Iwahori–spherical representations of $\mathrm{GSp}(4)$ and Siegel modular forms of degree 2 with square-free level

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Abstract. A theory of local old– and newforms for representations of GSp(4) over a p–adic field with Iwahori–invariant vectors is developed. The results are applied to Siegel modular forms of degree 2 with square-free level with respect to various congruence subgroups.

Introduction

For representations of GL(2) over a p–adic field F there is a well-known theory of local newforms due to Casselman, see [Cas]. This local theory together with the global strong multiplicity one theorem for cuspidal automorphic representations of GL(2) is reflected in the classical Atkin–Lehner theory for elliptic modular forms. On the other hand, there is currently no satisfactory theory of local newforms for the group $\mathrm{GSp}(4,F)$. As a consequence, there is no analogue of Atkin–Lehner theory for Siegel modular forms of degree 2. It is the goal of this paper to provide such theories for the "square-free" case. In the local context this means that the representations in question are assumed to have non-trivial Iwahori–invariant vectors. In the global context it means that we are considering various congruence subgroups of square-free level.

This paper is organized into three parts. In the first part we shall take from [ST] the complete list of irreducible, admissible representations of $\mathrm{GSp}(4,F)$ supported in the minimal parabolic subgroup and list their basic properties (Table 1). We shall describe the local Langlands correspondence for these representations and give all the local parameters and local factors (Table 2). Assuming the inducing characters are unramified, we shall compute the dimensions of the spaces of fixed vectors under any parahoric subgroup for each of these representations (Table 3).

In the second part of this paper we shall define local new– and oldforms with respect to a parahoric subgroup. Our main local result is Theorem 2.3.1, saying that, with respect to a fixed parahoric subgroup, a representation has either oldforms or newforms, but never both. In Table 3 the spaces of newforms have been indicated by writing their dimensions in bold face. We see that in almost all cases the space of newforms (with respect to a fixed parahoric subgroup) is one-dimensional, but there are two exceptions.

In the third part we will apply the previously obtained local results to prove several theorems on classical Siegel modular forms "of square-free level". We will need the spin (degree 4) L–function of GSp(4) as a global tool. Even though we only need the usual analytic properties of this L–function for global representations whose local components at finite places are all Iwahori–spherical, none of the current results on this L–function seems to satisfy all our needs. We shall therefore *assume* that an L–function theory with the desired properties exists. Under

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this assumption, we shall prove something similar to a "strong multiplicity one" result for certain cuspidal automorphic representations of GSp(4), but without actually knowing multiplicity one. We shall then define old– and newforms for Siegel modular forms with respect to three different congruence subgroups: The "minimal" congruence subgroup $U_\emptyset(N)$ (corresponding to the local Iwahori subgroups), the usual Hecke subgroup $\Gamma_0(N)$ (for systematic reasons here called $U_1(N)$), and the paramodular group $U_{02}(N)$. In each case we shall prove several results that would be expected from any reasonable notion of newforms. For example, if a newform is an eigenform at *almost all* good places, then it is an eigenform at *all* good places. We shall also describe Euler factors at bad places and define the completed spin L–function for these modular forms.

We shall now make some more comments on the local data given in Table 3. As mentioned above, if a dimension in this table is typed in bold face, then the space consists entirely of newforms, otherwise entirely of oldforms. We see that many representations have newforms with respect to two different parahoric subgroups. Amongst the unitary representations only those of type IIIa have a two-dimensional space of newforms with respect to P_1, the "Hecke" subgroup. In a sense, this can be naturally repaired in the global theory by considering a certain Hecke operator T_2, see section 3.3.

The signs in Table 3 indicate eigenvalues of the *Atkin–Lehner involution* where this makes sense, namely for the "symmetric" parahoric subgroups and for representations with trivial central character. The column "ε" gives the value of the ε–factor of the representation at 1/2. Investigating Table 3, we find an interesting relation between Atkin–Lehner eigenvalues and ε–factors. Roughly speaking, the trace of the Atkin–Lehner involution on the full space of newforms is closely related to the sign defined by the ε–factor. See Proposition 1.3.1 for a more precise statement.

There have been several attempts in the literature to define a good notion of old and new Siegel modular forms. The first one seems to be IBUKIYAMA [Ib1], who defines old– and newforms for the minimal congruence subgroup $B(p)$. Then there is [Ib2], where definitions for the paramodular group of prime level are given. In both cases the definitions coincide with ours. The motivation to single out newforms in [Ib1] and [Ib2] comes from the comparison of global dimension formulas, providing further evidence that these are the "correct" definitions. ANDRIANOV [An2] has defined newforms for $\Gamma_0(N)$ for any N, not only in the square-free case. Recently, a definition of newforms for $\Gamma_0(p)$ that is equivalent to ours has been given by RASTEGAR [Ra] in a more geometric setting.

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Notation

We shall realize the algebraic group GSp(4) as the set of matrices $g \in \mathrm{GL}(4)$ that satisfy

$$ {}^t gJg = \lambda(g)J \text{ for some } \lambda(g) \in \mathrm{GL}(1), \qquad \text{where } J = \begin{pmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{pmatrix}. $$

This defines a homomorphism $\lambda : \mathrm{GSp}(4) \to \mathrm{GL}(1)$, called the *multiplier homomorphism*, whose kernel is by definition the symplectic group Sp(4). As a minimal parabolic subgroup of GSp(4) we choose upper triangular matrices. There are two conjugacy classes of maximal parabolic subgroups, represented by the *Siegel parabolic subgroup* P, whose Levi factor is

$$ M_P = \left\{ \begin{pmatrix} A & \\ & uA' \end{pmatrix} : u \in \mathrm{GL}(1),\ A \in \mathrm{GL}(2) \right\} \simeq \mathrm{GL}(1) \times \mathrm{GL}(2), $$

where $A' := \begin{pmatrix} & 1\\ 1 & \end{pmatrix} {}^t A^{-1} \begin{pmatrix} & 1\\ 1 & \end{pmatrix}$, and the *Klingen parabolic subgroup* Q, whose Levi factor is

$$ M_Q = \left\{ \begin{pmatrix} u && \\ & A & \\ && u^{-1}\det(A) \end{pmatrix} : u \in \mathrm{GL}(1),\ A \in \mathrm{GL}(2) \right\} \simeq \mathrm{GL}(1) \times \mathrm{GL}(2). $$

Let F be a non-archimedean local field. We shall employ the notations of [ST] for representations of $\mathrm{GSp}(4,F)$. For characters χ_1, χ_2 and σ of F^* let $\chi_1 \times \chi_2 \rtimes \sigma$ be the representation of $G(F) = \mathrm{GSp}(4,F)$ induced from the character

$$ \begin{pmatrix} t_1 & * & * & * \\ & t_2 & * & * \\ && ut_2^{-1} & * \\ &&& ut_1^{-1} \end{pmatrix} \longmapsto \chi_1(t_1)\chi_2(t_2)\sigma(u) $$

of the Borel subgroup. The induction is always normalized, i.e., the standard space of $\chi_1 \times \chi_2 \rtimes \sigma$ consists of $\mathbb{C}$–valued functions on $\mathrm{GSp}(4,F)$ with the transformation property

$$ f\left(\begin{pmatrix} t_1 &&& * \\ & t_2 && \\ && ut_2^{-1} & \\ &&& ut_1^{-1} \end{pmatrix} g \right) = \chi_1(t_1)\,\chi_2(t_2)\,\sigma(u)\,|t_1^2 t_2|\,|u|^{-3/2} f(g). \qquad (1) $$

The central character of this representation is $\chi_1\chi_2\sigma^2$. Provided that $e(\chi_1) \geq e(\chi_2) > 0$, where $e(\chi_i)$ denotes the real number with $|\chi_i(x)| = |x|^{e(\chi_i)}$ (the *exponent*), let $L((\chi_1,\chi_2,\sigma))$ be the unique irreducible quotient (the *Langlands quotient*) of $\chi_1 \times \chi_2 \rtimes \sigma$ (see [ST], section 1). If π is a representation of $\mathrm{GL}(2,F)$ and σ a character of F^* let $\pi \rtimes \sigma$ be the representation of $\mathrm{GSp}(4,F)$ induced from the representation

$$ \begin{pmatrix} A & * \\ & uA' \end{pmatrix} \longmapsto \sigma(u)\pi(A) $$

of $P(F)$. The exponent $e(\pi)$ is the unique real number such that $||^{-e(\pi)}\pi$ is unitarizable. Provided that π is square integrable and $e(\pi) > 0$, the induced representation $\pi \rtimes \sigma$ has a unique Langlands quotient, denoted by $L((\pi, \sigma))$. Finally, assume that χ is a character of F^* and σ a representation of $\mathrm{GSp}(2, F) = \mathrm{GL}(2, F)$. Then $\chi \rtimes \sigma$ denotes the representation of $\mathrm{GSp}(4, F)$ induced from the representation

$$\begin{pmatrix} u & * & * \\ & A & * \\ & & u^{-1}\det(A) \end{pmatrix} \longmapsto \chi(u)\sigma(A)$$

of $Q(F)$. If $e(\chi) > 0$, there is a unique Langlands quotient $L((\chi, \sigma))$. For the familiar induced representation $\pi(\chi_1, \chi_2)$ of $\mathrm{GL}(2, F)$ we shall use the symbol $\chi_1 \times \chi_2$. Note that if $\mathrm{GL}(2)$ is considered as the group of symplectic similitudes $\mathrm{GSp}(2)$, then $\chi_1 \rtimes \chi_2 = \chi_1\chi_2 \times \chi_2$. As in [ST] we shall write $\nu(x) = |x|$ for the normalized absolute value on the local field F.

1 Representations supported in the minimal parabolic subgroup

By [Bo2], the Iwahori–spherical representations we are interested in are precisely the constituents of representations parabolically induced from an unramified character of the minimal parabolic subgroup. We shall therefore begin by making a complete list of such induced representations and document their basic properties. Most results are taken from [ST]. In addition we shall describe the local Langlands correspondence for these representations and compute all the local factors (Table 2). After that we will compute the dimensions of spaces of fixed vectors under each parahoric subgroup for each representation in our list. The results are summarized in Table 3, which is quite important for this paper.

1.1 The list of irreducible representations

The reducibilities of the representations of $\mathrm{GSp}(4, F)$ parabolically induced from a character of the minimal parabolic subgroup were all determined in the paper [ST]. This paper contains also a complete list of unitary, tempered and square integrable representations supported in the Borel subgroup. In the following we shall divide the irreducible representations of $\mathrm{GSp}(4, F)$ supported in the minimal parabolic subgroup into six groups I–VI and briefly describe each group.

Group I: Irreducible representations of the form $\chi_1 \times \chi_2 \rtimes \sigma$ with characters χ_1, χ_2, σ of F^*.

By [ST], Lemma 3.2, the induced representation $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible if and only if $\chi_1 \neq \nu^{\pm 1}$, $\chi_2 \neq \nu^{\pm 1}$ and $\chi_1 \neq \nu^{\pm 1}\chi_2^{\pm 1}$.

Group II: Constituents of $\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$, where $\chi \notin \{\nu^{\pm 1}, \nu^{\pm 3}\}$.

By [ST], Lemma 3.3 and Lemma 3.7, there are two constituents. The unique irreducible subrepresentation is $\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$, and the quotient is isomorphic to $\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$.

Group III: Constituents of $\chi \times \nu \rtimes \nu^{-1/2}\sigma$, where $\chi \notin \{\mathbf{1}, \nu^{\pm 2}\}$.

By [ST], Lemma 3.4 and Lemma 3.9, there are the two irreducible constituents $\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$ and $\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(1)}$, the latter one being the quotient.

Group IV: Constituents of $\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$.

By [ST], Lemma 3.5, we have (in the Grothendieck group)

$$\begin{aligned}\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma &= \nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-3/2}\sigma + \nu^{3/2}\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-3/2}\sigma \\ &= \nu^2 \rtimes \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)} + \nu^2 \rtimes \nu^{-1}\sigma\mathbf{1}_{\mathrm{GSp}(2)}.\end{aligned}$$

Each of the four representations on the right is reducible and has two irreducible constituents as shown in the following table. The quotients are on the bottom resp. on the right.

	$\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-3/2}\sigma$	$\nu^{3/2}\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-3/2}\sigma$
$\nu^2 \rtimes \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$L((\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)}))$
$\nu^2 \rtimes \nu^{-1}\sigma\mathbf{1}_{\mathrm{GSp}(2)}$	$L((\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma))$	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$

(2)

Group V: Constituents of $\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma$, where ξ_0 is a non-trivial quadratic character.

According to [ST] Lemma 3.6 we have

$$\begin{aligned}\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma &= \underbrace{\nu^{1/2}\xi_0\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma}_{\text{sub}} + \underbrace{\nu^{1/2}\xi_0\,\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma}_{\text{quot}} \\ &= \underbrace{\nu^{1/2}\xi_0\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \xi_0\nu^{-1/2}\sigma}_{\text{sub}} + \underbrace{\nu^{1/2}\xi_0\,\mathbf{1}_{\mathrm{GL}(2)} \rtimes \xi_0\nu^{-1/2}\sigma}_{\text{quot}}.\end{aligned}$$

Each of the representations on the right side has two constituents as indicated in the following table. The quotients appear on the bottom resp. on the right.

	$\nu^{1/2}\xi_0\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \xi_0\nu^{-1/2}\sigma$	$\nu^{1/2}\xi_0\,\mathbf{1}_{\mathrm{GL}(2)} \rtimes \xi_0\nu^{-1/2}\sigma$
$\nu^{1/2}\xi_0\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma$	$\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)$	$L((\nu^{1/2}\xi_0\,\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$
$\nu^{1/2}\xi_0\,\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma$	$L((\nu^{1/2}\xi_0\,\mathrm{St}_{\mathrm{GL}(2)}, \xi_0\nu^{-1/2}\sigma))$	$L((\nu\xi_0, \xi_0 \rtimes \nu^{-1/2}\sigma))$

(3)

Here $\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)$ is a square integrable representation.

Group VI: Constituents of $\nu \times \mathbf{1}_{F^*} \rtimes \nu^{-1/2}\sigma$.

By [ST] Lemma 3.8, we have

$$\begin{aligned}\nu \times \mathbf{1}_{F^*} \rtimes \nu^{-1/2}\sigma &= \underbrace{\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma}_{\text{sub}} + \underbrace{\nu^{1/2}\,\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma}_{\text{quot}} \\ &= \underbrace{\mathbf{1}_{F^*} \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}}_{\text{sub}} + \underbrace{\mathbf{1}_{F^*} \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}}_{\text{quot}},\end{aligned}$$

and each representation on the right side is again reducible. Their constituents are summarized in the following table. Again the quotients appear on the bottom resp. on the right.

