Explicit construction of spaces of Hilbert modular cusp forms using quaternionic theta series

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Abstract

After preliminary work [19], [21] in the 1950s, a paper by Martin Eichler appeared in [50] in 1972 in which the basis problem for modular forms was solved. Eichler describes explicitly how theta series attached to quaternion ideals can be used to construct a basis for a given space $S_k\big(\Gamma_0(N), 1\big)$ of cusp forms of even weight $k > 2$ and trivial character for the group $\Gamma_0(N)$.

In the same year an article by Hideo Shimizu ([65]) was published that can be viewed as a generalization of Eichler's work to the case of an arbitrary totally real number field F. Like Eichler, Shimizu is able to find a set of generators—although not necessarily a basis—for spaces of cusp forms. His result, however, is far less explicit than Eichler's. A reason is that in his proof Shimizu makes intensive use of the representation theory of the group $\mathrm{GL}_2(\mathbb{A}_F)$ over the adele ring $\mathbb{A}_F$, so that this group and its representations still play a crucial role in the description of the generating system that he constructs, and a connection to the classical theta series can hardly be seen.

Therefore, it is the aim of this thesis to prove an explicit version of Shimizu's Theorem. This means that we will show how Shimizu's "adelic theta series" can be transformed into classical theta series not unlike those in Eichler's work, which have the advantage of being explicit enough to be evaluated on a computer.

To this end, we begin by explaining the correspondence between adelic and classical modular forms, which is well-known in the case $F = \mathbb{Q}$, but only seldom treated in the case of general number fields. Afterwards we provide the basics of the theory of admissible representations of $\mathrm{GL}_2(\mathbb{A}_F)$ that are necessary for the understanding of Shimizu's result. With this background material we are then able to prove our Main Theorem 5.2.1, which is an explicit version of Shimizu's Theorem.

Finally, the results are used to construct theta series on a computer with the help of the computer algebra system MAGMA. We are thus able to find explicit sets of generators—in some cases even bases—for a selection of spaces of cusp forms.

Zusammenfassung

Nach Vorarbeiten [19], [21] aus den 1950er Jahren erschien 1972 in [50] ein Artikel Martin Eichlers über das Basisproblem für Modulformen, worin er explizit beschreibt, wie man aus Thetareihen zu gewissen Quaternionenidealen eine Basis eines vorgegebenen Raumes $S_k\big(\Gamma_0(N), 1)$ von Spitzenformen geraden Gewichts $k > 2$ und mit trivialem Charakter zur Gruppe $\Gamma_0(N)$ konstruiert.

Im selben Jahr veröffentlichte Hideo Shimizu einen Artikel ([65]), den man als Verallgemeinerung von Eichlers Arbeit auf den Fall eines beliebigen total reellen Zahlkörpers F auffassen kann. Auch ihm gelingt es, ein Erzeugendensystem — wenngleich nicht notwendigerweise eine Basis — für Räume von Spitzenformen anzugeben. Allerdings ist sein Resultat weit weniger explizit als Eichlers. Dies liegt in erster Linie daran, dass er in seinem Beweis intensiven Gebrauch von der Darstellungstheorie der Gruppe $\mathrm{GL}_2(\mathbb{A}_F)$ über dem Adelring $\mathbb{A}_F$ macht, so dass auch in der Beschreibung des von ihm angegebenen Erzeugendensystems der Gruppe $\mathrm{GL}_2(\mathbb{A}_F)$ und Darstellungen derselben eine tragende Rolle zukommt und zunächst keinerlei Ähnlichkeit mit klassischen Thetareihen erkennbar scheint.

Ziel der vorliegenden Arbeit ist es nun, eine explizite Version von Shimizus Theorem zu beweisen. Das heißt, es soll gezeigt werden, wie sich Shimizus ,,adelische Thetareihen“ in klassische Thetareihen übersetzen lassen, die — wie in Eichlers ursprünglicher Arbeit — so konkret angegeben werden können, dass dadurch eine Berechnung mit dem Computer möglich wird.

Zu diesem Zweck wird zunächst der Zusammenhang zwischen adelischen und klassischen Modulformen hergeleitet, der im Fall $F = \mathbb{Q}$ hinlänglich bekannt ist, im Fall allgemeinerer Zahlkörper in der Literatur aber kaum behandelt wird. Ferner werden diejenigen Aussagen über die Theorie der zulässigen Darstellungen von $\mathrm{GL}_2(\mathbb{A}_F)$ bereitgestellt, die für das Verständnis von Shimizus Resultat nötig sind. Mit diesen Hilfsmitteln kann dann der Hauptsatz 5.2.1, d. h. eine explizite Version von Shimizus Theorem, bewiesen werden.

Schließlich wird dieses Resultat benutzt, um mit Hilfe des Computeralgebrasystems MAGMA Thetareihen zu konstruieren und somit ganz konkret Erzeugendensysteme — und teilweise sogar Basen — einiger ausgewählter Räume von Spitzenformen zu berechnen.

Contents

Introduction

Let F be a totally real number field of degree $[F : \mathbb{Q}] = n > 1$ of arbitrary narrow class number h^+, and let $\mathbf{k} \in \mathbb{Z}^n$, $\mathbf{k} \geq 2$ be a (possibly non-parallel) weight vector.

Hilbert modular forms in the *classical* sense are holomorphic functions on the n-fold complex upper half plane with the well-known transformation rule

$$f|_{\mathbf{k}}\gamma = \chi(\gamma)f$$

for all γ in some congruence subgroup of $\mathrm{GL}_2^+(F)$ and some character χ. This definition is completely analogous to the elliptic situation, i. e. to the case of modular forms over $\mathbb{Q}$. But when delving deeper into the theory and trying to carry methods and proofs over from $\mathbb{Q}$ to the number field F, one will soon face more problems than expected. The theory of Hecke operators, for example, which is known to be a highly useful tool in the treatment of modular forms, does not generalize as easily to the number field situation, and it becomes particularly technical if F is of narrow class number $h^+ > 1$.

On the other hand, one may also define Hilbert modular forms in the *adelic* setting as certain automorphic forms on $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)$ (the precise definition is given in Definition 2.2.9). Although it looks rather unwieldy, the adelic approach turns out to be more promising for many purposes and theoretical results. For example, the adelic Hecke theory allows a uniform treatment of all number fields, so that a non-trivial narrow class group no longer causes complications. But above all, automorphic forms on $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)$ give naturally rise to admissible representations of $\mathrm{GL}_2(\mathbb{A}_F)$, which can then be approached by representation theoretic means as explained in Hervé Jacquet's and Robert Langlands's famous work [41].

A natural question to ask when examining a given space of modular forms for a certain weight, character and congruence subgroup is: What is the dimension of this space? Moreover, which forms constitute a basis?

In preliminary works [19], [21] and finally in [22],[23], Martin Eichler solved the so-called "basis problem" for $S_k\big(\Gamma_0(N), 1\big)$, the space of elliptic cusp forms for the group $\Gamma_0(N)$ of even weight k and trivial character $\chi = 1$. By proving that $S_k\big(\Gamma_0(N), 1\big)$ is the direct sum of spaces of newforms and translates thereof, each of which is spanned by theta series attached to a certain quaternion algebra, he gave an explicit description of a basis of the space $S_k\big(\Gamma_0(N), 1\big)$.

In the same year, a paper by Hideo Shimizu was published in which the author presents an alternative proof of Jacquet-Langlands [41, Thm. 14.4] and—as a byproduct—obtains generators for certain spaces $\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big)$ of adelic Hilbert modular cusp forms of weight $\mathbf{k} > 2$ and level $\mathfrak{n}$. His result can be viewed as a generalization of Eichler's theorem to the number field case in the sense that the space $\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big)$ decomposes into a direct sum of spaces $U(\mathfrak{m})$ of newforms and translates thereof, each of which is generated by functions that we will call, for lack of a better word, *adelic theta series*.

One drawback of Shimizu's result is that it does not present a basis for the space of cusp forms but only a set of generators, while the question of linear dependencies is not addressed. Moreover, the adelic theta series that Shimizu uses are defined in the language of admissible representations, so that the similarity to the classical theta series in Eichler's work is only visible to the expert. In particular, the definition of the adelic theta series (see Theorem 5.2.3) is so inexplicit that it is of hardly any use for computational purposes.

At the time that Shimizu's paper appeared it was far from practicable to calculate any Fourier coefficients of Hilbert modular forms on a computer. Therefore the motivation for finding a more explicit version of Shimizu's result was rather limited. But in the light of the ever increasing computer capacity and continually growing functionality of modern computer algebra systems it has become more and more attractive to find explicit examples of Hilbert modular forms.

Several groups of researchers are nowadays engaged with the construction of Hilbert modular forms, and consequently, considerable progress in the whole area has been made during the last few years. In [67], for example, the Fourier coefficients of Hilbert modular forms are constructed by examining Hecke operators on a certain space of functions on the double coset space $\mathfrak{O}^*_{\mathbb{A}}\backslash A^*_{\mathbb{A}}/A^*_F$, which parametrizes the $\mathfrak{O}$-left ideal classes of an Eichler order $\mathfrak{O}$ in a quaternion algbera A. The algorithm follows the ideas in [54]. In [16], [17], [18] it is further refined by finding an alternative description of the double coset space, so that the time-consuming computation of the $\mathfrak{O}$-left ideal classes can be avoided. The results in these papers are used for finding evidence for a general modularity conjecture between modular forms and elliptic curves (see also [14]).

But despite the significant progress that has been made, a satisfactory generalization of Eichler's solution of the "basis problem", i. e. an explicit version of Shimizu's Theorem in terms of *classical* theta series, has neither been formulated nor proven yet. It is the subject of the present thesis.

The aim of this thesis is threefold. Firstly, we found the literature dealing with the correspondence between classical and adelic modular forms somewhat insatisfactory. While over $\mathbb{Q}$ this correspondence has been explained in detail by several authors (among others [29, § 3], [8, Ch. 3], [49, § 1]), not much literature is available for the number field case ([28]). In particular, number fields of non-trivial narrow class number are rarely treated. One reason may be that the ideas behind this correspondence that work over $\mathbb{Q}$ can easily be adopted to the number field F, the only difference being that the statements and proofs become more cumbersome. To the expert, who is familiar with the elliptic case, it may be clear enough that most results are more or less the same over F as over $\mathbb{Q}$, and he may be content with the available literature. The non-expert, however, who is trying to find an introduction to Hilbert modular forms, may need some more explanation on this topic. We hope to fill a gap in the treatment of Hilbert modular forms by stating and proving the Correspondence Theorem 2.3.7 between classical and adelic modular forms in the general case of totally real number fields with arbitrary class number and for modular forms of arbitrary weight and character. This will be our aim in Chapters 1 and 2.

A second goal is to examine Shimizu's result and to make clear to what extend it is a generalization of Eichler's work. To this end, we will first explain the representation theoretic background and ideas behind Shimizu's work (Chapter 3). Then we have to delve into the theory of harmonic polynomials, which come into play when constructing theta series of weight $> \mathbf{2}$ (Chapter 4). Afterwards we will show how Shimizu's results can be carried over from the adelic side to the classical situation, and thus we derive our Main Theorem 5.2.1, which states that for certain spaces of classical Hilbert modular cusp forms, which we denote by $S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_l\mathfrak{d}, \mathfrak{n}), 1\big)$, a set of generators can be found that consists of quaternionic theta series of the form

$$\Theta(\mathbf{z}) = \sum_{\nu \in \mathfrak{o}_F^{*+}/\mathfrak{o}_F^2} \nu^{\frac{\mathbf{k}-\mathbf{2}}{2}} \sum_{a \in I} P(\overline{a}) \exp\big(2\pi i \mathrm{Tr}(\nu \mathrm{nrd}(a)\mathbf{z})\big)$$

for suitable, explicitly given quaternion ideals I and harmonic polynomials P. The beginning of the proof makes use of the Correspondence Theorem, while its main part is motivated by [80, § 2]. Although the outline of the proof is clear, its realization turns out to be rather cumbersome and technical, thus making up the biggest portion of Chapter 5.

The third aim of this thesis is to give explicit examples of quaternionic theta series in order to illustrate the theoretical results of the first chapters. All necessary computations were carried out with the help of the computer algebra system Magma [13]. Due to technical reasons, however, we could not use the newest version Magma V2.15. The version we did use, namely Magma V.2.13, was one of the first versions in which functions for dealing with quaternion algebras over number fields were included, and as often happens with new functionality in software packages, not all of it was working flawlessly. For this reason, we were not able to compute any arbitrary example we wanted because some of them resulted in unexpected error messages (see Section 6.1 for more details).

Most—though not all—explicit examples of Hilbert modular forms that can be found in literature are forms of parallel weight $\mathbf{k} = (2, \ldots, 2)$ over $\mathbb{Q}(\sqrt{5})$, or at least over a quadratic number field of narrow class number $h^+ = 1$. We are therefore glad that in spite of the problems with the software package we were able to compute examples for each of the following settings:

- over a number field of narrow class number $h^+ > 1$,
- over a number field of degree $n > 2$,
- of non-parallel weight $\mathbf{k} > 2$.

The MAGMA-group is working hard on improving the functionality for Hilbert modular forms, that has been included for the first time in the latest version V2.15. The algorithms they are using are based on [18] and very likely to be far more efficient than ours. But experience shows how easily mistakes can slip into even the most careful implementation, in particular when handling so complex a matter. Indeed, the long list of change logs concerning Hilbert modular forms that can be found on the MAGMA homepage demonstrates that the algorithms are not yet perfected and the development is still in progress. It is therefore always good to have independent results with which to compare the output, and our tables in Chapter 6 may serve as such. They are independent of the MAGMA-results in the sense that by implementing the theta series $\Theta(\mathbf{z})$ we use a completely different approach to the construction of the modular forms.

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Chapter 1

Hilbert modular forms in the classical setting

This introductory chapter provides all necessary definitions and elementary facts about classical Hilbert modular forms over a number field F of degree $[F : \mathbb{Q}] = n > 1$. Although we do not assume that the reader is familiar with this topic, some knowledge of elliptic modular forms might prove useful in order to recognize the analogies between the rational and the number field situation.

We will also introduce theta series attached to quaternion ideals (see Section 1.3). Not only are they a well-known example of classical Hilbert modular forms and thus worth mentioning, but we will see in later chapters that they form a set of generators of the whole space of Hilbert modular cusp forms and are therefore of special interest.

Before we are able to define these theta series we will need a few results concerning quaternion algebras and their ideal theory. Section 1.2 contains a brief summary hereof.

1.1 Introduction to Hilbert modular forms

Let us start by giving a brief introduction to classical Hilbert modular forms. The reader is referred to [27, Ch. I § 4], [28, Ch. I § 1.2 and 1.4], [71, Ch. I § 6], and [34, Ch. 2 § 3] for further details on the subject covered in this section.

Let F be a totally real algebraic number field of degree $n = [F : \mathbb{Q}] > 1$, and let $\mathfrak{o}_F$ be its ring of integers. Then $\mathfrak{o}_F^*$ is the group of units and we denote by $\mathfrak{o}_F^{*+}$ the subgroup of totally positive units. Archimedean places of F will usually be denoted by v, non-archimedean places by $\mathfrak{p}$. We also use the concise notation $v \mid \infty$ and $\mathfrak{p} < \infty$ when referring to archimedean and non-archimedean places, respectively. The localization of F, $\mathfrak{o}_F$ etc. with respect to $\mathfrak{p}$

will be denoted by $F_{\mathfrak{p}}, \mathfrak{o}_{\mathfrak{p}}$ etc.

Denote by $\mathrm{GL}_2^+(F)$ the group of (2×2)-matrices over F whose determinant is totally positive. A *congruence subgroup* of $\mathrm{GL}_2^+(F)$ is a subgroup of $\mathrm{GL}_2^+(F)$ containing the kernel of the canonical reduction map

$$\mathrm{SL}_2(\mathfrak{o}_F) \to \mathrm{SL}_2(\mathfrak{o}_F/\mathfrak{n})$$

for some integral ideal $\mathfrak{n}$ of F. Let Γ be such a congruence subgroup and χ a character on Γ. The complex upper half space will be denoted by $\mathbb{H}$. In $\mathbb{H}^n$, we fix the vector $\mathbf{i} := (i, \dots, i)$ with $i^2 = -1$, and denote by $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ a vector of integers.

These notations will be fixed throughout this thesis.

