}(2)}$ of $\mathrm{PGL}(2,F)\times\mathrm{PGL}(2,F)$. We now give two more reasons why the conjectural local Langlands correspondence should indeed attach the parameter $\rho_{\mathrm{St}}\oplus\rho_{\mathrm{St}}$ to $\tau(T,\nu^{-1/2})$.

- The image of the parameter $\rho_{\mathrm{St}}\oplus\rho_{\mathrm{St}}$ can be conjugated by a suitable Cayley transformation into the standard Siegel parabolic subgroup of $\mathrm{Sp}(4,\mathbb{C})$. The representation(s) of $\mathrm{PGSp}(4,F)$ corresponding to this parameter should therefore be induced from the Klingen parabolic subgroup. More precisely, after pulling back to $\mathrm{GSp}(4,F)$, the induced representation is $\mathbf{1}_{F^*}\rtimes\mathrm{St}$. The unique irreducible quotient of this representation is $\tau(T,\nu^{-1/2})$.
- With all the constituents of $\nu\times\mathbf{1}_{F^*}\rtimes\nu^{-1/2}$ being Iwahori–spherical, one can compute their local parameters attached by Kazhdan and Lusztig, see [10]. The result is that $\tau(S,\nu^{-1/2})$ and $\tau(T,\nu^{-1/2})$ constitute an L–packet with parameter $\rho_{\mathrm{St}}\oplus\rho_{\mathrm{St}}$.

The fact that the generic representation $\tau(S,\nu^{-1/2})$ has local parameter $\rho_{\mathrm{St}}\oplus\rho_{\mathrm{St}}$ is also supported by Theorem 4.1 of [36], where the Novodvorski L–factor of $\tau(S,\nu^{-1/2})$ is computed as $L(s,\tau(S,\nu^{-1/2}))=L(s+1/2,\mathbf{1}_{F^*})^2=L(s,\mathrm{St}_{\mathrm{GL}(2)})^2$.

Twists of the Steinberg representation

Let ξ be a character of F^* of order two (thus ξ is quadratic but non-trivial), and consider $\pi=\xi\,\mathrm{St}_{\mathrm{GL}(2)}$, a twist of the Steinberg representation. By Conjecture 6.1 i), the parameter $\rho_{\xi\,\mathrm{St}}\oplus\rho_{\mathrm{triv}}$ corresponds to the unique irreducible quotient of the induced representation $\nu^{1/2}\xi\,\mathrm{St}_{\mathrm{GL}(2)}\rtimes\nu^{-1/2}$. By [32], Lemma 3.6, this representation has length 2. It is reasonable to suspect that the subrepresentation has local parameter $\rho_{\xi\,\mathrm{St}}\oplus\rho_{\mathrm{St}}$. It is square integrable and is denoted by $\delta([\xi,\nu\xi],\nu^{-1/2})$ in [32]. We abbreviate this by $\delta(\xi)$. Another indication that $\delta(\xi)$ has local parameter $\rho_{\xi\,\mathrm{St}}\oplus\rho_{\mathrm{St}}$ comes from the paper [36], where the Novodvorski L–factor of $\delta(\xi)$ is computed as $L(s,\delta(\xi))=L(s+1/2,\mathbf{1}_{F^*})L(s+1/2,\xi)$.

But since $\delta(\xi)$ is generic, it is not equal to $\Pi(\xi\,\mathrm{St}\otimes\mathrm{St})$. By Conjecture 6.1 ii), the L-packet with parameter $\rho_{\xi\,\mathrm{St}}\oplus\rho_{\mathrm{St}}$ has two members, and by Proposition 5.8, our $\Pi(\xi\,\mathrm{St}\otimes\mathrm{St})=\theta((\xi\,\mathbf{1}_{D^*}\otimes\mathbf{1}_{D^*})^+)$ is the non-generic member of this packet. Hence the situation is as in (21).

By [40], Proposition 8, the theta lifting $\theta(\xi\,\mathbf{1}_{D^*})$ from D^* to $\widetilde{\mathrm{SL}}(2,F)$ is an "odd" Weil representation, which is supercuspidal. By the first occurrence principle, $\Pi(\xi\,\mathrm{St}\otimes\mathrm{St})=\theta'(\theta(\xi\,\mathbf{1}_{D^*}))$ is also supercuspidal. The supercuspidal representation of $\mathrm{GSp}(4,F)$ with local parameter $\rho_{\xi\,\mathrm{St}}\oplus\rho_{\mathrm{St}}$ was considered in the paper [12] and was identified as a representation of type θ_{10}.