	$\mathbf{1}_{F^*} \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\mathbf{1}_{F^*} \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$
$\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma$	$\tau(S,\nu^{-1/2}\sigma)$	$L((\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma))$
$\nu^{1/2}\,\mathbf{1}_{\mathrm{GL}(2)} \rtimes \nu^{-1/2}\sigma$	$\tau(T,\nu^{-1/2}\sigma)$	$L((\nu,\mathbf{1}_{F^*} \rtimes \nu^{-1/2}\sigma))$

(4)

The representations $\tau(S,\nu^{-1/2}\sigma)$ and $\tau(T,\nu^{-1/2}\sigma)$ are tempered but not square integrable.

Table 1 below summarizes the basic properties of the irreducible representations of $\mathrm{GSp}(4,F)$ supported in the minimal parabolic subgroup. Complete information on unitarizability can be found in [ST], Theorem 4.4. The same paper tells us which of the unitary representations are tempered or square–integrable. In the column labeled "g" we have indicated the generic representations. If the characters are in the "Langlands position", then these are always the subrepresentations, see [CS]. The last column of Table 1 indicates the *local Saito–Kurokawa liftings*. These are certain local functorial liftings from $\mathrm{PGL}(2)\times\mathrm{PGL}(2)$ coming from the standard embedding of L–groups

$$\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{Sp}(4,\mathbb{C}).$$

For the global theory it is interesting to know which local representations are Saito–Kurokawa lifts, because, as the name indicates, these are the local components of the classical (and some less classical) Saito–Kurokawa liftings. See [Sch3] and [Sch4] for more information.

		constituent of	representation	tempered	L^2	g	SK
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)		$\chi_i,\sigma\in(F^*)^\wedge$		•	
II	a	$\nu^{1/2}\chi\times\nu^{-1/2}\chi\rtimes\sigma$	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	$\chi,\sigma\in(F^*)^\wedge$		•	
	b	$(\chi^2\notin\{\nu^{\pm1},\nu^{\pm3}\})$	$\chi\mathbf{1}_{\mathrm{GL}(2)}\rtimes\sigma$				•
III	a	$\chi\times\nu\rtimes\nu^{-1/2}\sigma$	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\chi,\sigma\in(F^*)^\wedge$		•	
	b	$(\chi\notin\{\mathbf{1},\nu^{\pm2}\})$	$\chi\rtimes\sigma\mathbf{1}_{\mathrm{GSp}(2)}$				
IV	a		$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$\sigma\in(F^*)^\wedge$	•	•	
	b	$\nu^2\times\nu\rtimes\nu^{-3/2}\sigma$	$L((\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)}))$				
	c		$L((\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma))$				
	d		$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$				
V	a		$\delta([\xi_0,\nu\xi_0],\nu^{-1/2}\sigma)$	$\sigma\in(F^*)^\wedge$	•	•	
	b	$\nu\xi_0\times\xi_0\rtimes\nu^{-1/2}\sigma$	$L((\nu^{1/2}\xi_0\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma))$				•
	c	$(\xi_0^2=\mathbf{1},\ \xi_0\neq\mathbf{1})$	$L((\nu^{1/2}\xi_0\mathrm{St}_{\mathrm{GL}(2)},\xi_0\nu^{-1/2}\sigma))$				•
	d		$L((\nu\xi_0,\xi_0\rtimes\nu^{-1/2}\sigma))$				
VI	a		$\tau(S,\nu^{-1/2}\sigma)$	$\sigma\in(F^*)^\wedge$		•	
	b	$\nu\times\mathbf{1}_{F^*}\rtimes\nu^{-1/2}\sigma$	$\tau(T,\nu^{-1/2}\sigma)$	$\sigma\in(F^*)^\wedge$			•
	c		$L((\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma))$				•
	d		$L((\nu,\mathbf{1}_{F^*}\rtimes\nu^{-1/2}\sigma))$				

Table 1: Irreducible representations of GSp(4) supported in the minimal parabolic subgroup.

1.2 The local Langlands correspondence

The dual group of GSp(4) is the complex Lie group $\mathrm{GSp}(4,\mathbb{C})$, see [Bo1]. Hence, by the conjectural local Langlands correspondence, there is a parameterization of the set of equivalence classes of irreducible, admissible representations of $\mathrm{GSp}(4,F)$ by conjugacy classes of admissible homomorphisms

$$\varphi:\ W_F'\longrightarrow\mathrm{GSp}(4,\mathbb{C}),\tag{5}$$

where $W_F'=W_F\times\mathrm{SL}(2,\mathbb{C})$ is the Weil–Deligne group. To every local parameter φ as in (5) there is associated an L–factor $L(s,\varphi)$ and an ε–factor $\varepsilon(s,\varphi,\psi)$, the latter one also depending on the choice of an additive character ψ of F, see [Ta] (in this paper we shall not consider the more general factors involving a finite-dimensional representation of the dual group; this finite-dimensional representation is here always the "standard" representation $\mathrm{GSp}(4,\mathbb{C})\to\mathrm{GL}(4,\mathbb{C})$).

If φ corresponds to the representation π of $\mathrm{GSp}(4,F)$, then the factors associated to π are by definition $L(s,\pi) := L(s,\varphi)$ and $\varepsilon(s,\pi,\psi) := \varepsilon(s,\varphi,\psi)$. Giving a representation $\varphi : W_F' \to \mathrm{GSp}(4,\mathbb{C})$ is equivalent to giving a pair (ϱ, N), where $\varrho : W_F \to \mathrm{GSp}(4,\mathbb{C})$ is a homomorphism whose image consists of semisimple elements and where N is a nilpotent element of the Lie algebra of $\mathrm{GSp}(4,\mathbb{C})$ such that $\varrho(w)N = |w|\, N\varrho(w)$ for all $w \in W_F$. In the analogous situation for $\mathrm{GL}(2)$, the pair (ϱ, N) with

$$\varrho(w) = \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}, \qquad\qquad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{6}$$

is the local parameter for the Steinberg representation $\mathrm{St}_{\mathrm{GL}(2)}$. Since we shall only consider representations of $\mathrm{GSp}(4,F)$ that are supported in the minimal parabolic subgroup, we shall be exclusively concerned with parameters of the form (ϱ, N), where $\varrho = (\varrho_1, \varrho_2, \varrho_3, \varrho_4)$ is a quadruple of characters of W_F (identified with characters of F^*). This means that the semisimple part of the local parameter is given by $w \mapsto \mathrm{diag}(\varrho_1(w), \varrho_2(w), \varrho_3(w), \varrho_4(w))$. Conjugating N by this diagonal matrix must yield $|w|\, N$.

The local Langlands correspondence for $\mathrm{GSp}(4,F)$ remains a conjecture, but for the type of representations we are interested in (those supported in the minimal parabolic subgroup) it is easy to "guess" the local parameters. Constituents of the same induced representation should have the same semisimple part and only differ in the N part. The parameter with $N = 0$ should belong to the Langlands quotient. We have listed the information on local parameters in Table 2 below. The last column of this table shows the resulting L–factors. We note that for *generic* representations, the L–factors given in Table 2 coincide with those defined via Novodvorski integrals, see [Tak], Theorem 4.1. All we shall assume in our global applications is that there exists an L–function theory which assigns the local L–factors listed in Table 2 in the case of Iwahori–spherical representations. One can check that for Iwahori–spherical representations the local parameters listed coincide with the local parameters given in [KL]; hence it is very likely that the L–factors in Table 2 are the "correct" factors.

		ϱ	N	L–factor
I		$\chi_1\chi_2\sigma,\ \chi_1\sigma,\ \chi_2\sigma,\ \sigma$	0	$L(s,\sigma)L(s,\chi_1\sigma)L(s,\chi_2\sigma)L(s,\chi_1\chi_2\sigma)$
II	a	$\chi^2\sigma,\ \nu^{1/2}\chi\sigma,$	N_1	$L(s,\sigma)L(s,\nu^{1/2}\chi\sigma)L(s,\chi^2\sigma)$
	b	$\nu^{-1/2}\chi\sigma,\ \sigma$	0	$L(s,\sigma)L(s,\nu^{1/2}\chi\sigma)L(s,\nu^{-1/2}\chi\sigma)L(s,\chi^2\sigma)$
III	a	$\nu^{1/2}\chi\sigma,\ \nu^{-1/2}\chi\sigma,$	N_4	$L(s,\nu^{1/2}\sigma)L(s,\chi\nu^{1/2}\sigma)$
	b	$\nu^{1/2}\sigma,\ \nu^{-1/2}\sigma$	0	$L(s,\nu^{-1/2}\sigma)L(s,\chi\nu^{-1/2}\sigma)L(s,\nu^{1/2}\sigma)L(s,\chi\nu^{1/2}\sigma)$
IV	a		N_5	$L(s,\nu^{3/2}\sigma)$
	b	$\nu^{3/2}\sigma,\ \nu^{1/2}\sigma,$	N_4	$L(s,\nu^{-1/2}\sigma)L(s,\nu^{3/2}\sigma)$
	c	$\nu^{-1/2}\sigma,\ \nu^{-3/2}\sigma$	N_1	$L(s,\nu^{-3/2}\sigma)L(s,\nu^{1/2}\sigma)L(s,\nu^{3/2}\sigma)$
	d		0	$L(s,\nu^{-3/2}\sigma)L(s,\nu^{-1/2}\sigma)L(s,\nu^{1/2}\sigma)L(s,\nu^{3/2}\sigma)$
V	a		N_3	$L(s,\nu^{1/2}\sigma)L(s,\nu^{1/2}\xi_0\sigma)$
	b	$\nu^{1/2}\sigma,\ \nu^{1/2}\xi_0\sigma,$	N_1	$L(s,\nu^{-1/2}\sigma)L(s,\nu^{1/2}\sigma)L(s,\nu^{1/2}\xi_0\sigma)$
	c	$\nu^{-1/2}\xi_0\sigma,\ \nu^{-1/2}\sigma$	N_2	$L(s,\nu^{-1/2}\xi_0\sigma)L(s,\nu^{1/2}\sigma)L(s,\nu^{1/2}\xi_0\sigma)$
	d		0	$L(s,\nu^{-1/2}\sigma)L(s,\nu^{-1/2}\xi_0\sigma)L(s,\nu^{1/2}\sigma)L(s,\nu^{1/2}\xi_0\sigma)$
VI	a		N_3	$L(s,\nu^{1/2}\sigma)^2$
	b	$\nu^{1/2}\sigma,\ \nu^{1/2}\sigma,$	N_3	$L(s,\nu^{1/2}\sigma)^2$
	c	$\nu^{-1/2}\sigma,\ \nu^{-1/2}\sigma$	N_1	$L(s,\nu^{-1/2}\sigma)L(s,\nu^{1/2}\sigma)^2$
	d		0	$L(s,\nu^{-1/2}\sigma)^2L(s,\nu^{1/2}\sigma)^2$

Table 2: Local parameters.

For each representation we have listed the pair (ϱ, N), using the following abbreviations for the nilpotent part.

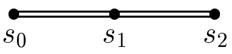
$$N_1=\begin{pmatrix}0&&&\\&0&1&\\&&0&\\&&&0\end{pmatrix},\qquad N_2=\begin{pmatrix}0&&&1\\&0&&\\&&0&\\&&&0\end{pmatrix},\qquad N_3=\begin{pmatrix}0&&&1\\&0&1&\\&&0&\\&&&0\end{pmatrix},$$

$$N_4=\begin{pmatrix}0&1&&\\&0&&\\&&0&-1\\&&&0\end{pmatrix},\qquad N_5=\begin{pmatrix}0&1&&\\&0&1&\\&&0&-1\\&&&0\end{pmatrix}.$$

There is one case of L–indistinguishability in Table 2, namely, the two tempered representations $\tau(S,\nu^{-1/2}\sigma)$ and $\tau(T,\nu^{-1/2}\sigma)$ (VIa and VIb) constitute a 2–element L–packet. Regarding the representation $\delta([\xi_0,\nu\xi_0],\nu^{-1/2}\sigma)$ of type Va, by [Pr], Theorem 7.1, there should exist a supercuspidal (and non-generic) representation of $\mathrm{GSp}(4,F)$ with the same local parameter (ϱ,N_3). This supercuspidal representation should be the θ_{10} type representation considered in [KPS].

1.3 Iwahori–spherical representations

Consider the Dynkin diagram of the affine Weyl group C_2:



We are going to realize the three generators s_0,s_1,s_2 for the affine Weyl group as the matrices

$$s_0=\begin{pmatrix}&&&-\varpi^{-1}\\&1&&\\&&1&\\\varpi&&&\end{pmatrix},\qquad s_1=\begin{pmatrix}&1&&\\1&&&\\&&&1\\&&1&\end{pmatrix},\qquad s_2=\begin{pmatrix}1&&&\\&&1&\\&-1&&\\&&&1\end{pmatrix}. \tag{7}$$

The elements s_1 and s_2 generate the usual 8–element Weyl group W. Consider further the element

$$\eta=\begin{pmatrix}1&&&\\&1&&\\&&-\varpi&\\&&&-\varpi\end{pmatrix}s_2s_1s_2=\begin{pmatrix}&&1&\\&&&1\\\varpi&&&\\&\varpi&&\end{pmatrix}\in\mathrm{GSp}(4,F). \tag{8}$$

Since conjugation by this matrix corresponds to classical Atkin–Lehner involutions, we call η also the *Atkin–Lehner element.* Note that

$$\eta s_0\eta^{-1}=s_2,\qquad\qquad \eta s_1\eta^{-1}=s_1,\qquad\qquad \eta s_2\eta^{-1}=s_0,$$

i.e., η induces the non-trivial automorphism of the Dynkin diagram. The *parahoric subgroups* P_S correspond to proper subsets S of $\{s_0,s_1,s_2\}$, the correspondence being that $P_S=\bigsqcup_{w\in\langle S\rangle}IwI$, where I is the Iwahori subgroup. We shall briefly describe each parahoric subgroup and introduce notations.

- $S=\{s_1,s_2\}$. This is the standard special maximal compact subgroup $\mathrm{GSp}(4,\mathfrak{o})$, which we also denote by K.

- $S=\{s_0,s_1\}$ defines the maximal compact subgroup P_{01} consisting of matrices of the block form $\begin{pmatrix}\mathfrak{o}&\mathfrak{p}^{-1}\\\mathfrak{p}&\mathfrak{o}\end{pmatrix}$. We have $P_{01}=\eta K\eta^{-1}$.

- $S=\{s_0,s_2\}$ defines another maximal compact subgroup P_{02} of smaller volume. It consists of all $g\in\mathrm{GSp}(4,F)$ such that

$$g\in\begin{pmatrix}\mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{p}^{-1}\\\mathfrak{p}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\\mathfrak{p}&\mathfrak{p}&\mathfrak{p}&\mathfrak{o}\end{pmatrix}\qquad\text{and}\qquad\det(g)\in\mathfrak{o}^*. \tag{9}$$

This parahoric subgroup is also called the *paramodular group*. In a classical context this group appears, for instance, in [IO]. The two groups K and P_{02} represent the two conjugacy classes of maximal compact subgroups of $\mathrm{GSp}(4,F)$.