Definition 1.1.1 ($\cdot|_{\mathbf{k}}$-operator, factor of automorphy). For $z \in \mathbb{H}^n$, $a, b, c, d \in \mathbb{R}^n$, $g = (g_i)_{i=1}^n = \left(\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\right)_{i=1}^n \in \mathrm{GL}_2^+(\mathbb{R})^n$ and a function $f : \mathbb{H}^n \to \mathbb{C}$, we define

- an operation of $\mathrm{GL}_2^+(\mathbb{R})^n$ on $\mathbb{H}^n$ by

$$gz := \left(\frac{a_i z_i + b_i}{c_i z_i + d_i}\right)_{i=1}^n \in \mathbb{H}^n \ ,$$

- the abbreviations

$$\mathrm{Tr}(az) := \sum_{i=1}^n a_i z_i \ , \qquad (cz+d)^{\mathbf{k}} := \prod_{i=1}^n (c_i z_i + d_i)^{k_i} \ , \qquad \det g^{\mathbf{k}} := \prod_{i=1}^n (\det g_i)^{k_i} \ ,$$

- the *factor of automorphy* by

$$j(g, z) := (cz + d) \det g^{-1/2} = \prod_{i=1}^n (c_i z_i + d_i)(\det g_i^{-1/2}) \ ,$$

- and the $\cdot|_{\mathbf{k}}$-operator by

$$(f|_{\mathbf{k}} g)(z) := j(g, z)^{-\mathbf{k}} f(gz) = \det g^{\mathbf{k}/2} (cz + d)^{-\mathbf{k}} f(gz) \ .$$

Definition 1.1.2. Let $\tau_1, \dots, \tau_n : F \hookrightarrow \mathbb{R}$ be the n embeddings of F into $\mathbb{R}$, and extend each τ_i to an embedding $\hat{\tau}_i : \mathrm{GL}_2^+(F) \hookrightarrow \mathrm{GL}_2^+(\mathbb{R})$ by applying τ_i to each matrix entry.

- For $c \in F$, we denote by $c^{(1)}, \dots, c^{(n)}$ the images of c under the embeddings $\tau_1, \dots, \tau_n$. Similarly, for $g \in \mathrm{GL}_2^+(F)$, we write $g^{(1)}, \dots, g^{(n)}$ for the images of g under $\hat{\tau}_1, \dots, \hat{\tau}_n$.
- By identifying $c \in F$ with the vector $(c^{(1)}, \dots, c^{(n)}) \in \mathbb{R}^n$ and $g \in \mathrm{GL}_2^+(F)$ with $(g^{(1)}, \dots, g^{(n)}) \in \mathrm{GL}_2^+(\mathbb{R})^n$, we may use all of the notation defined in Definition 1.1.1 also for elements in the number field F.

The properties of the factor of automorphy $j(\cdot\,,\,\cdot)$ and the $\cdot|_{\mathbf{k}}$-operator which we will most frequently use are stated in the following lemma.

Lemma 1.1.3. *For all $g, h \in \mathrm{GL}_2(F)$, $z \in \mathbb{H}^n$ and $f : \mathbb{H}^n \to \mathbb{C}$,*

$$j(gh, z) = j(g, hz)j(h, z) \qquad \text{and} \qquad f|_{\mathbf{k}}(gh) = \big(f|_{\mathbf{k}}g\big)|_{\mathbf{k}}h\ .$$

Proof. As in the case $F = \mathbb{Q}$, these identities can be verified by a straightforward calculation. □

Definition 1.1.4 (Hilbert modular form). A *Hilbert modular form of weight* $\mathbf{k}$ *for the group* Γ *with character* χ is a holomorphic function $f : \mathbb{H}^n \to \mathbb{C}$ that satisfies the transformation rule

$$f|_{\mathbf{k}}\gamma = \chi(\gamma)f \qquad \text{for all} \qquad \gamma \in \Gamma\ .$$

The space of all Hilbert modular forms of weight $\mathbf{k}$ for the group Γ with character χ will be denoted by $M_{\mathbf{k}}(\Gamma, \chi)$. If $\chi = 1$ we will sometimes write $M_{\mathbf{k}}(\Gamma)$ instead of $M_{\mathbf{k}}(\Gamma, 1)$.

Proposition 1.1.5 (Fourier expansion). *Let $f \in M_{\mathbf{k}}(\Gamma)$. The set*

$$\mathfrak{a} = \left\{ a \in F \;\middle|\; \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in \Gamma \right\}$$

is an ideal in F, and we denote by $\mathfrak{a}^\#$ its dual. Then $f(z)$ has a Fourier expansion *of the form*

$$f(z) = \sum_{\xi \in \mathfrak{a}^\#} a_\xi e^{2\pi i \mathrm{Tr}(\xi z)} \qquad \text{where} \qquad a_\xi = 0 \text{ unless all } \xi^{(i)} \geq 0\ .$$

The Fourier coefficients a_ξ *are given by the formula*

$$a_\xi = \frac{1}{\mathrm{vol}(\mathbb{R}^n/\mathfrak{a})} \int\limits_{\mathbb{R}^n/\mathfrak{a}} f(z) e^{-2\pi i \mathrm{Tr}(\xi z)}\, dx$$

where $z = (x_v + iy_v)_{v=1}^n$ and $dx = dx_1 \cdots dx_n$.

Proof. See [27, Ch. I, Lemma 4.1] or [28, section 1.2]. □

Remark 1.1.6.
(i) Clearly, either all or none of the conjugates $\xi^{(i)}$ of ξ are equal to zero. So the Fourier expansion of f has a constant term a_0, which may be zero, and all other coefficients a_ξ belong to totally positive elements $\xi \in \mathfrak{a}^\#$.

(ii) Note that our general assumption $[F : \mathbb{Q}] > 1$ is crucial for the validity of Proposition 1.1.5. In the case $F = \mathbb{Q}$, we would need the additional requirement that f be "regular at the cusps". This means, by definition, that there are no Fourier coefficients a_ξ for $\xi < 0$. If $[F : \mathbb{Q}] > 1$, however, the so-called *Koecher principle* guarantees that this regularity requirement is automatically satisfied, see for example [34, Theorem 3.3] or Koecher's original work [48].

□

Definition 1.1.7 (Cusp form). A Hilbert modular form is called *cusp form* if for all $\gamma \in \mathrm{GL}_2^+(F)$ the constant term in the Fourier expansion of $f|_{\mathbf{k}}\gamma$ vanishes. The space of all cusp forms in $M_{\mathbf{k}}(\Gamma, \chi)$ will be denoted by $S_{\mathbf{k}}(\Gamma, \chi)$.

Proposition 1.1.8.

(i) Hilbert modular forms of weight $\mathbf{0}$ *with trivial character are constant, i. e.*

$$M_{\mathbf{0}}(\Gamma) = \mathbb{C} \qquad \textit{and} \qquad S_{\mathbf{0}}(\Gamma) = \{0\} \ .$$

(ii) Hilbert modular forms with trivial character that are no cusp forms exist only for parallel *weight, i. e. for $k_1 = \ldots = k_n$. In other words,*

$$\mathbf{k} \textit{ not parallel} \implies M_{\mathbf{k}}(\Gamma) = S_{\mathbf{k}}(\Gamma) \ .$$

(iii) A necessary condition for the existence of a modular form $f \neq 0$ in the space $M_{\mathbf{k}}(\Gamma, \chi)$ is

$$\chi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \mathrm{sgn}(a)^{\mathbf{k}} \qquad \textit{for all} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \Gamma \ ,$$

where $\mathrm{sgn}(a) = (\mathrm{sgn}(a^{(1)}), \ldots, \mathrm{sgn}(a^{(n)}))$.

Proof. Part *(i)* and *(ii)* can be found in [27, I § 4]. The third assertion can be shown as in the case of elliptic modular forms: Let $0 \neq f \in M_{\mathbf{k}}(\Gamma, \chi)$ and let $\gamma \in \Gamma$ be a scalar matrix with diagonal (a, a), then

$$\chi(\gamma) f = f|_{\mathbf{k}}\gamma = (a^2)^{\mathbf{k}/2} a^{-\mathbf{k}} f = \mathrm{sgn}(a)^{\mathbf{k}} f \ .$$

□

1.2 Quaternion algebras and their ideals

Our aim in this section and the next is to present generalized theta series over number fields. The construction mainly follows [24] and makes use of quaternion algebras. For this reason, we begin with a short summary of the basic definitions and statements concerning ideals and orders in quaternion algebras. For more details, the reader is referred to [73].

Let A be a definite quaternion algebra over the totally real number field F with basis vectors $1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ satisfying

$$\mathfrak{i}^2 = \alpha, \quad \mathfrak{j}^2 = \beta, \quad \mathfrak{ij} = -\mathfrak{ji} = \mathfrak{k} \qquad \text{for some fixed } \alpha, \beta \in F^* \ .$$

Being definite means that A is ramified at all archimedean places of F. The non-archimedean places at which A is ramified are collected in the *discriminant* D_1 of A defined by

$$D_1 := \prod_{\substack{\mathfrak{p}<\infty \\ A \text{ is ramified at } \mathfrak{p}}} \mathfrak{p} \ .$$

Definition 1.2.1 (Quaternion ideals and orders).

(i) An *ideal of* A is a complete lattice in A, i. e. a finitely generated $\mathfrak{o}_F$-module I in A, such that there exists a basis $x_1, \ldots, x_4$ of A with

$$I \subseteq \mathfrak{o}_F x_1 + \ldots + \mathfrak{o}_F x_4 \qquad \text{and} \qquad FI = A \ .$$

(ii) An *order of* A is an ideal of A that is moreover a unitary ring. Equivalent is the definition: An *order of* A is a unitary ring $\mathfrak{O}$ consisting of integral elements in A such that $F\mathfrak{O} = A$. An order is called *maximal* if it is not properly contained in any larger order of A. An order is called an *Eichler order* if it is the intersection of two (not necessarily distinct) maximal orders.

(iii) For an ideal I, we call

$$\mathfrak{o}_l(I) := \{a \in A \mid aI \subseteq I\} \qquad \text{and} \qquad \mathfrak{o}_r(I) := \{a \in A \mid Ia \subseteq I\}$$

the *left order* and *right order* of I, respectively, and say that I is an $\mathfrak{o}_l(I)$*-left ideal* and an $\mathfrak{o}_r(I)$*-right ideal*. Both $\mathfrak{o}_l(I)$ and $\mathfrak{o}_r(I)$ are orders in A. We say that an ideal is *integral* if it is contained in its left (or equivalently in its right) order.

(iv) If $\mathfrak{O}$ is an order then two $\mathfrak{O}$-left (resp. right) ideals I, J are said to be in the same *ideal class* if there exists an $a \in A^*$ such that $I = Ja$ (resp. $I = aJ$).

Lemma and Definition 1.2.2. *If $\mathfrak{O}$ is an Eichler order in A then its localizations are of the form*

$$\begin{cases} \mathfrak{O}_\mathfrak{p} = \{a \in A_\mathfrak{p} \mid \operatorname{nrd}(a) \in \mathfrak{o}_\mathfrak{p}\} & \textit{if } \mathfrak{p} \mid D_1 \ , \\ \mathfrak{O}_\mathfrak{p} = a \begin{pmatrix} \mathfrak{o}_\mathfrak{p} & \mathfrak{o}_\mathfrak{p} \\ \mathfrak{p}^{k_\mathfrak{p}} & \mathfrak{o}_\mathfrak{p} \end{pmatrix} a^{-1} \quad \textit{for some } a \in A_\mathfrak{p}^* & \textit{else} \ . \end{cases}$$

The exponents $k_\mathfrak{p}$ are non-negative integers almost all of which are 0*. We put*

$$D_2 := \prod_{\mathfrak{p} \nmid D_1} \mathfrak{p}^{k_\mathfrak{p}} \ ,$$

and call the ideal $D_1 D_2$, or the pair (D_1, D_2), the level *of $\mathfrak{O}$. The order $\mathfrak{O}$ is maximal if and only if $D_2 = 1$.*

On the quaternion algebra A, consider the bilinear form $\beta(\cdot,\cdot)$ and the corresponding quadratic form $q(\cdot)$ given by

$$\beta(x,y) := \operatorname{tr}(x\overline{y}) \qquad \text{and} \qquad q(x) := \beta(x,x) = 2\operatorname{nrd}(x) \qquad \text{for all } x,y \in A\ ,$$

where $\operatorname{tr}(x) = x + \overline{x}$ and $\operatorname{nrd}(x) = x\overline{x}$ denote the (reduced) trace and norm, respectively. The *discriminant* of a quaternion ideal I with respect to $\beta(\cdot,\cdot)$ is the ideal $\operatorname{disc}(I) \subseteq F$ generated by

$$\left\{ \det\left(\beta(x_i,x_j)\right)_{i,j=1}^{4} \mid x_1,\dots,x_4 \in I \right\}\ .$$

The *dual* of I with respect to $\beta(\cdot,\cdot)$ is defined by

$$I^{\#} := \left\{ x \in A \mid \beta(x,I) \subseteq \mathfrak{o}_F \right\}\ .$$

Definition 1.2.3 (Norm, level). Let I be an ideal in A. Then the ideal in F that is generated by all $\operatorname{nrd}(x)$ with $x \in I$ is called the *(reduced) norm* of I and will be denoted by $\operatorname{nrd}(I)$. The *level* of I is

$$\mathfrak{N}(I) := \operatorname{nrd}(I)^{-1}\operatorname{nrd}(I^{\#})^{-1}\ .$$

Remark 1.2.4. Note that our definition of the level of I is equivalent to Eichler's definition in [24, § 2, Lemma 2] although we use a slightly different notion of the dual of I. □

Lemma 1.2.5. *Let $\mathfrak{O}$ be an Eichler order of level (D_1, D_2), and let I be an $\mathfrak{O}$-left ideal.*

(i) The number H of $\mathfrak{O}$-left (resp. right) ideal classes is finite and depends only on the level (D_1, D_2).

(ii) The discriminant of $\mathfrak{O}$ is $\operatorname{disc}(\mathfrak{O}) = D_1^2 D_2^2$.

(iii) The localization $I_\mathfrak{p}$ is a principal $\mathfrak{O}_\mathfrak{p}$-left ideal for every non-archimedean place $\mathfrak{p}$.

(iv) The dual of I has the discriminant $\operatorname{disc}(I^{\#}) = \operatorname{disc}(I)^{-1}$.

(v) The discriminant of I is $\operatorname{disc}(I) = \operatorname{nrd}(I)^4 \cdot \operatorname{disc}(\mathfrak{O})$. In particular, $\operatorname{disc}(I)$ is the square of an ideal of F.

(vi) The level of I is $\mathfrak{N}(I) = D_1 D_2$.

Proof. (i) See [73, I, § 4, Lemme 4.9].

(ii) For maximal orders, i. e. if $D_2 = 1$, see [56, Corollary (25.10)] or carry the proof given in [54, Prop. 1.1] over to the number field case. For an Eichler order $\mathfrak{O}$ contained in a maximal order $\mathfrak{M}$, use the fact that $\operatorname{disc}(\mathfrak{O}) = [\mathfrak{M} : \mathfrak{O}]^2 \operatorname{disc}(\mathfrak{M})$ and $[\mathfrak{M}_\mathfrak{p} : \mathfrak{O}_\mathfrak{p}] = (D_2)_\mathfrak{p}$ locally.

(iii) This is an exercise in [53, § 3], the proof is carried out in [68, Prop. 2.5.2].

(iv) See [52, § 82F].

(v) By *(iii)*, $I_\mathfrak{p} = \mathfrak{O}_\mathfrak{p} a_\mathfrak{p}$ for some $a_\mathfrak{p} \in A_\mathfrak{p}^*$. We can find some ideals $\mathfrak{a}_{i,\mathfrak{p}}$ in $F_\mathfrak{p}$ and suitable $u_i \in A_\mathfrak{p}^*$ such that $\mathfrak{O}_\mathfrak{p} = \mathfrak{a}_{1,\mathfrak{p}} u_1 + \ldots + \mathfrak{a}_{4,\mathfrak{p}} u_4$ (see [52, Thm. 81:3]). Then the ideal $I_\mathfrak{p}$ is $I_\mathfrak{p} = \mathfrak{a}_{1,\mathfrak{p}} u_1 a_\mathfrak{p} + \ldots + \mathfrak{a}_{4,\mathfrak{p}} u_4 a_\mathfrak{p}$. Because of

$$\beta(u_i a_\mathfrak{p},\, u_j a_\mathfrak{p}) = \mathrm{tr}(u_i a_\mathfrak{p} \overline{u_j a_\mathfrak{p}}) = \mathrm{tr}(u_i a_\mathfrak{p} \overline{a}_\mathfrak{p} \overline{u}_j) = \mathrm{nrd}(a_\mathfrak{p}) \mathrm{tr}(u_i \overline{u}_j) = \mathrm{nrd}(a_\mathfrak{p}) \beta(u_i, u_j) \ ,$$

the discriminant is $\mathrm{disc}(I_\mathfrak{p}) = \mathrm{nrd}(a_\mathfrak{p})^4 \cdot \mathrm{disc}(\mathfrak{O}_\mathfrak{p}) = \mathrm{nrd}(I_\mathfrak{p})^4 \cdot \mathrm{disc}(\mathfrak{O}_\mathfrak{p})$.