Supercuspidal representations

Now assume that π is a *supercuspidal* representation of $\mathrm{PGL}(2,F)$ with local parameter ρ. By Conjecture 6.1 i), the parameter $\rho\oplus\rho_{\mathrm{St}}$ corresponds to the Langlands quotient of the representation $\nu^{1/2}\pi\rtimes\nu^{-1/2}$. The following theorem concerning this induced representation is due to Shahidi (see [33], Theorem 5.1 and the examples in section 6).

7.1 Theorem. *The induced representation* $\nu^{1/2}\pi \rtimes \nu^{-1/2}$ *on* $\mathrm{GSp}(4,F)$ *has length* 2. *The subrepresentation is generic and square integrable. The Langlands quotient is unitary, non-tempered and not generic.*

Thus we see that our lift $\Pi(\pi \otimes \mathbf{1}) = L((\nu^{1/2}\pi, \nu^{-1/2}))$ is unitary and non-generic, as all the others before. The subrepresentation of $\nu^{1/2}\pi \rtimes \nu^{-1/2}$ should have parameter $\rho \oplus \rho_{\mathrm{St}}$ (supported by the L–function computation of [36]). However, it cannot be equal to our lifting $\Pi(\pi \otimes \mathrm{St})$ because the latter is supercuspidal by a similar reasoning as above invoking the first occurrence principle. Again we have the same situation as in (21) with $\Pi(\pi \otimes \mathrm{St}) = \theta((\pi^{\mathrm{JL}} \otimes \mathbf{1}_{D^*})^+)$ being the non-generic member of a (conjectural) L–packet.

Summary

The following table summarizes all the local liftings $\Pi(\pi \otimes \mathbf{1})$ and $\Pi(\pi \otimes \mathrm{St})$ we defined. All the PGSp(4) representations in the table are unitary and non-generic.

$\mathrm{PGL}(2) \times \mathrm{PGL}(2)$	$\mathrm{PGSp}(4)$	remarks
	principal series representations	
$\pi(\chi, \chi^{-1}) \otimes \mathbf{1}$	$L((\nu^{1/2}\chi,\, \nu^{1/2}\chi^{-1},\, \nu^{-1/2}))$ $\simeq \chi\, \mathbf{1}_{\mathrm{GL}(2)} \rtimes \chi^{-1}$	non-tempered
	$p-$adic	
$\mathrm{St} \otimes \mathbf{1}$	$L((\nu^{1/2}\mathrm{St}, \nu^{-1/2}))$	non-tempered
$\mathrm{St} \otimes \mathrm{St}$	$\tau(T, \nu^{-1/2})$	tempered
$\xi\, \mathrm{St} \otimes \mathbf{1},\ \mathrm{ord}(\xi) = 2$	$L((\nu^{1/2}\xi\, \mathrm{St}, \nu^{-1/2}))$	non-tempered
$\xi\, \mathrm{St} \otimes \mathrm{St},\ \mathrm{ord}(\xi) = 2$	$\theta((\xi\, \mathbf{1}_{D^*} \otimes \mathbf{1}_{D^*})^+)$	supercuspidal
$\pi \otimes \mathbf{1},\ \pi$ supercusp.	$L((\nu^{1/2}\pi, \nu^{-1/2}))$	non-tempered
$\pi \otimes \mathrm{St},\ \pi$ supercusp.	$\theta((\pi^{\mathrm{JL}} \otimes \mathbf{1}_{D^*})^+)$	supercuspidal
	real	
$\mathcal{D}(1) \otimes \mathbf{1}$	σ_2^-	non-tempered
$\mathcal{D}(1) \otimes \mathrm{St}$	σ_2^+	limit of disc. ser.
$\mathcal{D}(2k-3) \otimes \mathbf{1}$ $(k \geq 3)$	$L((\nu^{1/2}\, \mathcal{D}(2k-3), \nu^{-1/2}))$ $\simeq \sigma_k^-$	non-tempered
$\mathcal{D}(2k-3) \otimes \mathrm{St}$ $(k \geq 3)$	σ_k^+	holomorphic discrete series representation

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