- $S=\{s_1\}$. This is the Siegel congruence subgroup P_1 consisting of elements of the block form $\begin{pmatrix}\mathfrak{o} & \mathfrak{o}\\ \mathfrak{p} & \mathfrak{o}\end{pmatrix}$. It is the inverse image of the Siegel parabolic subgroup under the natural map $K\to\mathrm{GSp}(4,k)$, where $k=\mathfrak{o}/\mathfrak{p}$ is the residue field.
- $S=\{s_2\}$. This is the inverse image of the Klingen parabolic subgroup under the natural map $K\to\mathrm{GSp}(4,k)$. We denote it by P_2.
- $S=\{s_0\}$ defines a group P_0 which is conjugate to P_2 by η. It is not contained in K.
- $S=\emptyset$ defines the Iwahori subgroup which we denote by I. It consists of all matrices that are upper triangular mod $\mathfrak{p}$.

Let χ_1,χ_2,σ be *unramified* characters of F^* and consider the representation $\chi_1\times\chi_2\rtimes\sigma$ in its standard realization V. Table 3 further below lists the dimension of the space of fixed vectors under each parahoric subgroup in each irreducible constituent of $\chi_1\times\chi_2\rtimes\sigma$. Since some of these groups are conjugate we only have to consider K, P_{02}, P_1, P_2 and I.

We shall explain how the dimension information in this table was obtained, starting with type I representations. These are full induced representations, so the dimensions for I, P_1, P_2 and K are obtained by counting Weyl group elements. As for P_{02}–invariant vectors it is not hard to prove that a P_2–invariant function f in the standard model for $\chi_1\times\chi_2\rtimes\sigma$ is P_{02}–invariant if and only if

$$f(s_2s_1)=\chi_2(\varpi)q^{-1}f(s_1)\qquad\text{and}\qquad f(s_1s_2s_1)=\chi_1(\varpi)q^{-2}f(\mathbf{1}).\tag{10}$$

Thus we get dimension 2 for the P_{02}–invariant vectors. These arguments hold for every full induced representation, irreducible or not. The rest comes down to determining how these dimensions are distributed amongst the irreducible constituents. The dimensions for IIb and IIIb can also be determined by counting Weyl group elements. Subtracting from the dimensions for the full induced representations, we get the numbers for IIa and IIIa. For the other representations we observe the tables (2), (3) and (4), which tell us how the full induced representation decomposes. What we need is the information for just *one* representation in each table, and the rest will follow formally. As for type IV, the dimensions for $\sigma\mathbf{1}_{\mathrm{GSp}(4)}$ are 1 for each parahoric subgroup, and the rest follows. The hardest cases are V and VI, where additional work needs to be done. But this work was carried out in the paper [Sch4], where the dimensions for the Saito–Kurokawa representations Vb,c and VIb,c were determined.

		representation	K	P_{02}	P_2	P_1	I	ε	a
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	**1**	2 $+-$	4	4 $++$ $--$	8 $++++$ $----$	1	0
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	0	**1** $-$	2	**1** $-$	4 $+---$	$-(\sigma\chi)(\varpi)$	1
	b	$\chi\mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	**1**	1 $+$	2	3 $++-$	4 $+++-$	1	0
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	0	0	**1**	**2** $+-$	4 $++--$	1	2
	b	$\chi \rtimes \sigma\mathbf{1}_{\mathrm{GSp}(2)}$	**1**	2 $+-$	3	2 $+-$	4 $++--$	1	0
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	0	0	0	0	**1** $-$	$-\sigma(\varpi)$	3
	b	$L((\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)}))$	0	0	**1**	**2** $+-$	3 $++-$	1	2
	c	$L((\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma))$	0	**1** $-$	2	**1** $-$	3 $+--$	$-\sigma(\varpi)$	1
	d	$\sigma\mathbf{1}_{\mathrm{GSp}(4)}$	**1**	1 $+$	1	1 $+$	1 $+$	1	0
V	a	$\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)$	0	0	**1**	0	2 $+-$	-1	2
	b	$L((\nu^{1/2}\xi_0\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$	0	**1** $+$	1	**1** $+$	2 $++$	$\sigma(\varpi)$	1
	c	$L((\nu^{1/2}\xi_0\mathrm{St}_{\mathrm{GL}(2)}, \xi_0\nu^{-1/2}\sigma))$	0	**1** $-$	1	**1** $-$	2 $--$	$-\sigma(\varpi)$	1
	d	$L((\nu\xi_0, \xi_0 \rtimes \nu^{-1/2}\sigma))$	**1**	0	1	2 $+-$	2 $+-$	1	0
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	0	0	**1**	**1** $-$	3 $+--$	1	2
	b	$\tau(T, \nu^{-1/2}\sigma)$	0	0	0	**1** $+$	1 $+$	1	2
	c	$L((\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$	0	**1** $-$	1	0	1 $-$	$-\sigma(\varpi)$	1
	d	$L((\nu, \mathbf{1}_{F^*} \rtimes \nu^{-1/2}\sigma))$	**1**	1 $+$	2	2 $+-$	3 $++-$	1	0

Table 3: Invariant vectors.

The signs under some of the entries denote Atkin–Lehner eigenvalues, to be explained further below. The next-to-last column gives the signs defined by ε–factors, see also below. The numbers in bold face indicate newforms, to be defined in sections 2.2 and 2.3. The last column contains the exponent of the conductor of the local parameter (as listed in Table 2).

Atkin–Lehner eigenvalues

The parahoric subgroups normalized by the Atkin–Lehner element η (see (8)) are precisely the "symmetric" groups I, P_1 and P_{02}. Therefore, if H denotes one of these groups, then η acts on the space of H–invariant vectors, for any representation (π, V) of $\mathrm{GSp}(4,F)$. Let us assume in addition that π has trivial central character. Then $\pi(\eta)$ acts as an involution, because $\eta^2 = \varpi\mathbf{1}$. We call these operators *Atkin–Lehner involutions.* They split the space V^H of H–invariant vectors into ±1–eigenspaces V_+^H and V_-^H. The plus and minus signs under the dimensions of the spaces V^H in Table 3 indicate how these spaces split into Atkin–Lehner eigenspaces (provided the central character is trivial). The signs listed in Table 3 are correct if one assumes that

- in Group II, where the central character is $\chi^2\sigma^2$, the character $\chi\sigma$ is trivial.
- in Groups IV, V and VI, where the central character is σ^2, the character σ itself is trivial.

If these assumptions are not met, then one has to interchange the plus and minus signs in Table 3 to get the correct dimensions.

Now we shall explain how the information on Atkin–Lehner eigenvalues in Table 3 can be obtained. In a full induced representation, the distribution of the signs is as given in the type I row. This follows from direct computations in the standard induced model. If the induced representation is reducible, we have to see how these signs are distributed amongst irreducible constituents, for which we observe the tables (2), (3) and (4). The additional information we require comes from the trivial representation in case IV, and from the Saito–Kurokawa representations in cases V and VI. As for the latter, the necessary computations were carried out in [Sch4].

ε–factors

Let $\varepsilon(s,\pi,\psi)$ be the local ε–factor attached to an irreducible representation π of $\mathrm{GSp}(4,F)$ and an additive character ψ (and the standard representation of the L–group). Here we mean the local factors defined via the local Langlands correspondence and representations of the Weil–Deligne group, but these factors should coincide with the ones defined in [PS2] via local zeta integrals. We have the general relation

$$\varepsilon(s,\pi,\psi)\varepsilon(1-s,\hat{\pi},\psi) = \omega_\pi(-1), \tag{11}$$

where $\hat{\pi}$ is the contragredient representation and ω_π is the central character of π. It is known that if ω_π is trivial, then $\pi \simeq \hat{\pi}$. In this case it follows from (11) that $\varepsilon(1/2,\pi,\psi) \in \{\pm1\}$. By general properties of ε–factors, this sign is independent of the choice of ψ. Hence there is a sign $\varepsilon(1/2,\pi)$ *canonically* attached to any irreducible, admissible representation of $\mathrm{PGSp}(4,F)$ (provided we know the local Langlands correspondence).

If π is not square integrable, then the image of the local parameter $W_F' \to \mathrm{GSp}(4,\mathbb{C})$ lies in a Levi component of a proper parabolic subgroup and the ε–factor is easy to determine since it factorizes. For example, if $\pi = \chi_1 \times \chi_2 \rtimes \sigma$ is irreducible, then

$$\varepsilon(s,\pi,\psi) = \varepsilon(s,\sigma,\psi)\varepsilon(s,\sigma\chi_1,\psi)\varepsilon(s,\sigma\chi_2,\psi)\varepsilon(s,\sigma\chi_1\chi_2,\psi).$$

Provided all the characters are unramified and $\chi_1\chi_2\sigma^2 = 1$, it follows that $\varepsilon(1/2, \pi) = 1$. Using the information from Table 2 it is thus easy to compute the signs for most of the representations in our list. For the square-integrable representation of type Va, note that the image of the local parameter is not contained in a proper Levi subgroup of $\mathrm{GSp}(4, \mathbb{C})$. It is however contained in a Levi subgroup of $\mathrm{GL}(4, \mathbb{C})$, and hence the ε–factor still factorizes. The only representation in our list where this is not the case is the Steinberg representation (and its unramified quadratic twist). But there we can use the formula in section (4.1.6) of [Ta], which tells us that the sign is $-\sigma(\varpi)$.

In the next-to-last column of Table 3 we have listed the signs defined by ε–factors under the assumption that the central character is trivial and all inducing characters are unramified. The number a in the last column contains the exponent of the conductor of the local parameter (this number is denoted by $a(V)$ in section (4.1.6) of [Ta]). Its relevance is that the ε–factor is a constant multiple of q^{-as}.

In the next section we will define *newforms* with respect to a fixed parahoric subgroup P. If a representation contains such newforms with respect to P, we have indicated this in Table 3 by writing the corresponding dimension in bold face. For example, IIIa contains a one-dimensional space of newforms with respect to P_2, and a two-dimensional space of newforms with respect to P_1. Note that if there are newforms with respect to P_2 (resp. P_{12}), then there are also newforms with respect to the conjugate group P_0 (resp. P_{01}) which are not listed in the table.

For irreducible representations of $\mathrm{PGL}(2, F)$ the sign defined by the ε–factor coincides with the eigenvalue of the Atkin–Lehner involution on the one-dimensional space of local newforms; see section 3.2 of [Sch1]. We can observe a similar phenomenon in the present situation. We have distinguished 17 types of representations supported in the minimal parabolic subgroup. Types VIa and VIb constitute an L–packet, so let us instead talk about 16 types of L–packets that contain Iwahori–invariant vectors. Then we observe:

1.3.1 Proposition. *The following are equivalent for an L–packet π of* $\mathrm{PGSp}(4, F)$ *containing Iwahori–fixed vectors.*

i) The exponent a of the conductor of the L–packet π is even.

ii) The ε–factor does not change when the representations in π are twisted with ξ_0, the non-trivial unramified quadratic character of F^*.

iii) π contains newforms with respect to one of the "non–symmetric" groups K or P_2.

iv) The trace of the Atkin–Lehner involution on the full space of newforms is 0.

If these conditions are <u>not</u> fulfilled, then every local newform in π is an eigenvector for the Atkin–Lehner involution, and the eigenvalue coincides with $\varepsilon(1/2, \pi)$.

Proof: Everything follows by examining Table 3. The equivalence of i) and ii) also follows from the definitions of a and $\varepsilon(s, \pi)$. ∎

2 Local newforms

We shall now define local old– and newforms for the Iwahori–spherical representations. Our main tool is the Iwahori–Hecke algebra $\mathcal{I}$. Once we have chosen a suitable basis of the 8–dimensional space of Iwahori–fixed vectors of a full induced representation, we can compute the action of $\mathcal{I}$ explicitly. Then all our results follow essentially from elementary linear algebra.

2.1 The Iwahori–Hecke algebra

The Iwahori–Hecke algebra $\mathcal{I}$ of $\mathrm{GSp}(4,F)$ is the convolution algebra of left and right I–invariant functions on $\mathrm{GSp}(4,F)$. It acts on the space of I–invariant vectors V^I of any irreducible, admissible representation (π,V) of $\mathrm{GSp}(4,F)$. If $V^I\neq 0$, then this finite-dimensional representation determines the isomorphism class of π, see [Bo2].

The structure of $\mathcal{I}$ is as follows. The identity element e is the characteristic function of I. For $j=0,1,2$ let e_i be the characteristic function of Is_iI (see (7)). If η is as in (8), we denote the characteristic function of ηI again by η. Then $\mathcal{I}$ is generated by e_0,e_1,e_2 and η, and we have the following relations.

- $e_i^2=(q-1)e_i+qe$ for $i=0,1,2$.
- $\eta e_0\eta^{-1}=e_2,\quad \eta e_1\eta^{-1}=e_1,\quad \eta e_2\eta^{-1}=e_0.$
- $e_0e_1e_0e_1=e_1e_0e_1e_0,\quad e_1e_2e_1e_2=e_2e_1e_2e_1,\quad e_0e_2=e_2e_0.$

All of this follows from general structure theory. There are other relations, but we will not need them.

Let χ_1,χ_2,σ be unramified characters of F^*, and let V be the standard space of the induced representation $\chi_1\times\chi_2\rtimes\sigma$. We shall now explicitly compute the action of $\mathcal{I}$ on V^I. This 8–dimensional space has the basis f_w, $w\in W$, where f_w is the unique I–invariant function with $f_w(w)=1$ and $f_w(w')=0$ for $w'\in W$, $w'\neq w$. It is convenient to order the basis as follows:

$$f_e,\quad f_1,\quad f_2,\quad f_{21},\quad f_{121},\quad f_{12},\quad f_{1212},\quad f_{212},\tag{12}$$

where we have abbreviated $f_1=f_{s_1}$ and so on. Having fixed this basis, the operators e_0,e_1,e_2 and η on V^I become 8×8–matrices. These are given in the following lemma.