(vi) From what we have seen in the first parts of this lemma it follows that

$$\mathfrak{N}(I)^4 = \mathrm{nrd}(I)^{-4} \mathrm{nrd}(I^\#)^{-4} = \mathrm{disc}(I)^{-1} \mathrm{disc}(I^\#)^{-1} \mathrm{disc}(\mathfrak{O})^2 = D_1^4 D_2^4 \ .$$

□

Remark 1.2.6. Any Eichler order $\mathfrak{O}$ may also be viewed as the $\mathfrak{O}$-ideal $\mathfrak{O} \cdot 1$. By Definition 1.2.2 and Definition 1.2.3, we have two ways of defining the level of $\mathfrak{O}$. Fortunately, there will be no confusion since the previous lemma assures that the two definitions coincide. □

Lemma 1.2.7. *Let $\mathfrak{O}$ be an Eichler order of A. Then*

$$\mathrm{nrd}(A^*) = F^+ \ , \qquad \mathrm{nrd}(\mathfrak{O}_\mathfrak{p}^*) = \mathfrak{o}_\mathfrak{p}^* \ , \qquad \mathrm{nrd}(\mathfrak{O}_\mathbb{A}^*) = \prod_{v \mid \infty} \mathbb{R}_{>0} \times \prod_{\mathfrak{p} < \infty} \mathfrak{o}_\mathfrak{p}^* \ .$$

Proof. Use [52, 42:11, 63:19 and Thm. 66:3] for the first assertion and use Lemma 1.2.2 for the second and third. □

We fix an Eichler order $\mathfrak{O}$ of level (D_1, D_2), and for the $\mathfrak{O}$-right ideal classes in A, we fix a complete set of representatives

$$\{I_1, \ldots, I_H\} \qquad \text{with left orders } \mathfrak{o}_l(I_j) =: \mathfrak{O}_j \ .$$

Then the left orders $\mathfrak{O}_j$ are again Eichler orders of level (D_1, D_2), but note that they are not necessarily distinct.

Locally, every I_j is a principal ideal generated by some $y_{j,\mathfrak{p}}$ as we have seen in Lemma 1.2.5. The local-global-correspondence (see [73, III, § 5A, Prop. 5.1]) assures that we can choose $y_{j,v} = 1$ for all $v \mid \infty$ and $y_{j,\mathfrak{p}} = 1$ for almost every $\mathfrak{p} < \infty$. By collecting the local data we can therefore identify I_j with $y_j \mathfrak{O}_\mathbb{A}^*$ where $y_j = (y_{j,\mathfrak{p}})_\mathfrak{p} \in A_\mathbb{A}^*$. From this, we get a disjoint decomposition

$$A_\mathbb{A}^* = \coprod_{j=1}^{H} A^* y_j \mathfrak{O}_\mathbb{A}^* \ , \tag{1.1}$$

of which we will make extensive use in our calculations in Chapters 4 and 5.

1.3 An example: Theta series attached to quaternion ideals

Over $\mathbb{Q}$, theta series attached to a $2m$-dimensional quadratic space with a positive definite quadratic form $q(x) = x^t U^t U x$, and to a homogeneous harmonic polynomial P are functions of the form

$$\theta(z) = \sum_{x \in \mathbb{Z}^{2m}} P(Ux) \exp(\pi i q(x) z) \qquad \text{for } z \in \mathbb{H} \; .$$

They are known to be fundamental examples of elliptic modular forms of weight $\deg(P)+m$ for some character and some group that depend on the level and the discriminant of the quadratic form (see [60] or [51, Ch. VI, Thm. 20]).

The definition of theta series can be adapted to the case of a totally real number field F and will lead to a method for constructing Hilbert modular forms. The complications that arise are mostly due to the fact that we need to deal with all embeddings $F \hookrightarrow \mathbb{R}$ at the same time, but these difficulties are mainly of technical nature. For further details see [24, Chapter I].

As in the previous section let A be a definite quaternion algebra equipped with the quadratic form $q(x) = 2\mathrm{nrd}(x)$. With respect to the basis $\{e_1, \dots, e_4\} := \{1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k}\}$, the quadratic form $q(x)$ has the matrix

$$B := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2\alpha & 0 & 0 \\ 0 & 0 & -2\beta & 0 \\ 0 & 0 & 0 & 2\alpha\beta \end{pmatrix} \in M_4(F) \qquad \text{where } \mathfrak{i}^2 = \alpha, \ \mathfrak{j}^2 = \beta \text{ as before.}$$

Fix a $\mathbb{Q}$-basis $\eta_1, \dots, \eta_n$ for F. Then

$$\mathcal{B} := \{b_1, \dots, b_{4n}\} := \{\eta_1 e_1, \ \dots, \ \eta_1 e_4, \quad \dots, \quad \eta_n e_1, \ \dots, \ \eta_n e_4\}$$

is a $\mathbb{Q}$-basis of A, and we will always assume that the basis vectors are ordered in this fashion. For $x \in A$, we will momentarily write $x_{\mathcal{B}} \in \mathbb{Q}^{4n}$ for the coordinate vector of x with respect to $\mathcal{B}$.

Let I be a quaternion ideal. As $\mathfrak{o}_F$ is only a Dedekind domain but not necessarily a principal ideal ring, the ideal I might not possess an $\mathfrak{o}_F$-basis, but we can always find a $\mathbb{Z}$-basis $\mathcal{C} = \{c_1, \dots, c_{4n}\}$ for I. Let T be the transformation matrix between the bases $\mathcal{C}$ and $\mathcal{B}$, i. e.

$$T = (t_{ij}) \in \mathrm{GL}_{4n}(\mathbb{Q}) \qquad \text{such that} \qquad c_j = \sum_{i=1}^{4n} t_{ij} b_i \; .$$

Clearly, $x_{\mathcal{B}} = T x_{\mathcal{C}}$.

As mentioned at the beginning of this section, the main ingredients of theta series defined over $\mathbb{Q}$ are terms of the form $\exp(\pi i q(x) z) = \exp\big(\pi i (x^t U^t U x) z\big)$ for $z \in \mathbb{H}$ and x running through a suitable lattice, together with a factor $P(Ux)$ where P is a homogeneous harmonic polynomial. In the generalization to number fields, the expression $q(x)z$ has to be replaced by

$$\mathrm{Tr}\big(q(x)z\big) := \sum_{i=1}^{n} \big(q(x)\big)^{(i)} z_i \qquad \text{for } z \in \mathbb{H}^n \text{ and } x \in I \ ,$$

the superscript (i) indicating the i-th embedding into the real numbers, as usual. We also want to adapt the polynomial expression $P(Ux)$ to the number field case. To this end, we want to find a matrix, which we call $U(z)$, such that $\mathrm{Tr}\big(q(x)z\big) = x_{\mathcal{C}}^t U(z)^t U(z) x_{\mathcal{C}}$.

In order to deal conveniently with all embeddings $F \hookrightarrow \mathbb{R}$ at the same time, we put

$$G := \begin{pmatrix} \eta_1^{(1)} I_4 & \dots & \eta_n^{(1)} I_4 \\ \vdots & & \vdots \\ \eta_1^{(n)} I_4 & \dots & \eta_n^{(n)} I_4 \end{pmatrix} \in M_{4n}(\mathbb{R}) \ , \qquad I_4 \text{ the } (4\times 4)\text{-identity matrix} \ ,$$

and

$$M := \begin{pmatrix} z_1 B^{(1)} & & 0 \\ & \ddots & \\ 0 & & z_n B^{(n)} \end{pmatrix} \in M_{4n}(\mathbb{C}) \ .$$

Then, for $x = x_1 e_1 + \ldots + x_4 e_4 \in A$ with $x_i \in F$,

$$(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}) = x_{\mathcal{B}}^t G^t = x_{\mathcal{C}}^t T^t G^t \qquad \text{where } \mathbf{x}^{(i)} = (x_1^{(i)}, \dots, x_4^{(i)}) \ .$$

On the other hand

$$\big(q(x)\big)^{(i)} = \mathbf{x}^{(i)} B^{(i)} (\mathbf{x}^{(i)})^t \ ,$$

so that

$$\begin{aligned} \mathrm{Tr}\big(q(x)z\big) &= \sum_{i=1}^{n} \big(q(x)\big)^{(i)} z_i = \sum_{i=1}^{n} \mathbf{x}^{(i)} z_i B^{(i)} \mathbf{x}^{(i)t} = (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}) M \begin{pmatrix} \mathbf{x}^{(1)t} \\ \vdots \\ \mathbf{x}^{(n)t} \end{pmatrix} \\ &= x_{\mathcal{C}}^t \big(T^t G^t M G T\big) x_{\mathcal{C}} \ . \end{aligned}$$

Since A is positive definite, the elements α and β must be totally negative elements, so we may define

$$S^{(i)} := \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{-2\alpha^{(i)}} & 0 & 0 \\ 0 & 0 & \sqrt{-2\beta^{(i)}} & 0 \\ 0 & 0 & 0 & \sqrt{2\alpha^{(i)}\beta^{(i)}} \end{pmatrix} \in M_4(\mathbb{R}) \qquad \text{for all } i = 1, \dots, n \ .$$

Then the matrix

$$U(z) := \begin{pmatrix} \sqrt{-iz_1}\, S^{(1)} & & 0 \\ & \ddots & \\ 0 & & \sqrt{-iz_n}\, S^{(n)} \end{pmatrix} GT \in M_{4n}(\mathbb{C})$$

satisfies

$$\mathrm{Tr}\big(q(x)z\big) = i \cdot x_{\mathcal{C}}^t \big(U(z)^t U(z)\big) x_{\mathcal{C}} \ .$$

Now we can adapt the harmonic polynomial $P(Ux)$ in the following way to the number field case: For $i = 1, \ldots, n$, let P_i be homogeneous polynomials of degree $l_i \in \mathbb{N}_0$ in 4 variables. Assume further that they are harmonic, i. e. $\Delta P_i = 0$, where Δ is the Laplace operator. In section 4.1, we will discuss harmonic polynomials in greater detail. For the time being, it is sufficient to note that

$$P\big(X_1^{(1)}, \ldots, X_4^{(1)}, \quad \ldots, \quad X_1^{(n)}, \ldots, X_4^{(n)}\big) := \prod_{i=1}^{n} P_i\big(X_1^{(i)}, \ldots, X_4^{(i)}\big)$$

is again a homogeneous harmonic polynomial of degree $\deg P = l_1 + \ldots + l_n$. So, the natural generalization of the polynomial expression that we are looking for is

$$Q(x) := (-\mathbf{i}z)^{-\mathbf{l}/2} P(U(z)x_{\mathcal{C}}) \qquad \text{where } \mathbf{l} = (l_1, \ldots, l_n) \ .$$

On closer inspection, we see that Q does not depend on z. We may therefore take $z = \mathbf{i}$, put

$$S := \begin{pmatrix} S^{(1)} & & 0 \\ & \ddots & \\ 0 & & S^{(n)} \end{pmatrix} \in M_{4n}(\mathbb{R})$$

and recall that by construction $U(\mathbf{i})x_{\mathcal{C}} = SGTx_{\mathcal{C}} = SGx_{\mathcal{B}} = S(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)})^t$. Thus, we obtain a simpler expression for Q, namely

$$Q(x) = P\big(S(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)})^t\big) \tag{1.2}$$

where, as before, $\mathbf{x}^{(i)} = (x_1^{(i)}, \ldots, x_4^{(i)})$ for $x = x_1 1 + x_2\mathfrak{i} + x_3\mathfrak{j} + x_4\mathfrak{k} \in A$.

After these lenghty preparations, we are now ready to state the fundamental theorem. Let

$$\Gamma_0(\mathfrak{c}, \mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathfrak{o}_F & \mathfrak{c}^{-1} \\ \mathfrak{n}\mathfrak{c} & \mathfrak{o}_F \end{pmatrix} \;\Big|\; ad - bc \in \mathfrak{o}_F^{*+} \right\}$$

and denote by $\Gamma_0^1(\mathfrak{c}, \mathfrak{n})$ the normal subgroup of $\Gamma_0(\mathfrak{c}, \mathfrak{n})$ consisting of all elements of determinant 1.

Theorem 1.3.1. *Let I be an ideal in A whose left and right order are Eichler orders of level (D_1, D_2). For $i = 1, \ldots, n$, let P_i be a homogeneous harmonic polynomial of degree l_i in 4 variables and put $\mathbf{k} := (l_1 + 2, \ldots, l_n + 2)$. Let Q be defined as in (1.2), and denote by $\mathfrak{d}$ the different of F. Then the theta series*

$$\vartheta_I(z; Q) := \sum_{x \in I} Q(x) \exp\big(2\pi i \mathrm{Tr}(\mathrm{nrd}(x)z)\big)$$

is a Hilbert modular form for the group $\Gamma_0^1\big(\mathrm{nrd}(I)\mathfrak{d},\, D_1D_2\big)$ of weight $\mathbf{k}$ with character 1.

Proof. In [24, § 4, Thm. 1], the series ϑ_I is proven to be a Hilbert modular form of weight $\mathbf{k}$ for the group $\Gamma_0^1(\mathrm{nrd}(I)\mathfrak{d},\, \mathfrak{N}(I))$, which, by Lemma 1.2.5, is equal to $\Gamma_0^1(\mathrm{nrd}(I)\mathfrak{d},\, D_1D_2)$. The triviality of the character is due to the fact that the quadratic space $(A, 2\mathrm{nrd})$ has square discriminant. □

Remark 1.3.2. In Proposition 1.1.8 we saw that a modular form in $M_{\mathbf{k}}(\Gamma, \chi)$ is trivial unless its character satisfies $\chi\left(\left(\begin{smallmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{smallmatrix}\right)\right) = \mathrm{sgn}(\varepsilon)^{\mathbf{k}}$ for all $\left(\begin{smallmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{smallmatrix}\right) \in \Gamma$. In the situation of the previous theorem, this condition reads $\chi(\pm I_2) = (\pm 1)^{\mathbf{k}} = (\pm 1)^{k_1+\dots+k_n}$ (where I_2 denotes the (2×2)-identity matrix) because $\pm I_2$ are the only diagonal matrices in $\Gamma = \Gamma_0^1(\mathrm{nrd}(I)\mathfrak{d},\, D_1D_2)$. In the theorem, however, we construct a modular form of character 1. Consequently, $(-1)^{\mathbf{k}} = 1$ or $\vartheta_I(z; Q) = 0$. That this is indeed true, can also be seen by direct calculation: Assume $(-1)^{\mathbf{k}} \neq 1$, which implies that $\deg(Q) = \sum \deg(P_i)$ is odd. Then

$$\begin{aligned}\vartheta_I(z;Q) = \vartheta_{(-I)}(z;Q) &= \sum_{x\in I} Q(-x)\exp\bigl(2\pi i \mathrm{Tr}(\mathrm{nrd}(-x)z)\bigr)\\ &= (-1)^{\deg(Q)} \sum_{x\in I} Q(x)\exp\bigl(2\pi i \mathrm{Tr}(\mathrm{nrd}(x)z)\bigr) = -\vartheta_I(z;Q)\ .\end{aligned}$$

Hence, $\vartheta_I(\,\cdot\,; Q)$ vanishes. The bottom line is that $\vartheta_I(\,\cdot\,; Q) = 0$ unless the degree of the homogeneous harmonic polynomial Q is even. □

We conclude this chapter with a slight generalization of Theorem 1.3.1 that allows us to construct Hilbert modular forms for the full group $\Gamma_0\bigl(\mathrm{nrd}(I)\mathfrak{d},\, D_1D_2\bigr)$, so that we are not forced to restrict our attention to matrices of determinant 1.

Corollary 1.3.3. *Suppose that*

$$\mathrm{sgn}(\varepsilon)^{\mathbf{k}} = \prod_{i=1}^{n} \mathrm{sgn}(\varepsilon^{(i)})^{k_i} = 1 \qquad \text{for all} \qquad \varepsilon \in \mathfrak{o}_F^* \ .$$

In the situation of the last theorem, define another theta series by

$$\Theta_I(z;Q) := \sum_{\delta\in\mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}} \delta^{\frac{\mathbf{k}-2}{2}}\vartheta_I(\delta z;Q) = \sum_{\delta\in\mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}} \delta^{\frac{\mathbf{k}-2}{2}} \sum_{x\in I} Q(x)\exp\bigl(2\pi i \mathrm{Tr}(\delta \mathrm{nrd}(x)z)\bigr)\ ,$$

where δ runs through a set of representatives of totally positive units modulo squares. Then this theta series $\Theta_I(z;Q)$ is a Hilbert modular form for the group $\Gamma_0(\mathrm{nrd}(I)\mathfrak{d},\, D_1D_2)$ of weight $\mathbf{k}$ with character 1.

Remark 1.3.4. The factor group $\mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}$ of totally positive units in F modulo squares is in fact a finite group. Since F is totally real, its order is

$$[\mathfrak{o}_F^{*+} : \mathfrak{o}_F^{*2}] = \frac{[\mathfrak{o}_F^{*} : \mathfrak{o}_F^{*2}]}{[\mathfrak{o}_F^{*} : \mathfrak{o}_F^{*+}]} = \frac{2^n}{2^n h/h^+} = \frac{h^+}{h}$$

(cf. [42, VI, § 3, Thm. 3.1]). Here h and h^+ are the class number and narrow class number of F, respectively. □

Proof of Corollary 1.3.3. First note that the assumption $\mathrm{sgn}(\varepsilon)^{\mathbf{k}} = 1$ makes sure that the series $\Theta_I(\,\cdot\,; Q)$ does not depend on the choice of the elements δ modulo $\mathfrak{o}_F^{*2}$. Indeed, for $\varepsilon \in \mathfrak{o}_F^*$ we have

$$\begin{aligned}(\delta\varepsilon^2)^{\frac{\mathbf{k}-\mathbf{2}}{2}}\vartheta_I(\delta\varepsilon^2 z; Q) &= \delta^{\frac{\mathbf{k}-\mathbf{2}}{2}}\varepsilon^{\mathbf{k}-\mathbf{2}}\mathrm{sgn}(\varepsilon)^{\mathbf{k}-\mathbf{2}}\sum_{x\in I} Q(x)\exp\big(2\pi i\mathrm{Tr}(\delta\varepsilon^2\mathrm{nrd}(x)z)\big)\\ &= \delta^{\frac{\mathbf{k}-\mathbf{2}}{2}}\sum_{x\in I} Q(\varepsilon x)\exp\big(2\pi i\mathrm{Tr}(\delta\mathrm{nrd}(\varepsilon x)z)\big) = \delta^{\frac{\mathbf{k}-\mathbf{2}}{2}}\vartheta_I(\delta z; Q)\end{aligned}$$

since $\varepsilon I = I$. Now let $\gamma \in \Gamma_0\big(\mathrm{nrd}(I)\mathfrak{d},\, D_1 D_2\big)$ be an arbitrary matrix, not necessarily of determinant 1. So

$$\gamma = \tilde{\gamma}\begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix} \qquad \text{where} \qquad \varepsilon := \det\gamma \in \mathfrak{o}_F^{*+} \text{ and } \tilde{\gamma} \in \Gamma_0^1\big(\mathrm{nrd}(I)\mathfrak{d},\, D_1 D_2\big)\ .$$

Then

$$\big(\Theta_I|_{\mathbf{k}}\gamma\big)(z; Q) = \sum_{\delta\in\mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}} \delta^{\frac{\mathbf{k}-\mathbf{2}}{2}}\big(\vartheta_I|_{\mathbf{k}}\gamma\big)(\delta z; Q) = \sum_{\delta\in\mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}} \delta^{\frac{\mathbf{k}-\mathbf{2}}{2}}\left(\vartheta_I|_{\mathbf{k}}\begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix}\right)(\delta z; Q)$$

because $\tilde{\gamma}$ acts trivially on $\vartheta_I(\,\cdot\,; Q)$ by Theorem 1.3.1. Now observe that

$$\varepsilon^{\frac{\mathbf{k}}{2}} = \prod_{i=1}^{n}(\varepsilon^{(i)})^{\frac{k_i}{2}} = \prod_{i=1}^{n}\varepsilon^{(i)}\prod_{i=1}^{n}(\varepsilon^{(i)})^{\frac{k_i-2}{2}} = N_{\mathbb{Q}}^F(\varepsilon)\,\varepsilon^{\frac{\mathbf{k}-\mathbf{2}}{2}} = \varepsilon^{\frac{\mathbf{k}-\mathbf{2}}{2}}$$

because $\varepsilon \in \mathfrak{o}_F^{*+}$ is necessarily of norm 1. Hence

$$\big(\Theta_I|_{\mathbf{k}}\gamma\big)(z; Q) = \sum_{\delta\in\mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}} \delta^{\frac{\mathbf{k}-\mathbf{2}}{2}}\varepsilon^{\frac{\mathbf{k}}{2}}\,\vartheta_I(\varepsilon\delta z; Q) = \sum_{\delta\in\mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}} (\varepsilon\delta)^{\frac{\mathbf{k}-\mathbf{2}}{2}}\,\vartheta_I(\varepsilon\delta z; Q) = \Theta_I(z; Q)$$

since $\varepsilon\delta$ runs through all of $\mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}$ if δ does. $\square$

Chapter 2

Adelic Hilbert modular forms

Now that we have learned the basic properties and seen the fundamental examples of classical Hilbert modular forms, we turn to the task of introducing Hilbert modular forms in the adelic setting (Section 2.2). The correspondence between the classical and the adelic setting is well known, but somewhat cumbersome if F has narrow class number $h^+ > 1$. It will be explained in Section 2.3.

Since the adelic approach makes extensive use of representation theoretic methods, the first section gives a brief introduction to the representation theory that will be needed later on.

2.1 Group representations

We start by recalling some facts about topological groups, which can be found, for example, in [55, Ch. 1].

Let G be a topological group. We say that G is *locally compact* if it is Hausdorff and if every $g \in G$ has a compact neighbourhood. It is well known that every locally compact group admits an invariant measure, the so-called *Haar measure*, which is unique up to a constant factor. The topological group G is called *totally disconnected* if the identity element, and hence every element in G, is its own connected component.

Example 2.1.1. A $\mathfrak{p}$-adic field $F_\mathfrak{p}$ equipped with the usual topology is locally compact and totally disconnected. The same holds for $\mathrm{GL}_2(F_\mathfrak{p})$. Let $\mathfrak{o}_\mathfrak{p}$ denote the ring of integers in $F_\mathfrak{p}$. Then $\mathrm{GL}_2(\mathfrak{o}_\mathfrak{p})$ is a maximal compact subgroup of $\mathrm{GL}_2(F_\mathfrak{p})$, and all maximal compact subgroups of $\mathrm{GL}_2(F_\mathfrak{p})$ are its conjugates, i. e. of the form $g^{-1}\mathrm{GL}_2(\mathfrak{o}_\mathfrak{p})g$ for some $g \in \mathrm{GL}_2(F_\mathfrak{p})$ (see [63, II, Ch. IV, § 2, section 1]). All of these subgroups are not only compact but also open. The groups

$$\Gamma_1(\mathfrak{p}^s) := \left\{ A \in \mathrm{GL}_2(\mathfrak{o}_\mathfrak{p}) \;\middle|\; A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{p}^s \right\} \qquad \text{for } s \in \mathbb{N}_0$$

are compact open normal subgroups of $\mathrm{GL}_2(\mathfrak{o}_\mathfrak{p})$, they form a basis of the neighbourhoods of the identity (cf. [55, Lemma 1-17]).

Let G be a locally compact group. By a *representation* of G we will always mean a *topological representation*, i. e. a homomorphism $\pi : G \to \mathrm{GL}(E)$ of G into the automorphism group of some Hilbert space E such that

$$G \times E \to E \ , \qquad (g, x) \mapsto \pi(g)x$$

is continuous with respect to the product topology on $G \times E$. The representation π is *irreducible* if E has no closed π-invariant subspaces other than $\{0\}$ and E. It is *unitary* if $\pi(g)$ is unitary for all $g \in G$. Two representations $\pi : g \to E$ and $\tilde{\pi} : G \to \tilde{E}$ are *equivalent* if there exists a continuous isomorphism $\sigma : E \to \tilde{E}$ with continuous inverse such that $\sigma \circ \pi(g) = \tilde{\pi}(g) \circ \sigma$ for all $g \in G$.

Example 2.1.2. If G is a topological group and E some Hilbert space of continuous functions $G \to \mathbb{C}$ then it is well known that the *right regular representation* ρ and the *left regular representation* λ, defined by

$$\big(\rho(g)f\big)(h) = f(hg) \qquad \text{and} \qquad \big(\lambda(g)f\big)(h) = f(g^{-1}h) \ ,$$

are representations.

Definition 2.1.3 (Matrix coefficient, character). Let E be a Hilbert space with Hermitian inner product $\langle \cdot , \cdot \rangle$ and let $\pi : G \to \mathrm{GL}(E)$ be a representation.

(i) A *matrix coefficient* of π is any function of the form

$$G \to \mathbb{C} \ , \qquad g \mapsto \langle \pi(g)x, y \rangle \qquad \text{for some } x, y \in E$$

or equivalently of the form

$$G \to \mathbb{C} \ , \quad g \mapsto L\big(\pi(g)x\big)$$

for some $x \in E$ and some continuous linear functional $L : E \to \mathbb{C}$.

(ii) If $\dim E < \infty$ then the *character* of π is defined by

$$\chi_\pi : G \to \mathbb{C} \ , \qquad \chi_\pi(g) := \mathrm{Tr}\big(\pi(g)\big) \ .$$

(iii) If G is compact we denote by $\widehat{G}$ the set of equivalence classes of irreducible unitary representations of G.

Example 2.1.4. Up to equivalence, all irreducible representations of $\mathbb{R}^*$ and $\mathbb{R}_{>0}$ are one-dimensional, i. e. *quasi-characters*, and of the form

$$\begin{array}{lll} & \mathbb{R}^* \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^* \ , & x \mapsto |x|^r \mathrm{sgn}(x) \\ \text{and} & \mathbb{R}_{>0} \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^* \ , & x \mapsto x^r \end{array}$$

for some $r \in \mathbb{C}$. Likewise, the quasi-characters of a $\mathfrak{p}$-adic field $F_\mathfrak{p}^*$ are of the form

$$F_\mathfrak{p}^* \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^* \ , \qquad x \mapsto |x|_\mathfrak{p}^r \, \mu(x)$$

where $r \in \mathbb{C}$ is a complex number and μ is the pullback of a unitary character $\mathfrak{o}_\mathfrak{p}^* \to S^1$ (cf. [55, § 7.1] or [78, I.VII § 3]).

Theorem 2.1.5 (Schur orthogonality). *If $\pi : G \to \mathrm{GL}(E)$ is an irreducible representation of a compact group G on a finite-dimensional Hilbert space E with a π-invariant inner product $\langle \cdot , \cdot \rangle$ then*

$$\int_G \langle \pi(g)v_1, v_2 \rangle \overline{\langle \pi(g)w_1, w_2 \rangle} \, dg = \frac{1}{\dim E} \langle v_1, w_1 \rangle \overline{\langle v_2, w_2 \rangle} \ .$$

Proof. See for example [9, Thm. 2.4]. □

In later sections we will deal with representations on infinite dimensional Hilbert spaces. As usual it will be one of the main tasks to study their decomposition into irreducible subrepresentations. In this context it is crucial to distinguish between algebraic direct sums and Hilbert space direct sums, so let us recall that for a (possibly infinite) sequence of Hilbert spaces $E_1, E_2, \dots$ the *algebraic direct sum* is defined as

$$\bigoplus_i E_i := \{(e_i)_i \mid e_i \in E_i, \text{ and } e_i = 0 \text{ for almost all } i\}$$

whereas the *Hilbert space direct sum* is defined as

$$\widehat{\bigoplus_i} E_i := \{(e_i)_i \mid e_i \in E_i \text{ and } \sum_i \|e_i\| < \infty\} \ .$$

Proposition 2.1.6. *Let E be a Hilbert space with Hermitian inner product $\langle \cdot , \cdot \rangle$, and let $\pi : K \to \mathrm{GL}(E)$ be a representation of a compact group K on E. Then there exists a Hermitian inner product $\langle\!\langle \cdot , \cdot \rangle\!\rangle$ on E that induces the same topology as $\langle \cdot , \cdot \rangle$ and such that π is unitary with respect to $\langle\!\langle \cdot , \cdot \rangle\!\rangle$.*

Proof. Cf. [8, Lemma 2.4.3]. □

Theorem 2.1.7 (Peter-Weyl). *Let $\pi : K \to \mathrm{GL}(E)$ be a representation of a compact group K on a Hilbert space E.*

(i) If π is unitary then E decomposes into a Hilbert space direct sum of irreducible unitary representations.

(ii) If π is unitary and irreducible then it is finite-dimensional.

Proof. Cf. [9, Thm. 4.3]. □

Corollary 2.1.8. *Let G be a group with a maximal compact subgroup K, let $\pi : G \to \mathrm{GL}(E)$ be a representation of G on a Hilbert space E. Then there exists a decomposition*

$$E = \widehat{\bigoplus_{\sigma}} E_\sigma \quad \text{where all } \sigma : K \to \mathrm{GL}(E_\sigma) \text{ are irreducible and finite-dimensional.} \tag{2.1}$$

In particular, the Hilbert space direct sum is an algebraic direct sum if π is finite-dimensional.

Proof. In virtue of Proposition 2.1.6 we may assume that the representation $\pi|_K$ is unitary. The existence of a decomposition of E into irreducible K-invariant subspaces then follows by the Peter-Weyl theorem 2.1.7 as does the finiteness of each σ. □

Example 2.1.9. Consider the group $\mathrm{SO}_2(\mathbb{R})^n$ for $n \in \mathbb{N}$. We denote its elements by

$$r(\boldsymbol{\theta}) := r\big((\theta_1, \dots, \theta_n)\big) := \left(\begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix}_{j=1}^{n} \right) \qquad \text{where} \qquad 0 \le \theta_j < 2\pi \ .$$

Note that $\mathrm{SO}_2(\mathbb{R})^n$ is a maximal compact subgroup of $GL_2^+(\mathbb{R})^n$ and that it is abelian. Now let $\pi : \mathrm{SO}_2(\mathbb{R})^n \to \mathrm{GL}(E)$ be an irreducible finite-dimensional representation. Then $\dim(E) = 1$ by Schur's Lemma (see for example [43, Cor. 1.9]). More precisely, π is of the form

$$\pi(r(\boldsymbol{\theta})) = e^{i\mathrm{Tr}(\mathbf{m}\boldsymbol{\theta})} = e^{i(m_1\theta_1 + \ldots + m_n\theta_n)} \qquad \text{for some } \mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n \ ,$$

which is shown, for example, in [74, II § 1] for $n = 1$. The general case follows immediately by examining the restriction of π to the j-th component of $\mathrm{SO}_2(\mathbb{R})^n$.

The Peter-Weyl Theorem and its Corollary then state that all finite-dimensional representations of $\mathrm{GL}_2^+(\mathbb{R})^n \to \mathrm{GL}(E)$ can be decomposed into an algebraic direct sum of representations $\mathrm{SO}_2(\mathbb{R})^n \to \mathbb{C}^*$ of the type $\pi\big(r(\boldsymbol{\theta})\big) = e^{i\mathrm{Tr}(\mathbf{m}\boldsymbol{\theta})}$.

Definition 2.1.10 ((Locally) K-finite). Let G be a group with a maximal compact subgroup K, and let $\pi : G \to \mathrm{GL}(E)$ be a representation of G on a Hilbert space E.

(i) A vector $f \in E$ is called *K-finite* if

$$\dim \left\langle \pi(k)f \mid k \in K \right\rangle < \infty \,.$$

(ii) We say that the representation π is *locally K-finite* if every $f \in E$ is K-finite.

Remark 2.1.11. Let us write $f \in E$ according to (2.1) as $f = \sum_\sigma f_\sigma$ with $f_\sigma \in E_\sigma$. If all but finitely many f_σ's were equal to 0 then f would automatically be K-finite as all of its components lie in finite-dimensional K-invariant subspaces. But note that this argument does not hold in general since the decomposition in (2.1) is a *Hilbert space* direct sum and f may have infintely many non-zero components. □

Lemma 2.1.12.