2.1.1 Lemma. *Let notations be as above. With respect to the basis (12) of V^I, the action of the elements e_1 and e_2 on V^I is given by the following matrices.*

$$\pi(e_1)=\begin{pmatrix}0&q&&&&&&\\1&q-1&&&&&&\\&&0&q&&&&\\&&1&q-1&&&&\\&&&&0&q&&\\&&&&1&q-1&&\\&&&&&&0&q\\&&&&&&1&q-1\end{pmatrix},\quad \pi(e_2)=\begin{pmatrix}0&0&q&0&0&0&0&0\\0&0&0&0&0&q&0&0\\1&0&q-1&0&0&0&0&0\\0&0&0&0&0&0&0&q\\0&0&0&0&0&0&q&0\\0&1&0&0&0&q-1&0&0\\0&0&0&0&1&0&q-1&0\\0&0&0&1&0&0&0&q-1\end{pmatrix}$$

The action of η is given by

$$\pi(\eta)=\begin{pmatrix} &&&&&&&\gamma q^{3/2}\\ &&&&&&\gamma q^{3/2}&\\ &&&&&\beta\gamma q^{1/2}&&\\ &&&&\beta\gamma q^{1/2}&&&\\ &&&\alpha\gamma q^{-1/2}&&&&\\ &&\alpha\gamma q^{-1/2}&&&&&\\ &\alpha\beta\gamma q^{-3/2}&&&&&&\\ \alpha\beta\gamma q^{-3/2}&&&&&&& \end{pmatrix}.$$

The action of e_0 is given by the matrix $\pi(\eta)\pi(e_2)\pi(\eta)^{-1}$.

Proof: A standard system of representatives for Is_1I is given by

$$\begin{pmatrix}1&x&&\\&1&&\\&&1&-x\\&&&1\end{pmatrix}s_1,\qquad\qquad x\in\mathfrak{o}/\mathfrak{p},\tag{13}$$

and similarly for Is_2I. Using these representatives and the identity

$$\begin{pmatrix}1&\\ \lambda&1\end{pmatrix}=\begin{pmatrix}-\lambda^{-1}&\\&-\lambda\end{pmatrix}\begin{pmatrix}1&\lambda\\&1\end{pmatrix}\begin{pmatrix}&1\\-1&\end{pmatrix}\begin{pmatrix}1&\lambda^{-1}\\&1\end{pmatrix},\tag{14}$$

our claims follow by straightforward computations which are left to the reader. ∎

Let us introduce a partial ordering on the set of standard parahoric subgroups as follows:

$$\begin{tikzcd}
P_{01} \arrow[dd] \arrow[drr] \arrow[ddrr, bend left] & & & & P_{12} \arrow[dll] \arrow[ddll, bend right] \arrow[dd] \\
& & P_{02} \arrow[dll] \arrow[drr] & & \\
P_0 \arrow[drr] & & P_1 \arrow[d] & & P_2 \arrow[dll] \\
& & I & &
\end{tikzcd}\tag{15}$$

Groups on a higher level have a bigger volume. On top we have the special maximal compact subgroups $K=P_{12}$ and its η–conjugate. For parahoric subgroups R and R' let us write $R'\succ R$ if there is an arrow from R' to R.

2.1.2 Proposition. *Let (π,V) be an Iwahori–spherical unitary representation of $\mathrm{GSp}(4,F)$. Let $\langle\,,\,\rangle$ be a $\mathrm{GSp}(4,F)$–invariant scalar product on V. Then the elements e_0,e_1,e_2 of the Iwahori–Hecke algebra act as self–adjoint operators on V^I. If π has central character ω_π, then we further have*

$$\langle\pi(\eta)v,w\rangle=\omega_\pi(\varpi)\langle v,\pi(\eta)w\rangle\qquad\qquad \textit{for all } v,w\in V.\tag{16}$$

Proof: The last assertion is obvious since $\eta^2 = \varpi\mathbf{1}$. As for e_1, let us abbreviate the 4×4–matrix in (13) by $n(x)$. Then, since the scalar product is K–invariant,

$$\langle\pi(e_1)v,w\rangle = \sum_{x\in\mathfrak{o}/\mathfrak{p}}\langle\pi(n(x)s_1)v,w\rangle = \sum_{x\in\mathfrak{o}/\mathfrak{p}}\langle v,\pi(s_1n(-x))w\rangle.$$

If w is I–invariant, we can eliminate the $n(-x)$ in the last expression. If v is also I–invariant, we can insert a $\pi(n(-x))$ in front of the v. We can then use the K–invariance again and arrive at $\sum\langle v,\pi(n(x)s_1)w\rangle = \langle v,\pi(e_1)w\rangle$. The proof for e_2 is similar. The assertion for e_0 follows using $\eta e_2\eta^{-1} = e_0$ and (16). ■

2.1.3 Remark. Even if the induced representation $\pi_1\times\pi_2\rtimes\sigma$ is not unitary, it will be useful to consider the K–invariant scalar product

$$\int\limits_K f_1(g)\overline{f_2(g)}\,dg \tag{17}$$

on the standard space of this representation. If the measure is normalized to give I volume 1, then the matrix of this scalar product restricted to the space of I–invariant vectors with respect to the basis (12) is $\mathrm{diag}\big(1,\,q,\,q,\,q^2,q^3,q^2,q^4,q^3\big)$ (the exponents are the lengths of the Weyl group elements). The argument in the proof of Proposition 2.1.2 shows that e_1 and e_2 act as self–adjoint operators with respect to this scalar product.

Special elements in the Iwahori–Hecke algebra are the *projection operators*

$$d_S = \frac{1}{\mathrm{vol}(P_S)}\mathrm{char}(P_S), \tag{18}$$

where "char" stands for characteristic function. Here S is a subset of $\{s_0,s_1,s_2\}$ and P_S is the corresponding subgroup. The measure is normalized so that I has volume 1. In particular, we have $d_\emptyset = \mathrm{char}(I) = e$ and $d_i = \frac{1}{q+1}(e+e_i)$ for $i=0,1,2$. Since the projection operators satisfy $d_S^2 = d_S$, we have

$$V^I = \mathrm{im}(\pi(d_S))\oplus\ker(\pi(d_S)) \tag{19}$$

for any representation (π,V), where $\pi(d_S)$ is considered as an endomorphism of V^I. Thus the space of P_S–fixed vectors $V^{P_S} = \mathrm{im}(\pi(d_S))$ always has a natural complement in V^I. It follows from Proposition 2.1.2 that if π is a *unitary* representation, then $\ker(\pi(d_S))$ coincides with the *orthogonal* complement of V^{P_S} in V^I.

2.2 Newforms for I

Let (π,V) be an irreducible, admissible representation of $\mathrm{GSp}(4,F)$. For any of the parahoric subgroups R of $\mathrm{GSp}(4,F)$ we shall give a separate definition of "local newform with respect to R". The idea is that if there is a "bigger" parahoric subgroup R' such that $V^{R'}\neq0$, then certain elements of V^R will be "old" since they can be obtained in a simple way from $V^{R'}$. More precisely, we shall do the following.

- Whenever $R' \succ R$ (see diagram (15)), we shall define natural linear operators from $V^{R'}$ to V^R. If $R' \supset R$, then one such operator is the identity.
- The image of all these operators for all $R' \succ R$ is by definition the space of *oldforms* with respect to R. If π is unitary, we can define the space of *newforms* as the orthogonal complement of the space of oldforms.
- By Proposition 2.1.2, newforms can also be characterized as the kernel of certain linear operators. This leads to a definition of newforms that does not require unitarity.
- We shall prove that if there exists an R' with $R' \succ R$ and $V^{R'} \neq 0$, then V^R consists entirely of oldforms. Otherwise, by definition, V^R consists entirely of newforms.
- If V^R consists of newforms, then its dimension is 1 or 2. The second case can only happen for $R = P_1$, and in this case, if π has trivial central character, there are two linearly independent newforms that can be distinguished by their Atkin–Lehner eigenvalue.

As an illustration, let us define local newforms with respect to the Iwahori subgroup I. Since I is minimal parahoric, it is natural to consider an I–invariant vector "old" if it is invariant under some bigger parahoric subgroup. In other words, the subspace $V^{I_0}+V^{I_1}+V^{I_2}$ of V^I constitutes the space of oldforms, and if π is a unitary representation, we define its orthogonal complement as the space of newforms. In this case, by (19) and the remarks thereafter,

$$V^I = \big(V^{I_0}+V^{I_1}+V^{I_2}\big) \oplus \big(\ker(\pi(d_0)) \cap \ker(\pi(d_1)) \cap \ker(\pi(d_2))\big) \tag{20}$$

(orthogonal decomposition). Thus, newforms with respect to I can be characterized as the common kernel of the projection operators d_0, d_1, d_2. The following proposition shows that this leads to a very restricted set of representations containing local newforms with respect to I.

2.2.1 Proposition. *The following three conditions are equivalent for an irreducible, admissible representation* (π, V) *of* $\mathrm{GSp}(4,F)$.

i) π *is an unramified twist of the Steinberg representation.*

ii) There exists a non-zero $v \in V^I$ *such that*

$$d_0(v) = d_1(v) = d_2(v) = 0.$$

iii) $V^{P_0}+V^{P_1}+V^{P_2}$ *is a proper subspace of* V^I.

Proof: For *unitary* representations, ii) and iii) are equivalent by (20). In general it can be checked case by case using Table 3. Statement ii) says that the Iwahori–Hecke algebra acts by the *sign character* sending each e_i to -1. It is well known that this characterizes the Steinberg representation, see [Bo2] (it also follows by examining Table 3). ∎

The proposition says that it is only the unramified twists of the Steinberg representation that admit *local newforms* with respect to I. If we restrict to representations with trivial central character, then there are precisely two such representations, $\mathrm{St}_{\mathrm{GSp}(4)}$ and $\xi_0\,\mathrm{St}_{\mathrm{GSp}(4)}$, where ξ_0 is the non-trivial unramified quadratic character. These two representations can be distinguished by the eigenvalue of the Atkin–Lehner involution on the local newform. Hence the situation is completely analogous to GL(2). We note that condition ii) in Proposition 2.2.1 leads to a characterization of classical newforms in terms of Fourier coefficients, see section 3.3.

2.3 Newforms for P_1, P_2 and P_{02}

Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4,F)$. We consider the following natural linear operators between spaces of vectors fixed under parahoric subgroups. Their images will define oldforms.

- Whenever $R' \supset R$, we have an inclusion $V^{R'} \subset V^R$.
- There is a natural operator from V^K to V^{P_1} provided by the element $e_0e_1e_0$ of the Iwahori–Hecke algebra. Note that this element commutes with e_1. Symmetrically, we have the operator $e_2e_1e_2$ from $V^{P_{01}}$ to V^{P_1}.
- Since the element $e_1e_2e_1$ commutes with e_2, it provides a natural operator from $V^{P_{02}}$ to V^{P_2}. Similarly, $e_1e_0e_1$ defines an operator $V^{P_{02}} \to V^{P_0}$.
- From V^K to $V^{P_{02}}$ we have the "trace operator" d_0. Similarly we have $d_2 : V^{P_{01}} \to V^{P_{02}}$.

Now if R is any of the standard parahoric subgroups, we define the space of *oldforms* $(V^R)^{\mathrm{old}}$ with respect to R as the space spanned by the image of all these operators for all $R' \succ R$ (see diagram (15)). For *unitary* representations, the space of *newforms* $(V^R)^{\mathrm{new}}$ with respect to R is defined as the orthogonal complement of $(V^R)^{\mathrm{old}}$ within V^R. By Proposition 2.1.2, this orthogonal complement can be described as the intersection of the kernels of the operators given in the last column of table (21) below. It is this description as a common kernel that we take as our *definition* of $(V^R)^{\mathrm{new}}$ for an arbitrary representation.

2.3.1 Theorem. *Let (π, V) be an irreducible, admissible representation of* $\mathrm{GSp}(4,F)$*. Let R be one of the parahoric subgroups I, P_1, P_2 or P_{02}. We define subspaces $(V^R)^{\mathrm{old}}$ and $(V^R)^{\mathrm{new}}$ of the space V^R as in the following table.*

R	$(V^R)^{\mathrm{old}} =$	$(V^R)^{\mathrm{new}} =$ common kernel of
I	$V^{P_0} + V^{P_1} + V^{P_2}$	$d_0,\ d_1,\ d_2$
P_1	$V^{P_{01}} + e_2e_1e_2V^{P_{01}} + V^K + e_0e_1e_0V^K$	$d_{01},\ d_{01}e_2e_1e_2,\ d_{12},\ d_{12}e_0e_1e_0$
P_2	$V^K + V^{P_{02}} + e_1e_2e_1V^{P_{02}}$	$d_{12},\ d_{02},\ d_{02}e_1e_2e_1$
P_{02}	$d_2V^{P_{01}} + d_0V^K$	$d_{01}d_2,\ d_{12}d_0$

(21)

Then exactly one of the following alternatives is true.

i) There exists a parahoric subgroup R' such that $V^{R'} \neq 0$ and such that $R' \succ R$ (see diagram (15)). In this case $V^R = (V^R)^{\mathrm{old}}$ and $(V^R)^{\mathrm{new}} = 0$.

ii) There exists no parahoric subgroup R' as in i). In this case $V^R = (V^R)^{\mathrm{new}}$ and $(V^R)^{\mathrm{old}} = 0$.

Proof: The Iwahori subgroup has already been treated in the previous section. We shall deal with $R = P_1$, the other cases being similar. We may realize π as a subrepresentation of an induced representation $\chi_1 \times \chi_2 \rtimes \sigma$. Let us define $\alpha, \beta, \gamma \in \mathbb{C}^*$ by

$$\alpha = \chi_1(\varpi), \qquad \beta = \chi_2(\varpi), \qquad \gamma = \sigma(\varpi). \tag{22}$$

Using the basis (12), we identify the space of I–invariant vectors in $\chi_1 \times \chi_2 \rtimes \sigma$ with $\mathbb{C}^8$. The K–spherical vector is given by $v_0 = {}^t(1,1,1,1,1,1,1,1)$. Using Lemma 2.1.1, it is easy to compute the action of the Iwahori–Hecke algebra on v_0. The result is that $(V^{P_1})^{\rm old}$ is spanned by the first four columns of the following matrix.

$$\begin{pmatrix}
1 & \frac{q^2(\beta(q-1)+\alpha(1+\beta)(q-1)+q)}{\alpha\beta} & \gamma q^{3/2} & \alpha\beta\gamma q^{3/2} & 0 & 0 & 0 & -q \\
1 & \frac{q^2(\beta(q-1)+\alpha(1+\beta)(q-1)+q)}{\alpha\beta} & \gamma q^{3/2} & \alpha\beta\gamma q^{3/2} & 0 & 0 & 0 & 1 \\
1 & \frac{q(\alpha(1+\beta)(q-1)+\beta q)}{\alpha} & \beta\gamma q^{1/2} & \alpha\gamma q^{1/2}(\beta(q-1)+q) & 0 & 0 & -q & 0 \\
1 & \frac{q(\alpha(1+\beta)(q-1)+\beta q)}{\alpha} & \beta\gamma q^{1/2} & \alpha\gamma q^{1/2}(\beta(q-1)+q) & 0 & 0 & 1 & 0 \\
1 & \alpha(q-1+q\beta^{-1}) & \alpha\gamma q^{-1/2} & \gamma q^{1/2}(\alpha(1+\beta)(q-1)+\beta q) & 0 & 1 & 0 & 0 \\
1 & \alpha(q-1+q\beta^{-1}) & \alpha\gamma q^{-1/2} & \gamma q^{1/2}(\alpha(1+\beta)(q-1)+\beta q) & 0 & -q & 0 & 0 \\
1 & \alpha\beta & \alpha\beta\gamma q^{-3/2} & \gamma q^{1/2}((\alpha\beta+\alpha+\beta)(q-1)+q) & 1 & 0 & 0 & 0 \\
1 & \alpha\beta & \alpha\beta\gamma q^{-3/2} & \gamma q^{1/2}((\alpha\beta+\alpha+\beta)(q-1)+q) & -q & 0 & 0 & 0
\end{pmatrix}$$

The last four columns span the intersection of the kernels on V^I of the operators defining $(V^{P_1})^{\rm new}$. All of this is easily computed using Lemma 2.1.1 and a computer algebra program. We see that the intersection of $(V^{P_1})^{\rm new}$ and $(V^{P_1})^{\rm old}$ is always trivial. In fact, we observe that these two spaces are *orthogonal* with respect to the scalar product introduced in Remark 2.1.3. The determinant of the above matrix is given by $\alpha^{-1}\beta^{-1}\gamma^2q^{-1}(1+q)^4(\alpha-q)^2(\beta-q)^2(\alpha\beta-q)(\alpha-\beta q)$. This determinant vanishes only at points of reducibility, proving our assertion in case that $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible. Each of the remaining cases is also easily checked. ■

2.3.2 Remarks. i) Observing that $\eta K \eta^{-1} = P_{01}$ and $\eta P_2 \eta^{-1} = P_0$, we have similar statements for the groups P_{01} and P_0 which we shall not state explicitly.

ii) Fixing a parahoric subgroup R, the theorem says that a given representation has either newforms or oldforms with respect to R, or none of them, but never both.

iii) A given representation may have newforms for two different groups. For example, representations of type IIa have newforms for both P_1 and P_{02}, and representations of type IIIa have newforms for both P_1 and P_2.

iv) Our definition of old– and newforms for P_{02} coincides with the one given in [Ib2], §1, since the "trace operators" considered there coincide with our operators d_0 and d_2.

v) As mentioned in the proof of Theorem 2.3.1, $(V^{P_1})^{\rm old}$ and $(V^{P_1})^{\rm new}$ are orthogonal with respect to the scalar product introduced in Remark 2.1.3. This is also true for the groups I, P_2 and P_{02}, as explicit calculations show.

2.3.3 Remark. We consider the analogous situation for the group $\mathrm{GL}(2,F)$. Here we have the standard maximal compact subgroup $K=P_1:=\mathrm{GL}(2,\mathfrak{o})$ and its conjugate $P_0:=\eta P_1\eta^{-1}$, where $\eta=\begin{pmatrix} & 1\\ \varpi & \end{pmatrix}$. The Iwahori subgroup is $I=P_0\cap P_1$. Given a representation (π,V), the subspace $V^{P_0}+V^{P_1}$ of V^I constitutes the space of oldforms with respect to I. In a unitary representation its orthogonal complement can be described as the common kernel of $d_0=e+e_0$ and $d_1=e+e_1$. In a classical language, a modular form $f\in S_k(\Gamma_0(N))$ is a newform if and only if for each $p|N$ both f and $\eta_p f$ are annihilated by the trace operator at p. Here η_p is the classical Atkin–Lehner involution at p.

2.3.4 Remark. Instead of the operator $\mathrm{id}:\ V^{P_{01}}\to V^{P_1}$ which we considered when defining oldforms for P_1, we can as well take $\eta:\ V^K\to V^{P_1}$. Similarly, instead of $e_2e_1e_2:\ V^{P_{01}}\to V^{P_1}$ we may take $e_2e_1e_2\eta:\ V^K\to V^{P_1}$. Since $Is_2s_1s_2I/I\simeq P_1s_2s_1s_2P_1/P_1$, it is easy to see that this latter operator is given by

$$\begin{aligned}\pi(e_2e_1e_2\eta)v_0&=\sum_{\mu,\kappa,x\in\mathfrak{o}/\mathfrak{p}}\pi\left(\begin{pmatrix}1&&\mu&\kappa\\&1&x&\mu\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}\varpi&&&\\&\varpi&&\\&&1&\\&&&1\end{pmatrix}\right)v_0\\&=\sum_{g\in P_1\left(\begin{smallmatrix}\varpi\mathbf{1}&\\&\mathbf{1}\end{smallmatrix}\right)P_1/P_1}\pi(g)v_0\qquad\qquad(v_0\in V^{P_1}).\end{aligned}\tag{23}$$

We see from (23) that $e_2e_1e_2\eta$ corresponds to a Hecke operator which is sometimes used in the classical theory of Siegel modular forms. For our global applications we shall therefore list in the following table the eigenvalues of $e_2e_1e_2\eta$ on V^{P_1} for those representations that contain newforms. We shall also list the eigenvalues of the operator d_1d_{02} on $(V^{P_1})^{\mathrm{new}}$, and the eigenvalues of $d_{02}d_1$ on $(V^{P_{02}})^{\mathrm{new}}$. When represented as 8×8–matrices, the two operators d_1d_{02} and $d_{02}d_1$ turn out to have a surprisingly simple description (which we shall not state explicitly).

	$e_2e_1e_2\eta$ on V^{P_1}	$(1+q)^2d_1d_{02}$ on V^{P_1}	$(1+q)^2d_{02}d_1$ on $V^{P_{02}}$
IIa	$q\,(\chi\sigma)(\varpi)$	$\frac{q^2}{q+1}(q^{-1/2}-\alpha)(q^{-1/2}-\alpha^{-1})$	$\frac{q^2}{q+1}(q^{-1/2}-\alpha)(q^{-1/2}-\alpha^{-1})$
IIIa	$q\,(\chi\sigma)(\varpi),\ q\,\sigma(\varpi)$	$0,\ 0$	—
IVb	$q^2\sigma(\varpi),\ \sigma(\varpi)$	$0,\ 0$	—
IVc	$q\,\sigma(\varpi)$	$-(q-1)^2$	$-(q-1)^2$
Vb	$-q\,\sigma(\varpi)$	$2q$	$2q$
Vc	$q\,\sigma(\varpi)$	$2q$	$2q$
VIa	$q\,\sigma(\varpi)$	0	—
VIb	$q\,\sigma(\varpi)$	0	—
VIc	—	—	0

(24)

The number α in the first row of the table abbreviates $\chi(\varpi)$. Since α^2 is not allowed to take the values $q^{\pm1}$ or $q^{\pm3}$, we can see from the last column that one can distinguish newforms for P_{02}

by their eigenvalues under $d_{02}d_1$ and under the Atkin–Lehner involution. Moreover, knowing nothing more than these two numbers, one can write down the correct L–factor. Similarly, for a newform with respect to P_1, knowledge of the eigenvalues under $e_2e_1e_2\eta$ and d_1d_{02} allows to determine the L–packet and the L–factor (but we cannot distinguish types VIa and VIb). These facts will be exploited in our global applications, see Theorem 3.3.9.

We note that, given a representation in the above list with trivial central character, one can tell if the representation is of type IIIa or not by knowing the eigenvalues under $e_2e_1e_2\eta$. This is because all the other representations have eigenvalues ± 1, $\pm q$ or $\pm q^2$, while these values do not occur for IIIa (we have $\chi\sigma^2 = 1$ and $\chi \notin \{\mathbf{1}, \nu^{\pm 2}\}$). This observation will be used in the proof of Theorem 3.3.7.

3 Global newforms

We shall now apply the previously obtained local results to classical Siegel modular forms of degree 2. Assuming the existence of a suitable L–function theory, we will first prove a strong multiplicity one result for certain cusp forms with Iwahori–spherical local components. After recalling several basic facts on the relation between classical modular forms and automorphic representations of $\mathrm{GSp}(4)$, we will define classical newforms for various congruence subgroups of square-free level. Our local and global representation–theoretic results will yield a number of theorems on the newforms thus defined.

3.1 Strong multiplicity one results

In this section we shall prove results of the following kind. Let $\pi_1 = \otimes \pi_{1,v}$ and $\pi_2 = \otimes \pi_{2,v}$ be two cuspidal automorphic representations of $\mathrm{GSp}(4, \mathbb{A})$ of a certain kind. Assume that $\pi_{1,v} \simeq \pi_{2,v}$ for almost all v. Then $\pi_1 \simeq \pi_2$. It is presently not known in general if $\pi_1 \simeq \pi_2$ implies $\pi_1 = \pi_2$ as spaces of automorphic forms, but if this weak multiplicity one is true, then our results are special cases of what is called strong multiplicity one.

3.1.1 Lemma. *Let S be a finite set. For each $i \in S$ let q_i be a positive power of some prime number p_i. We assume that $p_i \neq q_i$ for $i \neq j$. Let $R_i \in \mathbb{C}(X)$ be rational functions such that*

$$\prod_{i \in S} R_i(q_i^s) = 1 \qquad \text{for all } s \in \mathbb{C}. \tag{25}$$

Then all the R_i are constant.

Proof: Left to the reader. ∎

3.1.2 Lemma. *Let F be a non-archimedean local field, and let π_1 and π_2 be irreducible, unitary representations of $\mathrm{GSp}(4, F)$ with non-zero Iwahori–fixed vectors. Assume that there exists $c \in \mathbb{C}^*$ and an integer m such that*

$$\frac{L(s, \pi_1)}{L(s, \pi_2)} = cq^{ms} \frac{L(1-s, \hat{\pi}_1)}{L(1-s, \hat{\pi}_2)}, \tag{26}$$

where $L(s,\pi_i)$ are the local L–factors as listed in Table 2. Assume also that π_1 and π_2 have the same central character. Then π_1 and π_2 are constituents of the same induced representation (from an unramified character of the Borel subgroup).

Proof: This can be checked case by case, going through all the possibilities for π_1 and π_2 that are listed in Table 2. Note that we can count out representations of type IVb and IVc, since by [ST], Theorem 4.4, they are not unitary. As an example we will treat the case that both representations are of type I, where we have to show that $\pi_1 \simeq \pi_2$.

By our hypothesis that both representations have non-trivial Iwahori–fixed vectors, all the characters used for the induction are unramified. Hence there are $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}^*$ such that

$$L(s,\pi_i) = \big((1-\gamma_i q^{-s})(1-\alpha_i\gamma_i q^{-s})(1-\beta_i\gamma_i q^{-s})(1-\alpha_i\beta_i\gamma_i q^{-s})\big)^{-1}.$$

It follows from (26) that there is an equality of rational functions

$$\frac{(1-\gamma_2X)(1-\alpha_2\gamma_2X)(1-\beta_2\gamma_2X)(1-\alpha_2\beta_2\gamma_2X)}{(1-\gamma_1X)(1-\alpha_1\gamma_1X)(1-\beta_1\gamma_1X)(1-\alpha_1\beta_1\gamma_1X)} =$$
$$cX^{-m}\frac{(1-\gamma_2^{-1}q^{-1}X^{-1})(1-\alpha_2^{-1}\gamma_2^{-1}q^{-1}X^{-1})(1-\beta_2^{-1}\gamma_2^{-1}q^{-1}X^{-1})(1-\alpha_2^{-1}\beta_2^{-1}\gamma_2^{-1}q^{-1}X^{-1})}{(1-\gamma_1^{-1}q^{-1}X^{-1})(1-\alpha_1^{-1}\gamma_1^{-1}q^{-1}X^{-1})(1-\beta_1^{-1}\gamma_1^{-1}q^{-1}X^{-1})(1-\alpha_1^{-1}\beta_1^{-1}\gamma_1^{-1}q^{-1}X^{-1})}.$$

Eliminating denominators and comparing zeros on both sides, we find that

$$\begin{aligned}&\{\gamma_1, \alpha_1\gamma_1,\, \beta_1\gamma_1,\, \alpha_1\beta_1\gamma_1,\, \gamma_2 q,\, \alpha_2\gamma_2 q,\, \beta_2\gamma_2 q,\, \alpha_2\beta_2\gamma_2 q\}\\ &\quad = \{\gamma_2,\, \alpha_2\gamma_2,\, \beta_2\gamma_2,\, \alpha_2\beta_2\gamma_2,\, \gamma_1 q,\, \alpha_1\gamma_1 q,\, \beta_1\gamma_1 q,\, \alpha_1\beta_1\gamma_1 q\},\end{aligned}$$

where these are multisets, meaning elements are allowed to appear more than once. First consider the tempered case, meaning all the constants have absolute value 1 (see Table 1). Then necessarily

$$\{\gamma_1,\, \alpha_1\gamma_1,\, \beta_1\gamma_1,\, \alpha_1\beta_1\gamma_1\} = \{\gamma_2,\, \alpha_2\gamma_2,\, \beta_2\gamma_2,\, \alpha_2\beta_2\gamma_2\}.$$

Again considering several cases, one can easily check that this condition, together with the equality $\alpha_1\beta_1\gamma_1^2 = \alpha_2\beta_2\gamma_2^2$, which is equivalent to the equality of the central characters, imply $\pi_1 \simeq \pi_2$. In the non-tempered case one argues similarly, but uses estimates on the absolute values of the inducing characters taken from [ST], Theorem 4.4 (this is where the unitarity condition is used). ∎

3.1.3 Remark. The statement of the lemma would be false without the hypothesis on the central character, as the following examples show. Let ξ_0 be a non-trivial quadratic character of F^*.

- The representations $\xi_0 \times \chi \rtimes \sigma$ and $\xi_0 \times \xi_0\chi \rtimes \sigma$, if irreducible, have the same L–functions, but are not isomorphic.
- π_1 is a constituent of $\xi_0 \times \nu \rtimes \nu^{-1/2}\sigma$ (type III) and π_2 is a constituent of $\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma$ (type V).

In the following we shall utilize the spin L–function for cuspidal automorphic representations of $\mathrm{GSp}(4)$ as a global tool. Any L–function theory that has the following properties would suffice.

3.1.4 L–Function Theory for $\mathrm{GSp}(4)$.

i) To every cuspidal automorphic representation π of $\mathrm{PGSp}(4,\mathbb{A})$ is associated a global L–function $L(s,\pi)$ and a global ε–factor $\varepsilon(s,\pi)$, both defined as Euler products, such that $L(s,\pi)$ has meromorphic continuation to all of $\mathbb{C}$ and such that a functional equation

$$L(s,\pi)=\varepsilon(s,\pi)L(1-s,\pi)$$

of the standard kind holds.

ii) For Iwahori–spherical representations, the local factors $L_v(s,\pi_v)$ coincide with the spin local factors as given in Table 2, and the factors $\varepsilon_v(s,\pi_v,\psi_v)$ coincide with the ε–factors as given in Table 3.