(i) In the situation of Definition 2.1.10, the representation π is locally K-finite if and only if E is an algebraic direct sum of finite-dimensional irreducible K-invariant subspaces. In this case,

$$E = \bigoplus_{\sigma \in \widehat{K}} E(\sigma)$$

where $E(\sigma)$ is the sum of all those subspaces of E that are equivalent to σ. The space $E(\sigma)$ is called the σ-isotypic component *of E.*

(ii) If, in particular, $G = \mathrm{GL}_2(F_\mathfrak{p})$ and $K = \mathrm{GL}_2(\mathfrak{o}_\mathfrak{p})$, then π is locally K-finite if and only if for every $x \in E$, the stabilizer

$$\mathrm{Stab}_G(x) := \{g \in G \mid \pi(g)x = x\}$$

is an open subgroup of G.

Proof. For part *(i)*, see Remark 2.1.11 and [44, Prop. 1.18]. For *(ii)*, let $G = \mathrm{GL}_2(F_\mathfrak{p})$, $K = \mathrm{GL}_2(\mathfrak{o}_\mathfrak{p})$, and $x \in E$. If $\mathrm{Stab}_G(x)$ is open in G then $\mathrm{Stab}_K(x) = K \cap \mathrm{Stab}_G(x)$ is open in the compact group K, so that $K/\mathrm{Stab}_K(x) \cong \{\pi(k)x \mid k \in K\}$ is finite (see for example [55, Prop. 1-4]) and therefore spans a finite-dimensional vector space.

Conversely, let π be locally K-finite and fix an $x \in E$. The space $E' := \left\langle \pi(k)x \mid k \in K \right\rangle$ is of dimension r, say, so that $\pi|_K : K \to \mathrm{GL}(E') \cong \mathrm{GL}_r(\mathbb{C})$. In $\mathrm{GL}_r(\mathbb{C})$, we can choose an open neighbourhood U of the identity which is so small that it does not contain any non-trivial subgroup of $\mathrm{GL}_r(\mathbb{C})$. The inverse image $(\pi|_K)^{-1}(U)$ is, by continuity, an open neighbourhood of 1 in K. By Example 2.1.1, there is an open subgroup H of K with $H \subseteq (\pi|_K)^{-1}(U)$. Note that since K is open in G, so is H. The image $\pi(H)$ is a subgroup of $\mathrm{GL}_r(\mathbb{C})$ contained in U and hence trivial. It follows that $H \subseteq \mathrm{Stab}_G(x)$, so $\mathrm{Stab}_G(x)$ is open (cf. [38, II, Prop. 6]). □

Definition 2.1.13 (Admissible representation). As before, let G be a group with a maximal compact subgroup K, and let $\pi : G \to \mathrm{GL}(E)$ be a representation of G on a Hilbert space E. We call the representation π *admissible* if it is locally K-finite and in any decomposition of the space E of the form (2.1) every isomorphism class of irreducible finite-dimensional representations of K appears only finitely many times.

Remark 2.1.14.

(i) It can be shown that the notion of admissibility does not depend on the choice of the decomposition (cf. [43, VIII § 2]).

(ii) For each isomorphism class σ of irreducible unitary representations of K, denote by $E(\sigma)$ the σ-isotypic part of E. The admissibility of π is then equivalent to all $E(\sigma)$ being finite-dimensional.

$\square$

Proposition 2.1.15. *Let $G = \mathrm{GL}_2(\mathbb{R})$ or $G = \mathrm{GL}_2(\mathbb{R})^+$. If $\pi : G \to \mathrm{GL}(E)$ is a representation on a Hilbert space E then*

$$\pi : \mathfrak{g} \to \mathrm{GL}(E^\infty)\ , \qquad \pi(X)f = \frac{d}{dt}\pi\big(\exp(tX)\big)f|_{t=0}$$

is a representation of the Lie algebra $\mathfrak{g}$ of G on the space

$$E^\infty := \{e \in E \mid G \to E,\ g \mapsto \pi(g)e \text{ is a } C^\infty\text{-map on } G\}$$

of smooth vectors *in E. It extends to a representation on E^∞ of the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$.*

Proof. See [43, Prop. 3.9]. $\square$

2.2 Hilbert modular forms in the adelic setting

The definitions and basic properties that were given in Section 1.1 are clearly the obvious generalizations of elliptic modular forms (over $\mathbb{Q}$) to Hilbert modular forms (over the number field F). However, the classical theory does no longer generalize as easily when leaving the basics behind. As mentioned in the introduction, it turns out to be extremely useful, if not unavoidable, to examine Hilbert modular forms also in a different, namely the adelic setting. This is what we will introduce in this section.

Let $\mathbb{A}_F$ be the adele ring over the number field F, so that $\mathbb{A}_F^*$ is the group of ideles. In $\mathbb{A}_F$ and $\mathbb{A}_F^*$, we distinguish the following subgroups

$$F_\infty := \prod_{v|\infty} \mathbb{R} \ , \qquad \widehat{\mathfrak{o}_F} := \prod_{\mathfrak{p}<\infty} \mathfrak{o}_\mathfrak{p} \ , \qquad F_\infty^+ := \prod_{v|\infty} \mathbb{R}_{>0} \ , \qquad \widehat{\mathfrak{o}_F^*} := \prod_{\mathfrak{p}<\infty} \mathfrak{o}_\mathfrak{p}^* \ .$$

For $x \in \mathbb{A}_F$, denote by

$$x^\infty \quad \text{the archimedean part} \qquad \text{and by} \qquad x^f \quad \text{the non-archimedean part.}$$

So we have $x = x^\infty x^f$. We will also use this notation for a global element $x \in F$, in this case we understand x to be embedded diagonally into $\mathbb{A}_F$.

We remind the reader of the space of so-called Schwartz-Bruhat (or simply Schwartz) functions, which play an essential role in the adelic theory because they admit an adelic Fourier expansion (cf. [29, § 6A] or [8, § 3.1]). For the time being, we will only encounter Schwartz-Bruhat functions on $\mathbb{A}_F$. Still we prefer to give a slightly more general definition because Schwartz-Bruhat functions on the group $\mathrm{GL}_2(\mathbb{A}_F)$ will come into play in later sections.

Definition 2.2.1 (Schwartz-Bruhat function). Let F be a totally real number field and let $X = \mathbb{A}_F^m$ be a finite-dimensional vector space over the adele ring $\mathbb{A}_F$. The space $\mathcal{S}(X)$ of *(adelic) Schwartz-Bruhat functions* is generated by all functions of the form

$$\phi : X \to \mathbb{C} \ , \qquad \phi(x) = \prod_{v|\infty} \phi_v(x_v) \prod_{\mathfrak{p}<\infty} \phi_\mathfrak{p}(x_\mathfrak{p})$$

where

(i) each ϕ_v and $\phi_\mathfrak{p}$ is a *Schwartz-Bruhat function* over the local vector space, i. e.

$$\begin{cases} \phi_v : X_v \to \mathbb{C} & \text{is infinitely differentiable and rapidly decreasing} \\ & \text{with all derivatives rapidly decreasing} \qquad \text{for } v \mid \infty \ , \\ \phi_\mathfrak{p} : X_\mathfrak{p} \to \mathbb{C} & \text{is locally constant with compact support} \qquad \text{for } \mathfrak{p} < \infty \ , \end{cases}$$

(ii) $\phi_\mathfrak{p}$ is the characteristic function of $\mathfrak{o}_\mathfrak{p}^m$ for almost every $\mathfrak{p}$.

We say that a function is a *Schwartz-Bruhat function* on $\mathbb{A}_F/F$ if it is in $\mathcal{S}(\mathbb{A}_F)$ and F-invariant.

In $\mathrm{GL}_2(\mathbb{A}_F)$, we define the following subgroups

$$\begin{aligned} G_\infty &:= \mathrm{GL}_2(\mathbb{R})^n \ , \\ G_\infty^+ &:= \mathrm{GL}_2^+(\mathbb{R})^n \ , \\ K_\infty &:= \mathrm{O}_2(\mathbb{R})^n \ , \\ K_\infty^+ &:= \mathrm{SO}_2(\mathbb{R})^n \ , \\ K &:= K_\infty K_f \qquad \text{where} \qquad K_f := \prod_{\mathfrak{p}<\infty} K_\mathfrak{p} \end{aligned}$$

and each $K_\mathfrak{p}$ is a maximal compact open subgroup of $\mathrm{GL}_2(F_\mathfrak{p})$ with $\det(K_\mathfrak{p}) = \mathfrak{o}_\mathfrak{p}^*$, and such that $K_\mathfrak{p} = \mathrm{GL}_2(\mathfrak{o}_\mathfrak{p})$ for almost every $\mathfrak{p}$.

A *größencharacter* of F is a unitary character of $\mathbb{A}_F^*$ that is trivial on F^*. We often identify $\mathbb{A}_F^*$ with the centre

$$Z(\mathbb{A}_F) := Z(\mathrm{GL}_2(\mathbb{A}_F)) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \;\middle|\; a \in \mathbb{A}_F^* \right\}$$

of $\mathrm{GL}_2(\mathbb{A}_F)$ and F_∞^+ with

$$Z_\infty^+ := \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \;\middle|\; a \in F_\infty^+ \right\} \subseteq G_\infty^+ \ .$$

Using this identification we can view any größencharacter also as a character on $Z(\mathbb{A}_F)$, trivial on $Z(\mathrm{GL}_2(F))$.

When dealing with matrices in $\mathrm{GL}_2^+(\mathbb{R})$ it is convenient to use the Iwasawa decomposition, which allows us to write $g \in \mathrm{GL}_2^+(\mathbb{R})$ uniquely as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} r(\theta) \ , \tag{2.2}$$

where

$$r(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and $u, y > 0,\ x \in \mathbb{R},\ \theta \in [0, 2\pi[$. The variables (a, b, c, d) and (u, x, y, θ) are connected by the following identities,

$$\det(g) = ad - bc = u^2 \ , \qquad \frac{ai + b}{ci + d} = x + iy \ , \qquad ci + d = uy^{-1/2}e^{i\theta} \ .$$

We denote by $\mathfrak{g}_\mathbb{C}$ the complexification of the Lie algebra $\mathrm{Lie}(G_\infty)$ and by $\mathfrak{z}$ the centre of the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. We remind the reader that $\mathrm{Lie}(G_\infty^+) = \mathrm{Lie}(G_\infty)$ because the Lie algebra depends only on the tangent space at 1 which is identical in both cases.

Definition 2.2.2.

(i) We say that a function $f : \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{C}$ is *slowly increasing* if there exist an integer N and a constant $C > 0$ such that

$$|f(g)| \le C\|g\|^N \qquad \text{for all } g \in \mathrm{GL}_2(\mathbb{A}_F)$$

where we identify g with $(g, \det(g)^{-1}) \in M_2(\mathbb{A}_F) \times \mathbb{A}_F \cong \mathbb{A}_F^5$ and put

$$\|g\|_v := \max\{|g_i|_v \mid i = 1, \dots, 5\} \qquad \text{and} \qquad \|g\| := \prod_v \|g\|_v \ .$$

(ii) A function $f : \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{C}$ is *right-K-finite* if the complex vector space spanned by all K-right translates of f is finite dimensional, i. e.

$$\dim \left\langle g \mapsto f(gk) \mid k \in K \right\rangle_{\mathbb{C}} < \infty \ .$$

Similarly, $f : G_\infty \to \mathbb{C}$ is *$\mathfrak{z}$-finite* if

$$\dim \left\langle Xf \mid X \in \mathfrak{z} \right\rangle_{\mathbb{C}} < \infty$$

where the elements of $\mathfrak{z}$ act as differential operators on f as explained in Proposition 2.1.15.

The notion of K- and $\mathfrak{z}$-finiteness will play an important role in the now following definition of automorphic forms. We will explain them a little further in Remark 2.2.8.

Definition 2.2.3 (Automorphic Form). Let G_∞, K be as above. Let $f : \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{C}$ be a function with the following properties

(A.1) there exists a größencharacter ω of F, the so-called *central character*, such that

$$f(zg) = \omega(z)f(g) \qquad \text{for all} \quad z \in Z(\mathbb{A}_F), \quad g \in \mathrm{GL}_2(\mathbb{A}_F) \ ,$$

(A.2) f is left-$\mathrm{GL}_2(F)$-invariant, i. e.

$$f(\gamma g) = f(g) \qquad \text{for all} \quad \gamma \in \mathrm{GL}_2(F), \quad g \in \mathrm{GL}_2(\mathbb{A}_F) \ ,$$

(A.3) f is right-K-finite,

(A.4) f is slowly increasing,

(A.5) f is smooth if viewed as a function on G_∞,

(A.6) f is $\mathfrak{z}$-finite if viewed as a function on G_∞.

Such an f is called an *(adelic) automorphic form on* $\mathrm{GL}_2(\mathbb{A}_F)$ and the space of all adelic automorphic forms for a fixed größencharacter ω will be denoted by $\mathcal{A}(\omega)$. If additionally, the following *cuspidality condition*

(A.7)

$$\int_{\mathbb{A}_F/F} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0 \qquad \text{for almost every } g \in \mathrm{GL}_2(\mathbb{A}_F)$$

holds then we call f *(adelic) cusp form.* The subspace of cusp forms will be denoted by $\mathcal{A}_0(\omega)$.

Proposition 2.2.4. *Let $L^2_0\big(\omega, \mathrm{GL}_2(\mathbb{A}_F)\big)$ be the space of all functions $f : \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{C}$ such that f satisfies* **(A.1)**, **(A.2)** *and* **(A.7)** *and such that $|f|$ is square-integrable on $Z(\mathbb{A}_F)\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)$. This space is a Hilbert space with respect to the inner product*

$$\langle f_1, f_2\rangle := \int\limits_{Z(\mathbb{A}_F)\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)} f_1(g)\overline{f_2(g)}\, dg \ .$$

The right regular representation of $\mathrm{GL}_2(\mathbb{A}_F)$ on $L^2_0\big(\omega, \mathrm{GL}_2(\mathbb{A}_F)\big)$ is unitary and $\mathcal{A}_0(\omega)$ is equal to the space of K-finite elements in $L^2_0\big(\omega, \mathrm{GL}_2(\mathbb{A}_F)\big)$.

Proof. See [65, § 3, No. 2] and [32, § 3, No. 1]. □

Proposition 2.2.5 (Adelic Fourier expansion). *Let ϕ be a Schwartz-Bruhat function on $\mathbb{A}_F/F$ and let τ be the standard character on $\mathbb{A}_F$ defined by*

$$\tau(x) = \prod_{v|\infty}\tau_v(x_v)\prod_{\mathfrak{p}<\infty}\tau_{\mathfrak{p}}(x_{\mathfrak{p}}) \qquad \textit{where} \qquad \begin{cases} \tau_v(x_v) = \exp(2\pi i x_v) \\ \tau_{\mathfrak{p}}(x_{\mathfrak{p}}) = \exp\big(-2\pi i(\lambda\circ\mathrm{Tr}_{F_{\mathfrak{p}}/\mathbb{Q}_p})(x_{\mathfrak{p}})\big) \end{cases}$$

where λ is the canonical map $\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z}$ and $\mathrm{Tr}_{F_{\mathfrak{p}}/\mathbb{Q}_p}$ is the trace function $F_{\mathfrak{p}} \to \mathbb{Q}_p$ if $\mathfrak{p}$ lies over the rational prime p. Then the Fourier series

$$\sum_{\xi\in F}\tau(x\xi)\int_{\mathbb{A}_F/F}\bar\tau(u\xi)\phi(u)\, du$$

converges absolutely and uniformly to $\phi(x)$.

Proof. See [28, Appendix A.2]. □

Definition 2.2.6 (Fourier coefficient). Let f be an automorphic form on $\mathrm{GL}_2(\mathbb{A}_F)$. For $\xi \in F$ we call the function

$$W_\xi : \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{C}$$

given by

$$W_\xi(g) := \int_{\mathbb{A}_F/F}\bar\tau(u\xi) f\left(\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix} g\right) du$$

the *ξ-th Fourier coefficient* of f.