Of course such an L–function theory is predicted by general conjectures over any number field. For our classical applications we shall only need it over $\mathbb{Q}$. Furthermore, we can restrict to the archimedean component being a lowest weight representation with scalar minimal K–type (a discrete series representation if the weight is ≥ 3). All we need to know about ε–factors is in fact that they are of the form cp^{ms} with a constant $c\in\mathbb{C}^*$ and an integer m. Unfortunately, none of the current results on the spin L–function (see [No], [PS2] or [An1]) fully serves our needs; hence, in what follows, we have to make assumptions.

3.1.5 Theorem. *Let $\pi_1=\otimes\pi_{1,v}$ and $\pi_2=\otimes\pi_{2,v}$ be two cuspidal automorphic representations of $\mathrm{GSp}(4,\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of some number field F. Let S be a finite set of finite places of F. Assume the following holds:*

i) Different elements of S divide different places of $\mathbb{Q}$.

ii) $\pi_{1,v}\simeq\pi_{2,v}$ for each $v\notin S$.

iii) For each $v\in S$, both $\pi_{1,v}$ and $\pi_{2,v}$ possess non-trivial Iwahori–invariant vectors.

iv) The central characters of π_1 and π_2 coincide.

Assume also that an L–function theory as in 3.1.4 exists.[1] *Then, for each $v\in S$, the representations $\pi_{1,v}$ and $\pi_{2,v}$ are constituents of the same induced representation.*

Proof: Let $L(s,\pi_i)=\prod_v L_v(s,\pi_{i,v})$ be the global L–function of π_i. By our L–function theory, we have meromorphic continuation to all of $\mathbb{C}$ and a functional equation

$$L(s,\pi_i)=\varepsilon(s,\pi_i)L(1-s,\hat{\pi}_i).$$

[1] For this theorem and its corollaries we do not need the assertion about ε–factors in 3.1.4 ii).

Here $\hat{\pi}_i$ is the contragredient of π_i, and $\varepsilon(s,\pi_i) = \prod_v \varepsilon_v(s,\pi_{i,v},\psi_v)$ is the global ε–factor. Dividing the two functional equations and observing hypothesis ii), we obtain a relation

$$\prod_{v\in S} \frac{L(s,\pi_{1,v})\,L(1-s,\hat{\pi}_{2,v})\,\varepsilon(s,\pi_{2,v},\psi_v)}{L(s,\pi_{2,v})\,L(1-s,\hat{\pi}_{1,v})\,\varepsilon(s,\pi_{1,v},\psi_v)} = 1.$$

Note that each quotient on the left side is a rational function in q_v^s, where q_v is the number of elements of the residue field of F_v. Hypothesis i) and Lemma 3.1.1 therefore imply that each factor in the product is constant. This shows that for each $v \in S$ there is a relation

$$\frac{L(s,\pi_{1,v})}{L(s,\pi_{2,v})} = c_v q^{m_v s} \frac{L(1-s,\hat{\pi}_{1,v})}{L(1-s,\hat{\pi}_{2,v})} \tag{27}$$

with a constant $c_v \in \mathbb{C}^*$ and an integer m_v. Since π_i is cuspidal, each of the local representations is unitary. Furthermore, the central characters of $\pi_{1,v}$ and $\pi_{2,v}$ coincide by hypothesis. The result therefore follows from Lemma 3.1.2. ∎

3.1.6 Corollary. *Let $\pi_1 = \otimes\pi_{1,p}$ and $\pi_2 = \otimes\pi_{2,p}$ be two cuspidal automorphic representations of* $\mathrm{PGSp}(4,\mathbb{A})$, *where $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$. Let S be a finite set of prime numbers such that:*

i) $\pi_{1,p} \simeq \pi_{2,p}$ for each $p \notin S$.

ii) For each $p \in S$, both $\pi_{1,p}$ and $\pi_{2,p}$ are generic representations with non-trivial Iwahori–invariant vectors.

Assume also that an L–function theory as in 3.1.4 exists. Then $\pi_1 \simeq \pi_2$.

Proof: Hypothesis i) of Theorem 3.1.5 is fulfilled because we are over $\mathbb{Q}$. Both representations are assumed to have trivial central character, so hypothesis iv) of Theorem 3.1.5 is also fulfilled. Hence we can apply this Theorem and obtain that for each $p \in S$ the representations $\pi_{1,p}$ and $\pi_{2,p}$ are constituents of the same induced representation. But each induced representation has only one generic constituent, so necessarily $\pi_{1,p} \simeq \pi_{2,p}$. ∎

3.1.7 Corollary. *Let $\pi_1 = \otimes\pi_{1,p}$ and $\pi_2 = \otimes\pi_{2,p}$ be two cuspidal automorphic representations of* $\mathrm{PGSp}(4,\mathbb{A})$, *where $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$. Let S be a finite set of prime numbers such that:*

i) $\pi_{1,p} \simeq \pi_{2,p}$ for each $p \notin S$.

ii) For each $p \in S$, the representation $\pi_{1,p}$ is K–spherical if and only if $\pi_{2,p}$ is K–spherical.

iii) For each $p \in S$ such that π_1 and π_2 are not K–spherical, the representation $\pi_{i,p}$ $(i = 1,2)$ contains a non-zero vector $v_{i,p}$ invariant under the local paramodular group P_{02} at p.

iv) For each $p \in S$ such that π_1 and π_2 are not K–spherical, the vectors $v_{1,p}$ and $v_{2,p}$ are eigenvectors for the Atkin–Lehner involution η_p with the same eigenvalue.

Assume also that an L–function theory as in 3.1.4 exists. Then $\pi_1 \simeq \pi_2$.

Proof: The hypotheses of Theorem 3.1.5 are fulfilled, so $\pi_{1,p}$ and $\pi_{2,p}$ are constituents of the same induced representation. But a look at Table 3 shows that two representations with P_{02}–invariant vectors in the same group can be distinguished by their Atkin–Lehner eigenvalues. ■

3.2 Classical modular forms

This section is to collect several definitions and conventions on classical Siegel modular forms. We shall only treat holomorphic scalar-valued modular forms, but since all our manipulations will be done at finite places, everything we are saying in the following generalizes immediately to vector-valued modular forms. Also, for the sake of simplicity, we refrain from considering modular forms with character (these could be considered except when we are talking about Atkin–Lehner involutions).

When speaking about classical modular forms, it is more convenient to realize symplectic groups using the symplectic form $\begin{pmatrix} & \mathbf{1} \\ -\mathbf{1} & \end{pmatrix}$, which we shall do from now on. For N a positive integer, global analogues of the local parahoric subgroups are defined as follows (notations as in [HI]).

$$B(N) := \mathrm{Sp}(4,\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix},$$

$$U_1(N) := \mathrm{Sp}(4,\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix},$$

$$U_2(N) := \mathrm{Sp}(4,\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix},$$

$$U_0(N) := \mathrm{Sp}(4,\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix},$$

$$U_{02}(N) := \mathrm{Sp}(4,\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

The group $U_1(N)$ is usually denoted $\Gamma_0(N)$. The group $U_{02}(N)$ is the *paramodular group* of level N and corresponds to the local maximal compact subgroup P_{02}. Note that

$$\eta_N\, U_2(N)\, \eta_N^{-1} = U_0(N), \qquad \text{where } \eta_N = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & N & & \\ N & & & \end{pmatrix}, \tag{28}$$

while $B(N)$, $U_1(N)$ and $U_{02}(N)$ are normalized by η_N. If Γ' is one of the above groups, we define $S_k(\Gamma')$ to be the space of Siegel modular forms of degree 2 and weight k with respect to the group Γ'. This space is a hermitian vector space with respect to the Petersson scalar product. In this paper we shall not consider non-cuspidal modular forms.

Generalities on lifting modular forms

Let $G = \mathrm{GSp}(4)$. For each prime number p let K_p be an open compact subgroup of $G(\mathbb{Z}_p)$ such that the multiplier map $K_p \to \mathbb{Z}_p^*$ is surjective. Then it follows from strong approximation for $\mathrm{Sp}(4)$ that

$$G(\mathbb{A}_\mathbb{Q}) = G(\mathbb{Q})G(\mathbb{R})^+K_f, \qquad K_f = \prod_{p<\infty} K_p, \tag{29}$$

where $G(\mathbb{R})^+$ is the group of elements of $G(\mathbb{R})$ with positive multiplier. Now let $f \in S_k(\Gamma')$ be a modular form for a subgroup Γ'. We assume that

$$\Gamma' = G(\mathbb{Q}) \cap G(\mathbb{R})^+K_f, \qquad K_f = \prod_{p<\infty} K_p,$$

with local subgroups K_p for which the above hypothesis on the multiplier map holds. We define a function $\Phi_f : G(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ as follows. By (29), it is possible to write a given $g \in G(\mathbb{A})$ as $g = \rho g_\infty h$ with $\rho \in G(\mathbb{Q})$, $g_\infty \in G(\mathbb{R})^+$, $h \in K_f$. Then we put

$$\Phi_f(g) = \lambda(g_\infty)^k j(g_\infty, I)^{-k} f(g_\infty\langle I\rangle). \tag{30}$$

Here λ denotes the multiplier map and $I = \begin{pmatrix} i & \\ & i \end{pmatrix}$. The symbol $j(g_\infty, I)$ stands for the usual modular factor, and $Z \mapsto g_\infty\langle Z\rangle$ is the action of $G(\mathbb{R})^+$ on the Siegel upper half plane $\mathbb{H}_2$. Using the transformation property of the modular form f, one checks easily that Φ_f is well-defined. The factor $\lambda(g_\infty)^k$ ensures that

$$\Phi_f(gz) = \Phi_f(g) \qquad \text{for all } g \in G(\mathbb{A}),\ z \in Z(\mathbb{A}) \simeq \mathbb{A}^*. \tag{31}$$

Here Z denotes the center of G. Since f is a cuspform, the function Φ_f is an element of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z(\mathbb{A}))$. Let π be the automorphic representation of $G(\mathbb{A})$ generated by Φ_f inside this L^2–space. It decomposes into a finite direct sum of irreducible representations, $\pi = \bigoplus_i \pi_i$. Let us write each π_i as a tensor product of local representations,

$$\pi_i = \bigotimes_{p\le\infty} \pi_{i,p}, \qquad \pi_{i,p} \text{ an irreducible representation of } G(\mathbb{Q}_p).$$

Since f is a modular form of weight k, all the archimedean components $\pi_{i,\infty}$ are isomorphic to a representation π_k^+ of $G(\mathbb{R})$ that has a lowest weight vector of weight (k,k) (it belongs to the discrete series if $k \ge 3$, see [AS] for more details). Let us now assume that f is a common eigenfunction for almost all the local (commutative) Hecke algebras $\mathcal{H}_p$. Then it follows easily that for all such p and all i,j we have $\pi_{i,p} \simeq \pi_{j,p}$. In the GL(2)–case we could now conclude by strong multiplicity one that π must be irreducible. Unfortunately, strong multiplicity one or even multiplicity one is currently not available for GSp(4).

But assume now that N is a square-free number and that the subgroup Γ' contains $B(N)$. Then each $\pi_{i,p}$ for $p|N$ has non-zero Iwahori–fixed vectors, and we can use the results of section 3.1 to show in several cases that all the π_i are globally isomorphic (see the next section). In these cases we can therefore associate a unique equivalence class π_f of automorphic representations with the modular form f.

Atkin–Lehner involutions

Let N be an integer and Γ' one of the groups B, U_1 or U_{02}. We shall define the Atkin–Lehner involutions on the space $S_k(\Gamma'(N))$. For a prime p dividing N let p^j be the exact power of p dividing N. Choose a matrix $\gamma_p \in \mathrm{Sp}(4,\mathbb{Z})$ such that

$$\gamma_p \equiv \begin{pmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{pmatrix} \bmod p^j \qquad \text{and} \qquad \gamma_p \equiv \begin{pmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1 \end{pmatrix} \bmod Np^{-j},$$

and define the *Atkin–Lehner element*

$$u_p := \gamma_p \begin{pmatrix} p^j&&&\\ &p^j&&\\ &&1&\\ &&&1 \end{pmatrix}.$$

A different choice of γ_p results in multiplying u_p from the left with an element of the principal congruence subgroup $\Gamma(N)$. Therefore the action of u_p on modular forms for $\Gamma(N)$ is unambiguously defined. One can check that u_p normalizes $\Gamma'(N)$. Consequently the map $F \mapsto F\big|u_p$ defines an endomorphism of $S_k(\Gamma'(N))$, which is an involution since $u_p^2 \in p^j\Gamma(N)$. This is the *Atkin–Lehner involution* at p. We also denote it by $f \mapsto \eta_p f$. A straightforward calculation shows that these η_p on classical modular forms are compatible with the local Atkin–Lehner involutions of the same name defined in section 1.3. More precisely, we have

$$\Phi_{\eta_p f}(g) = \Phi_f(g\eta_p) \tag{32}$$

for the associated adelic functions, where the η_p on the right is the local element as defined in (8) at the place p.

Some trace operators

We have defined the local projection operators d_i at the end of section 2.1. We will now introduce analogous operators of the same name on global modular forms. Let N be a square-free positive integer and p a prime dividing N. Let $f \in S_k(B(N))$. Then

$$(d_1(p)f)(Z) := \frac{1}{p+1} \sum_{h \in B(N)\backslash(B(Np^{-1})\cap U_1(p))} (f|_k h)(Z) \tag{33}$$

and

$$(d_2(p)f)(Z) := \frac{1}{p+1} \sum_{h \in B(N)\backslash(B(Np^{-1})\cap U_2(p))} (f|_k h)(Z). \tag{34}$$

Here $|_k$ is the usual classical operator. We further have $d_0(p)f := \eta_p^{-1}\, d_2(p)\, \eta_p f$, where η_p is the Atkin–Lehner involution. Note that since the definition of $d_i(p)$ also depends on the level N, it should more precisely be denoted by $d_i(N,p)$. Instead, to ease notation, we will sometimes also drop the p and simply write d_i, hoping that N and p are clear from the context. It is easily checked that these operators are compatible with the associated adelic functions in the sense that

$$\Phi_{d_i(p)f} = d_i(p)\Phi_f \qquad \text{for } i = 0,1,2. \tag{35}$$

On the right side of each equation we have the local operators at the place p defined in section 2.1, acting on the adelic function in the obvious way. Let

$$f(Z) = \sum_T c(T) e^{2\pi i\, \mathrm{tr}(TZ)} \tag{36}$$

be the usual Fourier expansion of f, where T runs over positive definite, half-integral matrices, and let

$$f(Z) = \sum_{m=1}^{\infty} f_m(\tau, z) e^{2\pi i m\tau'}, \qquad Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}, \tag{37}$$

be the Fourier–Jacobi expansion of f. Here f_m is a Jacobi form of index m and level N (meaning for the subgroup $\Gamma_0(N)$ of $\mathrm{SL}(2,\mathbb{Z})$). Then easy calculations show

$$(d_1(p)f)(Z) = \sum_T \tilde{c}(T) e^{2\pi i\, \mathrm{tr}(TZ)} \qquad \text{with} \quad \tilde{c}(T) = \sum_{\gamma \in \Gamma_0(N)\backslash\Gamma_0(Np^{-1})} c(\gamma T\, {}^t\gamma) \tag{38}$$

and

$$(d_2(p)f)(Z) = \sum_{m=1}^{\infty} \tilde{f}_m(\tau, z) e^{2\pi i m\tau'} \qquad \text{with} \quad \tilde{f}_m = \sum_{\gamma \in \Gamma_0(N)\backslash\Gamma_0(Np^{-1})} f_m|_k\gamma. \tag{39}$$

In both equations $\Gamma_0(N)$ and $\Gamma_0(Np^{-1})$ mean subgroups of $\mathrm{SL}(2,\mathbb{Z})$ (not of $\mathrm{Sp}(4,\mathbb{Z})$). In equation (39), the symbol $f|_k\gamma$ denotes the usual action of an element of $\mathrm{SL}(2,\mathbb{Z})$ on a Jacobi form, as in [EZ].