Remark 2.2.7. The Fourier expansion introduced in Proposition 2.2.5 cannot be applied to f directly, as it is not a function on $\mathbb{A}_F/F$. Instead we consider

$$\phi_g(x) := f\left(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} g\right) \qquad \text{for} \qquad g \in \mathrm{GL}_2(\mathbb{A}_F) \ .$$

The $\mathrm{GL}_2(F)$-invariance of f immediately implies the F-invariance of ϕ_g. Therefore, ϕ_g has a Fourier expansion given by

$$\begin{aligned}\phi_g(x) &= \sum_{\xi\in F} \tau(x\xi) \int_{\mathbb{A}_F/F} \bar{\tau}(u\xi)\phi_g(u)\,du = \sum_{\xi\in F} \tau(x\xi) \int_{\mathbb{A}_F/F} \bar{\tau}(u\xi) f\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) du \\ &= \sum_{\xi\in F} W_\xi(g)\tau(x\xi) \ .\end{aligned}$$

For $\xi = 0$ the Fourier coefficient $W_\xi(g)$ becomes the integral in **(A.7)**. So we see that, as in the classical setting, adelic cusp forms are forms whose 0-th Fourier coefficient vanishes. $\square$

We will be interested in a certain subspace of $\mathcal{A}(\omega)$ which is defined by replacing the conditions **(A.3)** and **(A.6)**, that is K- and $\mathfrak{z}$-finiteness, by stronger requirements. In order to motivate the exact definition of this subspace, we first add a few remarks concerning these conditions.

Remark 2.2.8.

(i) Let f be a right-K-finite function viewed as a function on G_∞ alone, which is then right-K_∞-finite, in particular, it is right-K_∞^+-finite. This implies that the right regular representation ρ of the compact abelian group $K_\infty^+ = \mathrm{SO}_2(\mathbb{R})^n$ on the space

$$E := \left\langle g \mapsto f(gk) \mid k \in K_\infty^+ \right\rangle$$

of K_∞^+-right translates of f decomposes into a finite number of one-dimensional subrepresentations $E_1, \ldots, E_r$, and that for each E_j there is a $\mathbf{m}_j \in \mathbb{Z}^n$ such that the K_∞^+-action is given by

$$\rho\big(r(\theta)\big) = e^{i\mathrm{Tr}(\mathbf{m}_j\theta)}$$

(cf. Example 2.1.9). We will shortly make the additional requirement that the representation $\rho : K_\infty^+ \to \mathrm{GL}(E)$ itself be irreducible and hence one-dimensional.

(ii) Consider the Lie algebra $\mathfrak{gl}_2(\mathbb{C})$, which is the complexification of the Lie algebra of $\mathrm{GL}_2(\mathbb{R})$. As a $\mathbb{C}$-vector space it is generated by the matrices

$$Z := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad L := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad R := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and the multiplication is the Lie bracket $[X, Y] = XY - YX$. The universal enveloping algebra $U(\mathfrak{gl}_2(\mathbb{C}))$ can be realized as the tensor algebra of $\mathfrak{gl}_2(\mathbb{C})$ modulo the ideal J generated by all

$$X \otimes Y - Y \otimes X - [X, Y] \qquad \text{where } X, Y \in \mathfrak{gl}_2(\mathbb{C}) \ ,$$

i. e., all elements in $U(\mathfrak{gl}_2(\mathbb{C}))$ are of the form

$$X_1 \otimes \ldots \otimes X_m \bmod J \qquad \text{for some} \qquad m \in \mathbb{N}_0,\ X_i \in \{Z, H, L, R\} \ . \tag{2.3}$$

Now Z clearly commutes with every element in $\mathfrak{gl}_2(\mathbb{C})$, so that we can rewrite (2.3) as $Z^{\otimes r} \otimes X \bmod J$ for some non-negative integer r and an element X that, when written in the form (2.3), contains no factor Z. Note that any such X is an element of $U(\mathfrak{sl}_2(\mathbb{C}))$ since $\mathfrak{sl}_2(\mathbb{C})$ is the subalgebra of $\mathfrak{gl}_2(\mathbb{C})$ spanned by H, L, R.

The centre $Z(U(\mathfrak{gl}_2(\mathbb{C})))$ is therefore generated as an algebra by Z and $Z(U(\mathfrak{sl}_2(\mathbb{C})))$. To determine the latter we consider the Casimir element C of $\mathfrak{sl}_2(\mathbb{C})$, given by

$$C = \frac{1}{2}H^2 + RL + LR \ .$$

It is a general result from basic Lie theory that the Casimir element lies in the center of the corresponding universal enveloping Lie algebra (cf. [9, § 10]). In our particular situation, however, C not only lies in $Z(U(\mathfrak{sl}_2(\mathbb{C})))$ but generates it. This can be seen by applying the Harish-Chandra isomorphism, which establishes an algebra isomorphism between $Z(U(\mathfrak{sl}_2(\mathbb{C})))$ and $U(\mathfrak{h})^W$, the latter being the algebra of Weyl-group invariants in the universal enveloping algebra of the maximal torus $\mathfrak{h}$ generated by H. In our case, it is not difficult to verify that the Casimir element C is mapped to a generator of $U(\mathfrak{h})^W$, so that C itself must be a generator of $Z(U(\mathfrak{sl}_2(\mathbb{C})))$. The explicit calculation together with further details on the Harish-Chandra isomorphism can be found in [43, VIII § 5] or [44, IV § 7].

Now, the usual representation of $\mathfrak{gl}_2(\mathbb{R})$ as left-invariant differential operators on $C^\infty(\mathrm{GL}_2^+(\mathbb{R}))$ extends to $U(\mathfrak{gl}_2(\mathbb{C}))$. In this interpretation, the Casimir element C corresponds to the Laplace operator Δ. More precisely, if we write an element $g \in \mathrm{GL}_2^+(\mathbb{R})$ according to (2.2) in the Iwasawa coordinates (u, x, y, θ) then the Laplace operator in these coordinates takes the form

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + y \frac{\partial^2}{\partial x \partial \theta}$$

(see [8, § 2.1, eq. (1.29)]) and it coincides with $-\frac{1}{2}C$ (see [8, Prop. 2.2.5]). Thus, we can identify

$$-4\Delta = H^2 + 2RL + 2LR \ .$$

Now let us return to the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G_\infty)$ and recall that G_∞ consists of n copies of $\mathrm{GL}_2(\mathbb{R})$, one for each archimedean place v of F. Hence, $\mathfrak{g}$ is isomorphic to n copies of $\mathfrak{gl}_2(\mathbb{R})$, so let us denote by Z_v and Δ_v the elements Z and Δ, respectively, belonging to the v-th component of $\mathfrak{g}$. From what we have seen above we deduce that $\mathfrak{z}$ is generated by all Z_v and Δ_v.

By **(A.1)**, we already know that the elements Z_v act as scalars on every automorphic form. Consequently, condition **(A.6)** of $\mathfrak{z}$-finiteness is really a condition on the Laplace operators Δ_v, requiring that $\dim_{\mathbb{C}}\langle \Delta_v^m f \mid m \in \mathbb{N}_0 \rangle < \infty$ for each archimedean place v.

In the space to be defined below we will demand that the spaces $\langle \Delta_v^m f \mid m \in \mathbb{N}_0 \rangle$ are not only finite- but one-dimensional and that every Δ_v acts as a certain specified scalar.

□

Definition 2.2.9 (Hilbert automorphic form). Let $G_\infty, K_f, K_\infty, K_\infty^+$ be as above. Fix a vector $\mathbf{k} \in \mathbb{N}_0^n$ of non-negative integers, and let $\chi : K_f \to \mathbb{C}^*$ be a character. For a function $\mathbf{f} : \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{C}$, consider the following set of conditions:

(H.1) there exists a größencharacter ω of F such that

$$\mathbf{f}(zg) = \omega(z)\mathbf{f}(g) \qquad \text{for all} \quad z \in Z(\mathbb{A}_F), \quad g \in \mathrm{GL}_2(\mathbb{A}_F)\ ,$$

(H.2) $\mathbf{f}$ is left-$Z_\infty^+\mathrm{GL}_2(F)$-invariant, i. e.

$$\mathbf{f}(\gamma g) = \mathbf{f}(g) \qquad \text{for all} \quad \gamma \in Z_\infty^+\mathrm{GL}_2(F), \quad g \in \mathrm{GL}_2(\mathbb{A}_F)\ ,$$

(H.3) $\mathbf{f}$ is an eigenfunction under right translation with $K_\infty \times K_f$, more precisely

$$\mathbf{f}(gkr(\theta)) = \chi(k)e^{-i\mathrm{Tr}(\mathbf{k}\theta)}\mathbf{f}(g) \qquad \text{for all} \qquad g \in \mathrm{GL}_2(\mathbb{A}_F), \quad k \in K_f, \quad r(\theta) \in K_\infty^+$$

(H.4) $\mathbf{f}$ is slowly increasing,

(H.5) as a function on G_∞, $\mathbf{f}$ is smooth,

(H.6) $\mathbf{f}$ is an eigenfunction of Δ_v for all archimedean places v, more precisely

$$\Delta_v\,\mathbf{f} = -\frac{k_v}{2}\left(\frac{k_v}{2} - 1\right)\mathbf{f} \qquad \text{for all archimedean places } v\ ,$$

(H.7) the *cuspidality condition*

$$\int_{\mathbb{A}_F/F} \mathbf{f}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0 \qquad \text{for almost every } g \in \mathrm{GL}_2(\mathbb{A}_F)\ .$$

Then

(i) we call $\mathbf{f}$ a *(holomorphic) Hilbert automorphic form for the group K_f of weight* $\mathbf{k}$ *with character* χ if it satisfies conditions **(H.2)**–**(H.6)**, and we denote the space of all such $\mathbf{f}$ by $\mathcal{H}_\mathbf{k}(K_f, \chi)$,

(ii) we say that $\mathbf{f} \in \mathcal{H}_\mathbf{k}(K_f, \chi)$ has *central character* ω if it satisfies **(H.1)**, and we denote the space of all such $\mathbf{f}$ by $\mathcal{H}_\mathbf{k}(K_f, \chi, \omega)$,

(iii) we say that $\mathbf{f} \in \mathcal{H}_\mathbf{k}(K_f, \chi)$ is a *(holomorphic) Hilbert automorphic cusp form* if it satisfies **(H.7)**. We denote the space of all such cusp forms by $\mathcal{H}_\mathbf{k}^0(K_f, \chi)$ and the space of all such cusp forms with central character ω by $\mathcal{H}_\mathbf{k}^0(K_f, \chi, \omega)$.

Remark 2.2.10. Not every $\mathbf{f} \in \mathcal{H}_\mathbf{k}(K_f, \chi)$ has a central character because the existence of a größencharacter as in **(H.1)** cannot be deduced from **(H.2)**–**(H.6)**. It can be shown, however, that $\mathcal{H}_\mathbf{k}(K_f, \chi)$ is a direct sum of finitely many spaces $\mathcal{H}_\mathbf{k}(K_f, \chi, \omega)$ where ω runs through a suitable set of größencharacters (see for example [28, § 3.1, Corollary, p. 95]). We will only prove this statement for a specific group K_f (see Proposition 2.2.13 below). □

Theorem 2.2.11. *The space of Hilbert automorphic forms is finite-dimensional,*

$$\dim \mathcal{H}_{\mathbf{k}}(K_f,\chi) < \infty \ .$$

Proof. See [6, § 4, 4.3]. □

From now on we will assume that K_f is the group $K_0(\mathfrak{d},\mathfrak{n})$ given by

$$K_0(\mathfrak{d},\mathfrak{n}) := \prod_{\mathfrak{p}<\infty} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathfrak{o}_\mathfrak{p} & \mathfrak{d}_\mathfrak{p}^{-1} \\ \mathfrak{n}_\mathfrak{p}\mathfrak{d}_\mathfrak{p} & \mathfrak{o}_\mathfrak{p} \end{pmatrix} \;\middle|\; ad-bc \in \mathfrak{o}_\mathfrak{p}^* \right\}$$

where $\mathfrak{d}$ is the different of the number field F and $\mathfrak{n}$ is an integral ideal of F. We also make a specific choice for the character χ under consideration: Fix a character

$$\chi_0 : (\mathfrak{o}_F/\mathfrak{n})^* \to \mathbb{C}^* \ .$$

The lower right entry d of a matrix in $K_0(\mathfrak{d},\mathfrak{n})$ is an element in $\prod_\mathfrak{p} \mathfrak{o}_\mathfrak{p}$ that is a unit at every $\mathfrak{p} \mid \mathfrak{n}$. Any such element determines a unique $d_\mathfrak{n} \in (\mathfrak{o}_F/\mathfrak{n})^*$ such that $d_\mathfrak{n} \equiv d_\mathfrak{p} \bmod \mathfrak{p}$ for all $\mathfrak{p} \mid \mathfrak{n}$. Thus χ_0 gives rise to a map

$$\chi : K_0(\mathfrak{d},\mathfrak{n}) \to \mathbb{C}^* \ , \qquad \chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \chi_0(d_\mathfrak{n} \bmod \mathfrak{n}) \ .$$

It is readily verified that this map is multiplicative and hence a character of $K_0(\mathfrak{d},\mathfrak{n})$. Note that $\chi = 1$ if and only if $\chi_0 = 1$. We will always assume that χ is of this form.

Lemma 2.2.12. *If $\mathcal{H}_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),\chi,\omega\big) \neq \{0\}$ then χ and ω satisfy the following conditions:*

(i) $\omega(z) = \chi\left(\left(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right)\right) = \chi_0(z_\mathfrak{n} \bmod \mathfrak{n})$ *for all* $z \in \widehat{\mathfrak{o}_F^*}$.

(ii) $\omega(z) = \operatorname{sgn}(z)^{\mathbf{k}}$ *for all* $z \in F_\infty^*$.

In particular, if $\chi = 1$ then ω is unramified at every non-archimedean place.

Proof. Part *(i)* follows from **(H.1)** and **(H.3)** since $z \in \widehat{\mathfrak{o}_F^*}$, viewed as a scalar matrix, lies in both $Z(\mathbb{A}_F)$ and K_f. In order to prove *(ii)* observe that ω is unitary, so every archimedean component ω_v must be either 1 or sgn (cf. Example 2.1.4). Apply **(H.1)** and **(H.3)** to elements of the form

$$r\big((0,\dots,0,\pi,0,\dots,0)\big) = (I_2,\dots,I_2,-I_2,I_2,\dots,I_2)$$

(I_2 the (2×2)-identity matrix). It then becomes evident that $\omega_v = \operatorname{sgn}^{k_v}$ for every archimedean place v. □

Proposition 2.2.13. *The spaces of Hilbert automorphic forms and Hilbert automorphic cusp forms can be decomposed in the following way*

$$\mathcal{H}_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),\chi\big) = \bigoplus_{\omega} \mathcal{H}_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),\chi,\omega\big)\ ,$$
$$\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),\chi\big) = \bigoplus_{\omega} \mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),\chi,\omega\big)$$

where the sums on the right hand side run over all größencharacters ω such that the conditions (i) and (ii) of the previous lemma are satisfied.

Proof. Let $r \geq 0$ be a non-negative integer. For any prime ideal $\mathfrak{p} \nmid \mathfrak{n}$, choose a prime element $\pi_{\mathfrak{p}} \in \mathfrak{o}_{\mathfrak{p}}$. We claim that for $\mathbf{f} \in \mathcal{H}_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),\chi\big)$ and $x \in \mathrm{GL}_2(\mathbb{A}_F)$, the expression $\mathbf{f}(\pi_{\mathfrak{p}}^r x)$ does not depend on the choice of $\pi_{\mathfrak{p}}$. Indeed, if $\pi_{\mathfrak{p}}$ and $\tilde{\pi}_{\mathfrak{p}}$ are prime elements in $\mathfrak{o}_{\mathfrak{p}}$, which we embed into $\mathbb{A}_F$, then $\pi_{\mathfrak{p}}^{-1}\tilde{\pi}_{\mathfrak{p}} = \varepsilon$ where $\varepsilon_{\mathfrak{p}} \in \mathfrak{o}_{\mathfrak{p}}^*$ and $\varepsilon_v = 1$ for all other places. Thus, the diagonal matrix $\left(\begin{smallmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{smallmatrix}\right)$ lies in $K_0(\mathfrak{d},\mathfrak{n})$ and $\chi\left(\left(\begin{smallmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{smallmatrix}\right)\right) = 1$ since $\mathfrak{p} \nmid \mathfrak{n}$. By property **(H.3)** it follows that $\mathbf{f}(\tilde{\pi}_{\mathfrak{p}}^r x) = \mathbf{f}(\pi_{\mathfrak{p}}^r x \varepsilon^r) = \mathbf{f}(\pi_{\mathfrak{p}}^r x)$ for all $\mathbf{f} \in \mathcal{H}_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),\chi\big)$ and all $x \in \mathrm{GL}_2(\mathbb{A}_F)$. Now consider the operator $S(\mathfrak{p}^r)$ on $\mathcal{H}_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),\chi\big)$ defined by

$$\big(\mathbf{f}|S(\mathfrak{p}^r)\big)(x) := \mathbf{f}(\pi_{\mathfrak{p}}^r x) \qquad \text{for all } x \in \mathrm{GL}_2(\mathbb{A}_F)$$

(cf. [66, § 2]). It is clear that $S(\mathfrak{p})$ maps $\mathcal{H}_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),\chi\big)$ onto itself, so it is well-defined. Furthermore, $S(\mathfrak{p})$ is of finite order because $S(\mathfrak{p})^{h^+} = S(\mathfrak{p}^{h^+}) = S(1)$, which is the identity operator. As $\mathcal{H}_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),\chi\big)$ is finite-dimensional, each $S(\mathfrak{p})$ must be diagonalizable.