In a similar way we can also define operators $d_{ij}(p)$ (or $d_{ij}(N,p)$, or simply d_{ij}) on $S_k(B(N))$ that are compatible with the local operators d_{ij} defined in (18) and used in section 2.3. We shall refrain from giving explicit formulas here, but these operators will have some significance for the newform theory with respect to $U_1(N)$ and $U_{02}(N)$ in the next section.

3.3 Classical newforms

In this section we will use our previous representation–theoretic results to develop a theory of old– and newforms for Siegel cusp forms of degree 2 with square-free level. We will obtain different theories for the "minimal" subgroup $B(N)$, the Hecke subgroup $U_1(N)$ and the parahoric subgroup $U_{02}(N)$.

Newforms for $B(N)$

Let N be a positive integer, decomposed as $N = N_1N_2$ with coprime N_1, N_2. Then $B(N) \subset B(N_1) \cap U_1(N_2)$, so $S_k\big(B(N_1) \cap U_1(N_2)\big)$ is a subspace of $S_k(B(N))$. Similarly we have the subspaces $S_k\big(B(N_1) \cap U_2(N_2)\big)$ and $S_k\big(B(N_1) \cap U_0(N_2)\big)$. The definition of newforms for $B(N)$ we shall now give is designed to be compatible with the local definition given in section 2.2. It is also the same definition as given in the papers [Ib1] and [HI].

3.3.1 Definition. *Let N be a square-free positive integer. In $S_k(B(N))$ we define the subspace of* **oldforms** *$S_k(B(N))^{\rm old}$ to be the sum of the spaces*

$$S_k\big(B(N_1) \cap U_0(N_2)\big) + S_k\big(B(N_1) \cap U_1(N_2)\big) + S_k\big(B(N_1) \cap U_2(N_2)\big),$$

where N_1, N_2 run through all positive integers such that $N_1N_2 = N$, $(N_1, N_2) = 1$ and $N_2 > 1$. The subspace of **newforms** *$S_k(B(N))^{\rm new}$ is defined as the orthogonal complement of the space $S_k(B(N))^{\rm old}$ inside $S_k(B(N))$ with respect to the Petersson scalar product.*

Thus, a modular form for $B(N)$ is considered to be *old* if it is invariant under $\Gamma'(p)$ for some $p|N$ and some Γ' defined at p by a parahoric subgroup different from the minimal one.

3.3.2 Theorem. *Let N be a square-free positive integer, and let $f \in S_k(B(N))^{\rm new}$. We assume that f is an eigenform for the local Hecke algebras $\mathcal{H}_p$ for almost all primes p. Assuming that an L–function theory as in 3.1.4 exists,[2] the following holds.*

i) The corresponding adelic function Φ_f as defined in (30) generates a multiple of an automorphic representation π_f of ${\rm PGSp}(4, \mathbb{A}_\mathbb{Q})$.

ii) f is an eigenfunction for the local Hecke algebras $\mathcal{H}_p$ for <u>all</u> *primes $p \nmid N$.*

iii) Let W_f be the subspace of $S_k(B(N))^{\rm new}$ spanned by all eigenforms that have the same Satake parameters as f for almost all p. Then

$$\dim_\mathbb{C}(W_f) = {\rm mult}(\pi_f), \tag{40}$$

where the right side denotes the multiplicity of the automorphic representation π_f defined in i) within the space of all cusp forms. In particular, if multiplicity one holds, then a newform is determined, up to multiples, by almost all of its Satake parameters.

iv) f is an eigenform for the Atkin–Lehner involution η_p for each $p|N$.

v) For each $p|N$, the local component of π_f at p is given by

$$\pi_{f,p} = \begin{cases} {\rm St}_{{\rm GSp}(4,\mathbb{Q}_p)} & \text{if } \eta_p f = -f, \\ \xi_0\,{\rm St}_{{\rm GSp}(4,\mathbb{Q}_p)} & \text{if } \eta_p f = f. \end{cases}$$

Here ξ_0 is the unique non-trivial unramified quadratic character of $\mathbb{Q}_p^*$.

[2] We need to assume 3.1.4 ii) only for the Steinberg representation. We also need to assume that our L–function theory produces the Langlands local factors at the archimedean place, since otherwise the factor given in (42) would not be the correct one.

vi) *For primes $p \nmid N$ we define local spin L–factors as usual. With ε_p being the Atkin–Lehner eigenvalue at $p|N$, we further define*

$$L_p(s,f) = \left(1+\varepsilon_p p^{-3/2-s}\right)^{-1} \qquad \text{for } p|N, \tag{41}$$

$$L_\infty(s,f) = 4(2\pi)^{-2s+1-k}\Gamma\left(s+k-\frac{3}{2}\right)\Gamma\left(s+\frac{1}{2}\right). \tag{42}$$

Then the spin L–function $L(s,f)=\prod_{p\le\infty}L_p(s,f)$ has meromorphic continuation to all of $\mathbb{C}$ and satisfies the functional equation

$$L(s,f)=\varepsilon(s,f)L(1-s,f) \qquad \text{with } \varepsilon(s,f):=(-1)^k\Big(\prod_{p|N}\varepsilon_p\Big)N^{3(1/2-s)}. \tag{43}$$

If $N>1$, then $L(s,f)$ is holomorphic.

Proof: i) As explained above, Φ_f generates a representation π which we decompose into irreducibles π_i. If we decompose each π_i into a tensor product $\otimes\pi_{i,p}$ of local representations, then all the $\pi_{i,\infty}$ will be isomorphic. Moreover, by hypothesis, there is a finite set S of primes (containing the primes dividing N) such that for each $p\notin S$ all the $\pi_{i,p}$ are isomorphic. It follows from the definition of newforms and Proposition 2.2.1 that $\pi_{i,p}$ is an unramified twist of the Steinberg representation, for each $p|N$ and each i. In particular, the local components of π_i at *every* finite place are generic. We can therefore apply Corollary 3.1.6 to conclude that all the π_i are isomorphic.

ii) follows immediately from i).

iii) The dimension $\dim(W_f)$ is obviously the number of linearly independent $g\in S_k(B(N))^{\rm new}$ that can be extracted from the direct sum of all cuspidal automorphic representations that are isomorphic to π_f. Equation (40) therefore is equivalent to the fact that in each local representation $\pi_{i,p}$ there is exactly one linearly independent *local newform*. But this is obvious, since the space of Iwahori–invariant vectors of the Steinberg representation is one-dimensional; see Table 3. (We are also using the fact that in the lowest weight representations π_k^+ at the archimedean place the space of lowest weight vectors is one-dimensional.)

iv) and v) We already saw that $\pi_{f,p}$ is an unramified twist of the Steinberg representation. Since the central character is trivial, there are only the two possibilities listed in v). In either case we have a one-dimensional space of Iwahori–invariant vectors, proving iv). Which of the two representations actually appears is decided by the Atkin–Lehner eigenvalue, see Table 3.

vi) A look at Table 2 shows that for $p|N$ the L–factor is given as in (41). The ε–factor of $\sigma{\rm St}_{{\rm GSp}(4)}$ for unramified σ (and choice of a suitable unramified additive character ψ) is given by $-\sigma(\varpi)q^{3(1/2-s)}$, as can be determined from [Ta]. Thus $\varepsilon(s,\pi_{f,p},\psi)=\varepsilon_p p^{3(1/2-s)}$ for $p|N$. The archimedean factor in (42) is, up to a constant and up to a shift in the argument, just the usual Andrianov Γ–factor (which in degree 2 coincides with the archimedean Langlands L–factor, see [Sch2]). Since the archimedean ε–factor is $(-1)^k$, the global ε–factor is given by $(-1)^k(\prod_{p|N}\varepsilon_p)N^{3(1/2-s)}$. Now all the claimed analytic properties of $L(s,f)$ follow from our L–function theory and from [PS2], Theorem 5.3 (if $N=1$ then f may be a Saito–Kurokawa lifting in which case $L(s,f)$ would have poles). ∎

Remark: It was mentioned in [HI], p. 38, that (for any degree) the local components at $p|N$ of the automorphic representations associated to newforms in $S_k(B(N))$ are special representations.

In view of (38) and (39), the next result shows that newforms for $B(N)$ can be characterized in terms of their Fourier and Fourier–Jacobi expansions.

3.3.3 Corollary. *An element $f \in S_k(B(N))$ is a newform if and only if*

$$d_1(p)(f) = d_2(p)(f) = d_2(p)(\eta_p f) = 0 \qquad \text{for all } p|N,$$

where $d_1(p)$ and $d_2(p)$ are the operators defined in (33) resp. (34).

Proof: This follows from Theorem 3.3.2 iv), Proposition 2.2.1 and equation (35). ■

3.3.4 Corollary. *If a cusp form $f \in S_k(\mathrm{Sp}(4,\mathbb{Z}))$ is an eigenfunction for almost all Hecke algebras $\mathcal{H}_p$, then it is an eigenfunction for all those Hecke algebras.*[3]

Proof: Theorem 3.3.2 applies with $N = 1$. ■

Newforms for $U_1(N)$

Let N be a square-free positive integer. To describe newforms for the Hecke subgroup $U_1(N)$ (usually called $\Gamma_0(N)$) we shall begin by describing, for $p|N$, four endomorphisms $T_0(p)$, $T_1(p)$, $T_2(p)$, $T_3(p)$ of $S_k(U_1(N))$ which are analogous to some of the local operators considered in section 2.3. The operator $T_0(p)$ is simply the identity. We define $T_1(p) := \eta_p$, the Atkin–Lehner involution. Note that if $f \in S_k(U_1(N))$ happens to be a modular form for $U_1(Np^{-1})$, then

$$(T_1(p)f)(Z) = p^k f(pZ) \qquad (Z \in \mathbb{H}_2).$$

We define $T_2(p)$ by

$$\begin{aligned}(T_2(p)f)(Z) &= \sum_{x,\mu,\kappa \in \mathbb{Z}/p\mathbb{Z}} \left(f\Big|_k \begin{pmatrix} 1 &&& \\ & 1 && \\ && p & \\ &&& p \end{pmatrix} \begin{pmatrix} 1 && x & \mu \\ & 1 & \mu & \kappa \\ && 1 & \\ &&& 1 \end{pmatrix} \right)(Z) \\ &= \sum_{g \in P_1 \backslash P_1 \left(\begin{smallmatrix} \mathbf{1} & \\ & p\mathbf{1} \end{smallmatrix}\right) P_1} \left(f\big|_k g\right)(Z) \end{aligned} \tag{44}$$

(cf. Remark 2.3.4). In terms of Fourier expansions, if $f(Z) = \sum_{n,r,m} A(n,r,m) e^{2\pi i(n\tau + rz + m\tau')}$, then

$$(T_2(p)f)(Z) = \sum_{n,r,m} A(np, rp, mp) e^{2\pi i(n\tau + rz + m\tau')}, \qquad Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}. \tag{45}$$

[3] In this completely unramified case we can work with the Andrianov L–function and need not make any additional assumptions on the existence of a suitable L–function theory.

Finally, we define $T_3(p) := \eta_p \circ T_2(p)$. It is easy to check that with these definitions

$$\Phi_{T_0(p)f} = e\Phi_f, \qquad \Phi_{T_1(p)f} = \eta\Phi_f \qquad \Phi_{T_2(p)f} = e_2e_1e_2\eta\Phi_f, \qquad \Phi_{T_3(p)f} = e_0e_1e_0\Phi_f \quad (46)$$

holds for the associated adelic functions. On the right side of each equation we have elements of the local Iwahori–Hecke algebra acting on the adelic functions in the obvious way.

3.3.5 Definition. *Let N be a square-free positive integer. In $S_k(U_1(N))$ we define the subspace of* **oldforms** *$S_k(U_1(N))^{\rm old}$ to be the sum of the spaces*

$$T_i(p)S_k\big(U_1(Np^{-1})\big), \qquad i = 0,1,2,3, \quad p|N.$$

We define the subspace of **newforms** *$S_k(U_1(N))^{\rm new}$ to be the orthogonal complement of the space $S_k(U_1(N))^{\rm old}$ inside $S_k(U_1(N))$ with respect to the Petersson scalar product.*

We remark that, for this congruence subgroup, the same definition of the space of oldforms as the image of four linear operators has been given in [Ra].

3.3.6 Remark. It follows from Theorem 2.3.1 that the space $S_k(U_1(N))^{\rm new}$ can be characterized as the common kernel of the operators d_{12}, $d_{12}e_0e_1e_0$, $d_{12}\eta$ and $d_{12}\eta e_2e_1e_2$ for all $p|N$ inside $S_k(U_1(N))$. This is analogous to Corollary 3.3.3.

Some attempts to define L–functions for modular forms $f \in S_k(U_1(N))$ are based on eigenfunctions for $T_2(p)$ at the places $p|N$ (see [An3]). Along these lines we can prove the following result.

3.3.7 Theorem. *Let N be a square-free positive integer and let $f \in S_k(U_1(N))^{\rm new}$. We assume that f is an eigenform for the local Hecke algebras $\mathcal{H}_p$ for almost all primes p. We further assume that*

$$T_2(p)f = \lambda_p f \qquad \text{with } \lambda_p \neq \pm p \text{ for all } p|N. \quad (47)$$

Then:

i) The corresponding adelic function Φ_f as defined in (30) generates a multiple of an automorphic representation π_f of ${\rm PGSp}(4,\mathbb{A}_\mathbb{Q})$.

ii) f is an eigenfunction for the local Hecke algebras $\mathcal{H}_p$ for <u>all</u> primes $p \nmid N$.

iii) For primes $p \nmid N$ we define local spin L–factors as usual. We further define

$$L_p(s,f) = \big((1-\lambda_p p^{-3/2-s})(1-\lambda_p^{-1}p^{1/2-s})\big)^{-1} \qquad \text{for } p|N, \quad (48)$$

and the archimedean factor as in (42).[4] *Then the spin L–function $L(s,f) = \prod_{p\le\infty} L_p(s,f)$ has meromorphic continuation to all of $\mathbb{C}$ and satisfies the functional equation*

$$L(s,f) = \varepsilon(s,f)L(1-s,f) \qquad \text{with } \varepsilon(s,f) := (-1)^k N^{1-2s}. \quad (49)$$

If $N > 1$, then $L(s,f)$ is holomorphic.