The operators $S(\mathfrak{p})$ for different prime ideals $\mathfrak{p} \nmid \mathfrak{n}$ commute. Hence, $\mathcal{H}_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),\chi\big)$ has a basis of simultaneous eigenfunctions of all such $S(\mathfrak{p})$. By [66, Prop. 2.1], any such eigenfunction is an automorphic form for some größencharacter ω of finite order. Thus, $\mathcal{H}_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),\chi\big)$ is the direct sum of the spaces $\mathcal{H}_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),\chi,\omega\big)$, which are $\{0\}$ whenever the conditions *(i)* and *(ii)* of Lemma 2.2.12 are not fulfilled.

The above argumentation remains valid when considering the subspace of cusp forms. □

Corollary 2.2.14. *The spaces of Hilbert automorphic forms and Hilbert automorphic cusp forms of trivial character can be decomposed in the following way*

$$\mathcal{H}_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big) = \bigoplus_{\omega} \mathcal{H}_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1,\omega\big)\ ,$$
$$\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big) = \bigoplus_{\omega} \mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1,\omega\big)$$

where the sums on the right hand side run over all größencharacters ω such that

$$\omega_v = \mathrm{sgn}^{k_v} \text{ for all } v \mid \infty\ , \qquad \omega_{\mathfrak{p}} \text{ is unramified for all } \mathfrak{p} < \infty\ .$$

Proof. Apply the previous proposition to $\chi = 1$. □

2.3 Correspondence between classical and adelic Hilbert modular forms

We have defined classical Hilbert modular forms as certain functions on $\mathbb{H}^n$ and Hilbert automorphic forms as functions on $\mathrm{GL}_2(\mathbb{A}_F)$. The aim in this section is to describe the relationship between both. Readers familiar with the theory of elliptic modular forms will know that with the help of the strong approximation theorem for $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$,

$$\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2^+(\mathbb{R})K_0 \qquad \text{for a suitable } K_0 \subseteq \prod_p \mathrm{GL}_2(\mathbb{Z}_p) \ ,$$

a modular form $f \in S_k(\Gamma, \chi)$ induces the automorphic form $\phi_f : \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ given by

$$\phi_f(g) := j(g_\infty, i)^{-k}\chi(k_0)f(g_\infty \cdot i) \qquad \text{for } g = \gamma g_\infty k_0 \in \mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2^+(\mathbb{R})K_0$$

(see [29, Prop. 3.1]). In the case of an arbitrary number field F, the approximation theorem becomes more complicated since it must take into account the possibly nontrivial class number of F. As a consequence, we will encounter some technical difficulties in establishing the correspondence between classical and adelic automorphic forms.

The main reference for this section is [66, § 2].

As usual, let F be a totally real number field of narrow class number h^+. For an integral ideal $\mathfrak{n}$ and a fractional ideal $\mathfrak{c}$ of F, we put

$$\Gamma_0(\mathfrak{c}, \mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathfrak{o}_F & \mathfrak{c}^{-1} \\ \mathfrak{n}\mathfrak{c} & \mathfrak{o}_F \end{pmatrix} \;\middle|\; ad - bc \in \mathfrak{o}_F^{*+} \right\} \ .$$

Further, let

$\mathfrak{c}_1, \dots, \mathfrak{c}_{h^+}$	be integral representatives of the narrow ideal classes of F ,
$t_1, \dots, t_{h^+} \in \mathbb{A}_F^*$	generators of $\mathfrak{c}_j$ with $t_j^\infty = 1$,

and we fix the matrices

$$x_j = \begin{pmatrix} 1 & 0 \\ 0 & t_j \end{pmatrix} \ , \qquad \text{so that} \qquad (\det(x_j)) = (t_j) = \mathfrak{c}_j \qquad \text{for all } j = 1, \dots, h^+ \ .$$

Theorem 2.3.1 (Strong Approximation Theorem).
We have the following decompositions

$$\mathbb{A}_F = F + (F_\infty \times \widehat{\mathfrak{o}_F}) \ ,$$

$$\mathbb{A}_F^* = \coprod_{j=1}^{h^+} t_j F^*(F_\infty^+ \times \widehat{\mathfrak{o}_F^*}) \ ,$$

$$\mathrm{GL}_2(\mathbb{A}_F) = \coprod_{j=1}^{h^+} \mathrm{GL}_2(F)x_j\big(\mathrm{G}_\infty^+ \times K_0(\mathfrak{d}, \mathfrak{n})\big) \ .$$

Proof. The approximation theorem for the additive group $\mathbb{A}_F$ can be found in [55, Thm. 5-8]. For the multiplicative groups $\mathbb{A}_F^*$ and $\mathrm{GL}_2(\mathbb{A}_F)$ see [71, I § 7]. The proof of a more general version of the Strong Approximation Theorem for an arbitrary linear algebraic group is given in [45], see also [46]. □

Corollary 2.3.2. *The factor group $\mathbb{A}_F/F$ is isomorphic to*

$$\mathbb{A}_F/F \cong (F_\infty \times \widehat{\mathfrak{o}_F})/\mathfrak{o}_F \ .$$

Proof. Use the first part of the previous theorem and the fact that $F \cap (F_\infty \times \widehat{\mathfrak{o}_F}) = \mathfrak{o}_F$. □

Lemma 2.3.3. *Every $g \in \mathrm{GL}_2(\mathbb{A}_F)$ can be written as*

$$g = \gamma \begin{pmatrix} 1 & 0 \\ 0 & t_j \end{pmatrix} g_\infty k_0 \qquad \text{for some} \qquad \gamma \in \mathrm{GL}_2(F), \quad g_\infty \in \mathrm{G}_\infty^+, \quad k_0 \in K_0(\mathfrak{d}, \mathfrak{n}) \ ,$$

where j and the coset $\gamma\Gamma_0(\mathfrak{c}_j\mathfrak{d}, \mathfrak{n})$ are uniquely determined by g. More precisely, if we have

$$g = \gamma \begin{pmatrix} 1 & 0 \\ 0 & t_j \end{pmatrix} g_\infty k_0 = \tilde{\gamma} \begin{pmatrix} 1 & 0 \\ 0 & t_j \end{pmatrix} \tilde{g}_\infty \tilde{k}_0 \ ,$$

then there exists an element $\gamma_0 = \gamma_0^\infty \gamma_0^f \in \Gamma_0(\mathfrak{c}_j\mathfrak{d}, \mathfrak{n})$ such that

$$\tilde{\gamma} = \gamma\gamma_0^{-1} \ , \qquad \tilde{g}_\infty = \gamma_0^\infty g_\infty \qquad \text{and} \qquad \tilde{k}_0 = (x_j^{-1}\gamma_0 x_j)^f k_0 \ .$$

Proof. The existence of the decomposition as well as the uniqueness of j follow from the Strong Approximation Theorem 2.3.1. If

$$g = \gamma x_j g_\infty k_0 = \tilde{\gamma} x_j \tilde{g}_\infty \tilde{k}_0 \qquad \text{where } x_j = \begin{pmatrix} 1 & 0 \\ 0 & t_j \end{pmatrix}$$

then it is immediate that $\tilde{\gamma}^{-1}\gamma = x_j \tilde{g}_\infty \tilde{k}_0 (g_\infty k_0)^{-1} x_j^{-1}$ lies in

$$\mathrm{GL}_2(F) \cap x_j \left(\mathrm{G}_\infty^+ \times K_0(\mathfrak{d}, \mathfrak{n})\right) x_j^{-1} = \Gamma_0(\mathfrak{c}_j\mathfrak{d}, \mathfrak{n}) \ .$$

Let us call this element γ_0, so $\tilde{\gamma} = \gamma\gamma_0^{-1}$, and hence $\tilde{g}_\infty \tilde{k}_0 = x_j^{-1}\gamma_0 x_j g_\infty k_0$. In this last expression, we compare the archimedean and non-archimedean parts, respectively, recalling that x_j^∞ is trivial, to get the desired result. □

For the rest of this section, we fix a character

$$\chi_0 : (\mathfrak{o}_F/\mathfrak{n})^* \to \mathbb{C}^*$$

and the corresponding character

$$\chi : K_0(\mathfrak{d},\mathfrak{n}) \to \mathbb{C}^* \ , \qquad \chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \chi_0(d_{\mathfrak{n}} \bmod \mathfrak{n})$$

of $K_0(\mathfrak{d},\mathfrak{n})$. For all $j = 1,\dots,h^+$ put

$$\chi_j(\gamma) := \chi\big((x_j^{-1}\gamma x_j)^f\big)^{-1} = \chi(x_j^{-1}\gamma^f x_j)^{-1} \qquad \text{for } \gamma \in \Gamma_0(\mathfrak{c}_j\mathfrak{d},\mathfrak{n}) \ .$$

It is clear that each χ_j is a well-defined character of $\Gamma_0(\mathfrak{c}_j\mathfrak{d},\mathfrak{n})$ since $x_j^{-1}\gamma^f x_j \in K_0(\mathfrak{d},\mathfrak{n})$ whenever $\gamma \in \Gamma_0(\mathfrak{c}_j\mathfrak{d},\mathfrak{n})$. The characters χ_0, χ and χ_j for $j = 1,\dots,h^+$ will be fixed throughout the rest of this section.

After these preparatory remarks, we are now ready to establish the correspondence between automorphic and classical Hilbert modular forms.

Proposition 2.3.4. *Let $\mathbf{f} \in \mathcal{H}_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),\chi\big)$ be a Hilbert automorphic form for the group $K_0(\mathfrak{d},\mathfrak{n})$ of weight $\mathbf{k}$ with character χ. For $j \in \{1,\dots,h^+\}$ put*

$$f_j : \mathbb{H}^n \to \mathbb{C} \ , \qquad z \mapsto j(g_\infty,\mathbf{i})^{\mathbf{k}}\,\mathbf{f}(x_j g_\infty)$$

where $g_\infty \in G_\infty^+$ is an arbitrary element satisfying $g_\infty\mathbf{i} = z$. Then

$$f_j \in M_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_j\mathfrak{d},\mathfrak{n}),\chi_j\big) \qquad \textit{for all } j = 1,\dots,h^+ \ .$$

Moreover, if $\mathbf{f} \in \mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),\chi\big)$ then $f_j \in S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_j\mathfrak{d},\mathfrak{n}),\chi_j\big)$ for all $j = 1,\dots,h^+$.

Proof. Let $z \in \mathbb{H}^n$. First, we need to show that the definition of f_j does not depend on the particular choice of g_∞. Suppose that $g_\infty, h_\infty \in G_\infty^+$ are such that $g_\infty\mathbf{i} = h_\infty\mathbf{i} = z$. Then $g_\infty^{-1}h_\infty$ stabilizes $\mathbf{i}$ and must therefore be an element $r(\theta)$ in $\mathrm{SO}_2(\mathbb{R})^n = K_\infty^+$. Thus,

$$j(h_\infty,\mathbf{i})^{\mathbf{k}}\,\mathbf{f}(x_j h_\infty) = j\big(g_\infty, r(\theta)\mathbf{i}\big)^{\mathbf{k}} j\big(r(\theta),\mathbf{i}\big)^{\mathbf{k}}\,\mathbf{f}\big(x_j g_\infty r(\theta)\big) = j(g_\infty,\mathbf{i})^{\mathbf{k}} e^{i\mathbf{k}\theta} e^{-i\mathbf{k}\theta}\mathbf{f}(x_j g_\infty) \ ,$$

which shows that f_j is well-defined. Now, take an element $\gamma = \gamma^\infty\gamma^f \in \Gamma_0(\mathfrak{c}_j\mathfrak{d},\mathfrak{n})$. Then $\gamma z = \gamma g_\infty\mathbf{i} = \gamma^\infty g_\infty\mathbf{i}$, hence

$$\begin{aligned}\big(f_j|_{\mathbf{k}}\gamma\big)(z) &= j(\gamma,z)^{-\mathbf{k}} f_j(\gamma z) = j(\gamma,z)^{-\mathbf{k}} j(\gamma g_\infty,\mathbf{i})^{\mathbf{k}}\,\mathbf{f}(x_j\gamma^\infty g_\infty) \\ &= j(\gamma,z)^{-\mathbf{k}} j(\gamma,g_\infty\mathbf{i})^{\mathbf{k}} j(g_\infty,\mathbf{i})^{\mathbf{k}}\,\mathbf{f}(x_j\gamma^\infty g_\infty) = j(g_\infty,\mathbf{i})^{\mathbf{k}}\,\mathbf{f}(x_j\gamma^\infty g_\infty) \ .\end{aligned}$$

Since the archimedean part of x_j is trivial, x_j commutes with $\gamma^\infty g_\infty$. Further note that $\mathbf{f}$ is left-$\mathrm{GL}_2(F)$-invariant and that $x_j^{-1}\gamma^f x_j \in K_0(\mathfrak{d},\mathfrak{n})$. Thus,

$$\begin{aligned}\big(f_j|_{\mathbf{k}}\gamma\big)(z) &= j(g_\infty,\mathbf{i})^{\mathbf{k}}\,\mathbf{f}(\gamma^{-1}\gamma^\infty g_\infty x_j) = j(g_\infty,\mathbf{i})^{\mathbf{k}}\,\mathbf{f}\big((\gamma^f)^{-1} g_\infty x_j\big) \\ &= j(g_\infty,\mathbf{i})^{\mathbf{k}}\,\mathbf{f}\big(g_\infty x_j (x_j^{-1}\gamma^f x_j)^{-1}\big) = \chi(x_j^{-1}\gamma^f x_j)^{-1}\, j(g_\infty,\mathbf{i})^{\mathbf{k}}\,\mathbf{f}(g_\infty x_j) \\ &= \chi_j(\gamma)\, j(g_\infty,\mathbf{i})^{\mathbf{k}}\,\mathbf{f}(x_j g_\infty) = \chi_j(\gamma) f_j(z) \ .\end{aligned}$$

To complete the proof of the first assertion, it remains to show that the functions f_j are holomorphic. This is a consequence of $\mathbf{f}$ being smooth and satisfying the Laplace differential equation. We will not carry out the details since they would require a lengthy discussion of the weight lowering operator (cf. for example [8, § 2.1, Eq. (1.2)]). However, the proof is analogous to the case $F = \mathbb{Q}$, which can be found in [29, Proof of Prop. 2.1] and [30, Ch. I, § 4.5].

Finally suppose that $\mathbf{f}$ is a cusp form. We need to verify that

$$\int_{\mathbb{R}^n/\mathfrak{o}_F} (f_j|_{\mathbf{k}}\gamma)(x+iy)\,dx = 0 \qquad \text{for all } j = 1,\dots,h^+ \text{ and all } \gamma \in \mathrm{GL}_2^+(F) \ .$$

Here, y is an arbitrary element in $\mathbb{R}^n_{>0}$. We choose $y = (1,\dots,1)$ and put

$$g_\infty := \left(\begin{pmatrix} 1 & x_v \\ 0 & 1 \end{pmatrix}\right)_{v=1}^{n} , \qquad \text{so that} \qquad g_\infty \mathbf{i} = x + iy =: z \ .$$

Note that $j(g_\infty, \mathbf{i}) = 1$. Now fix $j \in \{1,\dots,h^+\}$ and some $\gamma \in \mathrm{GL}_2(F)$. A short calculation, similar to those above, yields $(f_j|_{\mathbf{k}}\gamma)(z) = \mathbf{f}(x_j\gamma g_\infty)$. Splitting γ into its archimedean part γ^∞ and its non-archimedean part γ^f and making use of **(H.2)**, we can even show

$$\big(f_j|_{\mathbf{k}}\gamma\big)(z) = \mathbf{f}(g_\infty h) \qquad \text{where} \qquad h := (\gamma^f)^{-1} x_j \gamma^f \in \mathrm{GL}_2(\mathbb{A}) \ .$$

Thus we can apply Corollary 2.3.2 to derive the following identity from the cuspidality condition **(H.7)** on $\mathbf{f}$

$$\int_{\mathbb{R}^n/\mathfrak{o}_F} \big(f_j|_{\mathbf{k}}\gamma\big)(z)\,dx = \int_{\mathbb{R}^n/\mathfrak{o}_F} \mathbf{f}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h\right) dx = \mathrm{const}\cdot \int_{\mathbb{A}_F/F} \mathbf{f}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h\right) dx = 0 \,,$$

which proves that each f_j is a cusp form. □

Proposition 2.3.5. *For every $j = 1,\dots,h^+$, let there be given a Hilbert modular form*

$$f_j \in M_{\mathbf{k}}(\Gamma_0(\mathfrak{c}_j\mathfrak{d},\mathfrak{n}),\chi_j) \ .$$

The function $\mathbf{f} : \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{C}$ defined by

$$\mathbf{f}(\gamma x_j g_\infty k_0) := \chi(k_0)\,(f_j|_{\mathbf{k}} g_\infty)\,(\mathbf{i}) = \chi(k_0) j(g_\infty,\mathbf{i})^{-\mathbf{k}} f_j(g_\infty \mathbf{i}) \ ,$$

where the notation is as in Lemma 2.3.3, is an element of $\mathcal{H}_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),\chi\big)$. Moreover, if all f_j are cusp forms then so is $\mathbf{f}$.