[4] We need to assume that our L–function theory really assigns this (Langlands) local L–factor to the discrete series representation at the archimedean place.

Proof: We argue essentially as in Theorem 3.3.2. Since f is a newform, every $\pi_{i,p}$ for $p|N$ must contain local newvectors with respect to P_1 in the sense of Theorem 2.3.1. The operator T_2 corresponds to the element $e_2e_1e_2\eta$ of the local Iwahori–Hecke algebra at p, see (46). In view of table (24) and the remarks following it, the hypothesis (47) implies that $\pi_{i,p}$ is of type IIIa. In particular it is generic. We can therefore invoke Corollary 3.1.6 to prove i) (and ii)).

We now know for $p|N$ that $\pi_{f,p} = \chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$ with unramified characters χ and σ of $\mathbb{Q}_p^*$ such that $\chi\sigma^2 = 1$. By table (24) we have $\lambda_p = \sigma(p)$ or $\lambda_p = \sigma(p)^{-1}$. Hence a look at Table 2 shows that the L–factor is given as in (48). The ε–factor at $p|N$ is given by

$$\begin{aligned}\varepsilon(s, \chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}, \psi) &= \varepsilon(s, \sigma\mathrm{St}_{\mathrm{GL}(2)}, \psi)\varepsilon(s, \chi\sigma\mathrm{St}_{\mathrm{GL}(2)}, \psi)\\ &= (-\sigma(p)p^{1/2-s})(-(\chi\sigma)(p)p^{1/2-s}) = p^{1-2s}\end{aligned}$$

(choosing some unramified additive character ψ). Here we have used the fact (see, e.g., [Sch1]) that $\varepsilon(s, \sigma\mathrm{St}_{\mathrm{GL}(2)}, \psi) = -\sigma(\varpi)p^{1/2-s}$ for an unramified local character σ. Including the archimedean place, the global ε–factor is therefore given by $(-1)^k N^{1-2s}$. Now the analytic properties of $L(s, f)$ follow from our L–function theory and from [PS2], Theorem 5.3. The condition $N > 1$ ensures that f is not a Saito–Kurokawa lifting. ∎

If one of the $T_2(p)$ eigenvalues is $\pm p$, then the situation becomes more complicated because we cannot distinguish representations of type IIa, Vb,c and VIa,b (representations of type IVb,c are irrelevant since they are not unitary by [ST], Theorem 4.4; the trivial representation does also not appear as a local component of a global cuspidal automorphic representation, which follows from the existence of global generalized Whittaker models, see [PS2]). In this case we need more information to determine the type of local representation. Such information can be obtained by requiring that our modular form is also an eigenfunction for the operator

$$T_4(p) := (1 + p)^2 d_1(p)d_{02}(p) \tag{50}$$

for each $p|N$. Investigating table (24) we find that knowing the eigenvalues under $T_2(p)$ and $T_4(p)$ we can determine the local representation, except that we cannot distinguish types VIa and VIb. But VIa and VIb constitute an L–packet, so for defining the correct L–factor it is not necessary to distinguish these two representations.

Unfortunately, we cannot see an easy description for the Hecke operator $T_4(p)$ in (50) in terms of Fourier coefficients, in contrast to the simple formula (45) for $T_2(p)$. However, as already mentioned towards the end of section 2.3, the corresponding local operator is represented by a surprisingly simple 8×8–matrix.

3.3.8 Proposition. *Let N be square-free. The space $S_k(U_1(N))^{\mathrm{new}}$ has a basis consisting of common eigenfunctions for the operators $T_2(p)$ and $T_4(p)$, all $p|N$, and for the unramified Hecke algebras at all good places $p \nmid N$.*

Proof: The assertion follows from the fact that for each local representation (π, V) containing newforms with respect to P_1 the space V^{P_1} has a common eigenbasis for $e_2e_1e_2\eta$ and d_1d_{02}. This in turn is evident from a look at table (24). ∎

3.3.9 Theorem. *Let N be a square-free positive integer and let $f \in S_k(U_1(N))^{\rm new}$. We assume that f is an eigenform for the local Hecke algebras $\mathcal{H}_p$ for almost all primes p. We further assume that f is an eigenfunction for $T_2(p)$ and $T_4(p)$ for all $p|N$,*

$$T_2(p)f = \lambda_p f, \qquad T_4(p)f = \mu_p f \qquad \text{for } p|N. \tag{51}$$

Assuming that an L–function theory as in 3.1.4 exists, the following holds.

i) f is an eigenfunction for the local Hecke algebras $\mathcal{H}_p$ for <u>*all*</u> *primes $p \nmid N$.*

ii) Only the combinations of λ_p and μ_p as given in the following table can occur. Here ε is ± 1.

λ	μ	rep.	$L_p(s,f)^{-1}$	$\varepsilon_p(s,f)$
$-\varepsilon p$	$\notin \{0,2p\}$	IIa	$(1+\varepsilon(p+1)(p-\mu)p^{-3/2-s}+p^{-2s})(1+\varepsilon p^{-1/2-s})$	$\varepsilon p^{1/2-s}$
$\neq \pm p$	0	IIIa	$(1-\lambda p^{-3/2-s})(1-\lambda^{-1}p^{1/2-s})$	p^{1-2s}
$-\varepsilon p$	$2p$	Vb, c	$(1-\varepsilon p^{1/2-s})(1-p^{-1/2-s})(1+p^{-1/2-s})$	$\varepsilon p^{1/2-s}$
$-\varepsilon p$	0	VIa, b	$(1+\varepsilon p^{-1/2-s})^2$	p^{1-2s}

(52)

(We skipped some indices p.)

iii) We define the archimedean L–factor as in (42) and the archimedean ε–factor by $(-1)^k$. We further define the unramified spin Euler factors for $p \nmid N$ as usual, and the ε–factors to be 1. For places $p|N$ we define L– and ε–factors according to table (52). Then the resulting L–function has meromorphic continuation to the whole complex plane and satisfies the functional equation

$$L(s,f) = \varepsilon(s,f)L(1-s,f), \tag{53}$$

where $L(s,f) = \prod_{p\le\infty} L_p(s,f)$ and $\varepsilon(s,f) = (-1)^k \prod_{p|N} \varepsilon_p(s,f)$.

iv) $L(s,f)$ has at most two simple poles at $s = 3/2$ and $s = -1/2$. If $\lambda_p \neq \pm p$ or $\mu_p \notin \{0,2p\}$ for some $p|N$, then $L(s,f)$ is holomorphic everywhere.

Proof: i) We argue as before, considering the global representation $\pi_f = \oplus\pi_i$. Since f is a newform, every $\pi_{i,p}$ for $p|N$ must contain local newvectors with respect to P_1 in the sense of Theorem 2.3.1. In the present case we cannot conclude that all the irreducible components π_i must be isomorphic, because, as mentioned above, the eigenvalues in (51) cannot tell apart local representations VIa and VIb. This is however the only ambiguity, so that we can at least associate a *global L–packet* with f. In particular, we obtain i) by a familiar reasoning.

ii) The possible combinations for λ and μ follow immediately from the data given in table (24).

iii) The L–factors can be easily determined from Table 2 and the values given in table (24). The ε–factors are also easily computed from the local parameters given in Table 2. The functional equation then follows from our L–function theory.

iv) By [PS2], the L–function $L(s,f)$ has at most two simple poles at $s=3/2$ and $s=-1/2$. If $\lambda_p \neq \pm p$ or $\mu_p \notin \{0,2p\}$ for some $p|N$, then, according to table (52), we have a local component of type IIa or IIIa, hence a generic representation. In particular, our representations are not Saito–Kurokawa liftings, which implies $L(s,f)$ is holomorphic (see also part 4) of [PS2], Theorem 5.3). ■

Newforms for $U_{02}(N)$

We have defined trace operators d_0, d_1, d_2 in section 3.2, and we shall use these to define newforms for the paramodular group $U_{02}(N)$ of level N. We remark that in case $N=p$ is a prime the same definition has already been given in [Ib2]. The trace operators used there essentially coincide with our d operators.

3.3.10 Definition. *Let N be a square-free positive integer. In $S_k(U_{02}(N))$ we define the subspace of* **oldforms** *$S_k(U_{02}(N))^{\mathrm{old}}$ to be the sum of the spaces*

$$d_0(p)S_k\big(U_{02}(Np^{-1})\big) + d_2(p)\eta_p S_k\big(U_{02}(Np^{-1})\big), \qquad p|N.$$

The subspace of **newforms** *$S_k(U_{02}(N))^{\mathrm{new}}$ is defined as the orthogonal complement of the space $S_k(U_{02}(N))^{\mathrm{old}}$ inside $S_k(U_{02}(N))$ with respect to the Petersson scalar product.*

3.3.11 Remark. Just as in the previous cases we can characterize newforms as the kernel of certain operators. In fact, by Theorem 2.3.1 an element $f \in S_k(U_{02}(N))$ is a newform if and only if it is annihilated by the operators $d_{12}\eta$ and $d_{12}\eta d_2\eta$ for all $p|N$.

In the following theorem we will make use of the Hecke operators

$$T_5(p) := (1+p)^2 d_{02}(p) d_1(p) \tag{54}$$

on $S_k(U_{02}(N))^{\mathrm{new}}$ for $p|N$ (the theorem will show that global newforms are composed of local newforms at every place, so it is obvious that $T_5(p)$ acts on $S_k(U_{02}(N))^{\mathrm{new}}$). This operator is analogous to $T_4(p)$ introduced in (54) and serves a similar purpose. Again, we did not find a simple description in terms of Fourier coefficients, but the local representation of $T_5(p)$ as an 8×8–matrix has a simple shape.

3.3.12 Theorem. *Let N be a square-free positive integer, and $f \in S_k(U_{02}(N))^{\mathrm{new}}$. We assume that f is an eigenform for the local Hecke algebras $\mathcal{H}_p$ for almost all primes p and for the Atkin–Lehner involutions ε_p for all $p|N$. Assuming that an L–function theory as in 3.1.4 exists, the following holds.*

i) The corresponding adelic function Φ_f as defined in (30) generates a multiple of an automorphic representation π_f of $\mathrm{PGSp}(4,\mathbb{A}_\mathbb{Q})$.

ii) f is an eigenfunction for the local Hecke algebras $\mathcal{H}_p$ for <u>all</u> primes $p \nmid N$.

iii) *Let W_f be the subspace of $S_k(B(N))^{\rm new}$ spanned by all eigenforms that have the same Satake parameters as f for almost all p, and the same Atkin–Lehner eigenvalue for all $p|N$. Then*

$$\dim_{\mathbb{C}}(W_f) = \mathrm{mult}(\pi_f), \tag{55}$$

where the right side denotes the multiplicity of the automorphic representation π_f defined in i) within the space of all cuspforms.

iv) *f is an eigenfunction for the Hecke operator $T_5(p)$, for each $p|N$. Let μ_p be the eigenvalue.*

v) *For each $p|N$, the local component of π_f at p is one of the unitary representations of $\mathrm{PGSp}(4,\mathbb{Q}_p)$ listed in the following table. Which type of representation it is can be decided by the value of μ_p.*

μ_p	rep.	$L_p(s,f)^{-1}$	$\varepsilon_p(s,f)$
$\notin \{0,2p\}$	IIa	$(1+\varepsilon_p(p+1)(p-\mu_p)p^{-3/2-s}+p^{-2s})(1+\varepsilon_p p^{-1/2-s})$	$\varepsilon_p p^{1/2-s}$
$2p$	Vb, c	$(1-\varepsilon_p p^{1/2-s})(1-p^{-1/2-s})(1+p^{-1/2-s})$	
0	VIc	$(1+\varepsilon_p p^{1/2-s})(1+\varepsilon_p p^{-1/2-s})$	

(56)

In this table ε_p is the eigenvalue of the Atkin–Lehner involution at p.

vi) *If we define spin L–factors for $p|N$ as in table (56), then the global L–function $L(s,f) = \prod_{p\le\infty} L_p(s,f)$ has meromorphic continuation to all of $\mathbb{C}$ and satisfies the functional equation*

$$L(s,f) = \varepsilon(s,f)L(1-s,f), \qquad \textit{with } \varepsilon(s,f) = (-1)^k\Big(\prod_{p|N}\varepsilon_p\Big)N^{1/2-s}. \tag{57}$$

Here $L_\infty(s,f)$ is defined as in (42), and the unramified spin Euler factors for $p\nmid N$ are the usual ones.

vii) *$L(s,f)$ has at most two simple poles at $s=3/2$ and $s=-1/2$. If $\mu_p \notin \{0,2p\}$ for some $p|N$, then $L(s,f)$ is holomorphic everywhere.*

Proof: The argument for i), ii) and iii) is similar to the one in the proof of Theorem 3.3.2. Instead of Corollary 3.1.6 we are using Corollary 3.1.7. The fact that f is a newform assures that hypothesis ii) of Corollary 3.1.7 is satisfied.

iv) and v) A look at Table 3 shows that only the representations listed in (56) are unitary and have newforms with respect to P_{02}. In each case the dimension of the space of P_{02}–invariant vectors is one-dimensional, proving iv). The data given in the last column of table (24) shows the relation between the eigenvalue μ_p and the type of representation.

vi) and vii) The L–factors for $p|N$ can be read off from Table 2, and the ε–factors are easily seen to be $\varepsilon_p p^{1/2-s}$ in each case. Now we can refer to our L–function theory and [PS2], Theorem 5.3, again. If $\mu_p \notin \{0,2p\}$, then $\pi_{f,p}$ is of type IIa, hence generic, and the holomorphy follows since π_f is not a Saito–Kurokawa representation. ∎

3.3.13 Remark. For simplicity assume $N = p$ is a prime and consider the following linear maps:

$$S_k(U_1(p))^{\text{new}} \underset{d_1}{\overset{d_{02}}{\rightleftarrows}} S_k(U_{02}(p))^{\text{new}} \tag{58}$$

We see from Table 3 that the occurrence of representations of type IIIa accounts for the kernel of $d_{02}(p)$, and the occurrence of representations of type VIc accounts for the kernel of $d_1(p)$. The composite maps $d_1 \circ d_{02}$ and $d_{02} \circ d_1$ are, up to normalization, the Hecke operators T_4 and T_5 we used in the newform theory for U_1 resp. U_{02}.

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