Proof. Lemma 2.3.3 guarantees that $\mathbf{f}$ is well-defined since for $f_j \in M_{\mathbf{k}}(\Gamma_0(\mathfrak{c}_j\mathfrak{d},\mathfrak{n}),\chi_j)$ and $\gamma_0 \in \Gamma_0(\mathfrak{c}_j\mathfrak{d},\mathfrak{n})$, the identities

$$\chi((x_j^{-1}\gamma_0 x_j)^f k_0) = \chi_j(\gamma_0)^{-1}\chi(k_0) \quad \text{and} \quad f|_{\mathbf{k}}(\gamma_0^\infty g_\infty) = (f|_{\mathbf{k}}\gamma_0^\infty)|_{\mathbf{k}} g_\infty = \chi_j(\gamma_0) f|_{\mathbf{k}} g_\infty$$

hold. Now, verify the conditions **(H.2)**–**(H.6)** of Definition 2.2.9:

(H.2) The $\mathrm{GL}_2(F)$-invariance is obvious. Let $z \in Z_\infty^+$. If $g = \gamma x_j g_\infty k_0$ as before then

$$\mathbf{f}(zg) = \chi(k_0) j(zg_\infty, \mathbf{i})^{-\mathbf{k}} f_j(zg_\infty \mathbf{i}) = z^{-\mathbf{k}} (z^2)^{\frac{\mathbf{k}}{2}} \chi(k_0) j(g_\infty, \mathbf{i})^{-\mathbf{k}} f_j(g_\infty \mathbf{i}) = \mathbf{f}(g)$$

since z is totally positive.

(H.3) For $k \in K_0(\mathfrak{d}, \mathfrak{n})$, $r(\theta) \in \mathrm{SO}_2(\mathbb{R})^n$ and $g = \gamma x_j g_\infty k_0$ as before,

$$\begin{aligned}\mathbf{f}(gkr(\theta)) &= \chi(k_0 k) j(g_\infty r(\theta), \mathbf{i})^{-\mathbf{k}} f_j(g_\infty r(\theta)\mathbf{i}) \\ &= \chi(k_0)\chi(k) j(g_\infty, \mathbf{i})^{-\mathbf{k}} j(r(\theta), \mathbf{i})^{-\mathbf{k}} f_j(g_\infty \mathbf{i}) = \chi(k) e^{-i\mathbf{k}\theta} \mathbf{f}(g) \ .\end{aligned}$$

(H.4) We have already seen in Remark 1.1.6 that the functions f_j are regular at the cusps. From this, it follows that $\mathbf{f}$ is slowly increasing (cf. [49, § 2]).

(H.5) Smoothness is clear since the f_j are holomorphic.

(H.6) For an arbitrary archimedean place v, we view $\mathbf{f}$ as a function on the v-th component of $G_\infty^+ \subseteq \mathrm{GL}_2(\mathbb{A}_F)$. So for an element $g = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})^+$,

$$\mathbf{f}(g) = C(ci + d)^{-k_v} \det(g)^{k_v/2} \tilde{f}_j(gi)$$

where $\tilde{f}_j$ is the function f_j viewed as a function on the v-th component of $\mathbb{H}^n$ alone and $C = \chi(k_0) j(g_\infty, \mathbf{i})^{-\mathbf{k}} (ci + d)^{k_v} \det(g)^{-k_v/2}$ is a constant independent of g. In Iwasawa coordinates (u, x, y, θ), as introduced in (2.2), the function $\mathbf{f}$ becomes

$$\mathbf{f}(u, x, y, \theta) = C(uy^{-1/2} e^{i\theta})^{-k_v} u^{k_v} \tilde{f}_j(x + iy) = C y^{k_v/2} e^{-ik_v\theta} \tilde{f}_j(x + iy) \ .$$

In order to evaluate $\Delta_v \mathbf{f}$ we need the second order partial derivatives $\partial^2/\partial x^2$, $\partial^2/\partial y^2$ and $\partial^2/\partial x \partial\theta$ of $\mathbf{f}$. It is easily verified that they are

$$\begin{aligned}&\frac{\partial^2}{\partial x^2}\mathbf{f}(u, x, y, \theta) = C\, y^{k_v/2}\, e^{-ik_v\theta}\, \tilde{f}_j''(z) \ , \\ &\frac{\partial^2}{\partial y^2}\mathbf{f}(u, x, y, \theta) = Ce^{-ik_v\theta} \left(\frac{k_v}{2}\left(\frac{k_v}{2} - 1\right) y^{\frac{k_v}{2}-2} \tilde{f}_j(z) + ik_v y^{\frac{k_v}{2}-1} \tilde{f}_j'(z) - y^{\frac{k_v}{2}} \tilde{f}_j''(z)\right) \ , \\ &\frac{\partial^2}{\partial x \partial\theta}\mathbf{f}(u, x, y, \theta) = -C\, y^{k_v/2}\, ik_v e^{-ik_v\theta}\, \tilde{f}_j'(z) \ ,\end{aligned}$$

where $z = x + iy$. Hence the Laplacian satisfies

$$\Delta_v \mathbf{f}(u, x, y, \theta) = -\frac{k}{2}\left(\frac{k}{2} - 1\right) \mathbf{f}(u, x, y, \theta) \ .$$

That $\mathbf{f}$ is a cusp form if all f_j's are cusp forms can be verified by a calculation similar to the one in the proof of Proposition 2.3.4: Let $h \in \mathrm{GL}_2(\mathbb{A}_F)$ and $x \in \mathbb{A}_F$. Choose an arbitrary element $r \in F^*$ satisfying $\mathrm{sgn}(r_v) = \mathrm{sgn}(\det h_v)$ for all archimedean places $v \mid \infty$. Because of the $\mathrm{GL}_2(F)$-invariance of $\mathbf{f}$ we get

$$\mathbf{f}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h\right) = \mathbf{f}\left(\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h\right) = \mathbf{f}\left(\begin{pmatrix} 1 & rx \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} h\right) \ .$$

By replacing h by $\left(\begin{smallmatrix} r & 0 \\ 0 & 1 \end{smallmatrix}\right) h$ we may therefore assume without loss of generality that the archimedean part h^∞ of h has a totally positive determinant, i. e. $h^\infty \in G_\infty^+$. Using a similar argument and the Strong Approximation Theorem 2.3.1 we may further assume that

$$x = x^\infty + x^f \qquad \text{where} \qquad x^\infty \in F_\infty,\ x^f \in \widehat{\mathfrak{o}_F}\ .$$

As before write

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h = \gamma x_j g_\infty k_0 \qquad \text{where} \qquad \gamma \in \mathrm{GL}_2(F),\ g_\infty \in G_\infty^+,\ k_0 \in K_0(\mathfrak{d}, \mathfrak{n})\ ,$$

so that

$$\begin{pmatrix} 1 & x^\infty \\ 0 & 1 \end{pmatrix} h^\infty = \gamma^\infty g_\infty \qquad \text{and} \qquad \begin{pmatrix} 1 & x^f \\ 0 & 1 \end{pmatrix} h^f = \gamma^f x_j k_0\ .$$

In fact, $\gamma \in \mathrm{GL}_2^+(F)$ because of our assumption on h^∞. It is easily verified that

$$\begin{aligned} j(g_\infty, \mathbf{i}) &= j\left(\gamma^{-1} \begin{pmatrix} 1 & x^\infty \\ 0 & 1 \end{pmatrix} h^\infty, \mathbf{i}\right) = j(\gamma^{-1}, z) j\left(\begin{pmatrix} 1 & x^\infty \\ 0 & 1 \end{pmatrix}, h^\infty \mathbf{i}\right) j(h^\infty, \mathbf{i}) \\ &= j(\gamma^{-1}, z) j(h^\infty, \mathbf{i}) \qquad \text{where} \qquad z := \begin{pmatrix} 1 & x^\infty \\ 0 & 1 \end{pmatrix} h^\infty \mathbf{i} \in \mathbb{H}^n\ . \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{f}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h\right) &= \chi(k_0) j(g_\infty, \mathbf{i})^{-\mathbf{k}} f_j(g_\infty \mathbf{i}) \\ &= \chi(k_0) j(\gamma^{-1}, z)^{-\mathbf{k}} j(h^\infty, \mathbf{i})^{-\mathbf{k}} f_j(\gamma^{-1} z) \\ &= \chi(k_0) j(h^\infty, \mathbf{i})^{-\mathbf{k}} \big(f_j|_{\mathbf{k}} \gamma^{-1}\big)(z)\ . \end{aligned}$$

Note that $j(h^\infty, \mathbf{i})$ does not depend on x, and neither does $\chi(k_0)$ because χ is trivial on integral matrices of the form $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$. Therefore

$$\int_{\mathbb{A}_F/F} \mathbf{f}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h\right) dx = \mathrm{const} \cdot \int_{\mathbb{R}^n/\mathfrak{o}_F} \big(f_j|_{\mathbf{k}} \gamma^{-1}\big)(z)\, dx = 0\ .$$

Which concludes the proof that $\mathbf{f}$ is a cusp form. □

Remark 2.3.6. Clearly, if all f_j are 0 then so will be $\mathbf{f}$. Hence, if we wish to construct a non-trivial $\mathbf{f}$ we need to make sure that at least one of the f_j's does not vanish. This, according to Proposition 1.1.8*(iii)*, can only be the case if there is at least one $j \in \{1, \dots, h^+\}$ such that

$$\mathrm{sgn}(\varepsilon)^{\mathbf{k}} = \chi_j\left(\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}\right) \qquad \text{for all } \varepsilon \in \mathfrak{o}_F^*\ .$$

But this really amounts to a condition on χ_0 because of the relation

$$\chi_j\left(\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}\right) = \chi\left(\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}^f\right)^{-1} = \chi_0(\varepsilon \bmod \mathfrak{n})^{-1} \qquad \text{for all } \varepsilon \in \mathfrak{o}_F^*\ .$$

As $\mathrm{sgn}(\varepsilon) = \mathrm{sgn}(\varepsilon^{-1})$ we may even omit the exponent -1 and so obtain the equation

$$\mathrm{sgn}(\varepsilon)^{\mathbf{k}} = \chi_0(\varepsilon \bmod \mathfrak{n}) \qquad \text{for all } \varepsilon \in \mathfrak{o}_F^* \ ,$$

which must be satisfied if $\mathbf{f} \neq 0$. Recall that we have encountered this condition before as it is implicitly stated in Lemma 2.2.12. Now it becomes clear that this lemma can be viewed as the adelic version of Proposition 1.1.8*(iii)*. □

We finish this section by summarizing the results of the previous propositions and thus obtain the final version of a correspondence theorem between classical and adelic Hilbert modular forms.

Theorem 2.3.7. *The map* $(f_1, \ldots, f_{h^+}) \mapsto \mathbf{f}$ *constructed in Proposition 2.3.5 induces* $\mathbb{C}$-*vector space isomorphisms*

$$\bigoplus_{j=1}^{h^+} M_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_j\mathfrak{d}, \mathfrak{n}), \chi_j\big) \xrightarrow{\ \sim\ } \mathcal{H}_{\mathbf{k}}\big(K_0(\mathfrak{d}, \mathfrak{n}), \chi\big)$$

$$\textit{and} \qquad \bigoplus_{j=1}^{h^+} S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_j\mathfrak{d}, \mathfrak{n}), \chi_j\big) \xrightarrow{\ \sim\ } \mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d}, \mathfrak{n}), \chi\big) \ .$$

The inverse map is described in Proposition 2.3.4.

Proof. Proposition 2.3.5 shows that the map $(f_1, \ldots, f_{h^+}) \mapsto \mathbf{f}$ is a well-defined homomorphism between the given spaces, which maps tuples of cusp forms to cusp forms. It is clear, by construction, that the map $\mathbf{f} \mapsto (f_1, \ldots, f_{h^+})$ introduced in Proposition 2.3.4 is its inverse. □

Chapter 3

The action of the Hecke algebra on automorphic forms

Our next aim is to decompose the space of Hilbert modular forms into subspaces of newforms. Whether we consider classical or adelic modular forms is irrelevant because after the discussion in the previous chapter we know how to translate the one into the other. However, it turns out that our task becomes easier in the adelic setting because here we can make full use of representation theoretic methods. More precisely, the decomposition into spaces of newforms will be achieved by studying the operation of the Hecke algebra on the space of adelic modular forms and its irreducible subrepresentations. This will be explained in detail in Section 3.4.

As a preparation, the first three sections of this chapter are devoted to a short overview of the local and global Hecke algebra of GL_2. Most of the results can be found in [41, § 2] and [32, § 1].

3.1 The local Hecke algebra (non-archimedean case)

As usual, let $F_\mathfrak{p}$ be a $\mathfrak{p}$-adic field. The group $\mathrm{GL}_2(F_\mathfrak{p})$ has a Haar measure dg, which we normalize in such a way that $K := \mathrm{GL}_2(\mathfrak{o}_\mathfrak{p})$, the standard maximal compact open subgroup of $\mathrm{GL}_2(F_\mathfrak{p})$, has volume 1.

Definition 3.1.1 (Hecke algebra of $\mathbf{GL_2(F_\mathfrak{p})}$). We put

$$\mathfrak{H}_\mathfrak{p} := \{f : \mathrm{GL}_2(F_\mathfrak{p}) \to \mathbb{C} \mid f \text{ is locally constant with compact support}\} .$$

Equipped with the convolution product

$$(f_1 * f_2)(g) := \int_{\mathrm{GL}_2(F_\mathfrak{p})} f_1(h) f_2(h^{-1}g)\, dh \ ,$$

the space $\mathfrak{H}_\mathfrak{p}$ becomes an associative algebra, which we call the *Hecke algebra* of $\mathrm{GL}_2(F_\mathfrak{p})$.

Definition 3.1.2 (Admissible representation of $\mathfrak{H}_\mathfrak{p}$). Let $\pi : \mathfrak{H}_\mathfrak{p} \to \mathrm{GL}(E)$ be a representation on a complex vector space E. We say that π is *admissible* if the following two conditions are satisfied:

(i) For any $x \in E$, there exists an $f \in \mathfrak{H}_\mathfrak{p}$ such that $\pi(f)x = x$.

(ii) For any $f \in \mathfrak{H}_\mathfrak{p}$, the space $\pi(f)E$ is finite-dimensional.

Definition 3.1.3 (Induced representation of $\mathfrak{H}_\mathfrak{p}$). For a locally K-finite representation $\pi : \mathrm{GL}_2(F_\mathfrak{p}) \to \mathrm{GL}(E)$ on a Hilbert space E, we define a representation $\tilde{\pi} : \mathfrak{H}_\mathfrak{p} \to \mathrm{GL}(E)$ by

$$\tilde{\pi}(f)x := \int\limits_{\mathrm{GL}_2(F_\mathfrak{p})} f(g)\pi(g)x\, dg \qquad \text{for all } f \in \mathfrak{H}_\mathfrak{p},\ x \in E\ .$$

Remark 3.1.4. Since π is locally K-finite, the stabilizer $\mathrm{Stab}_{\mathrm{GL}_2(F_\mathfrak{p})}(x)$ is an open subgroup of $\mathrm{GL}_2(F_\mathfrak{p})$ for every $x \in E$, as was shown in Lemma 2.1.12. Thus, $g \mapsto \pi(g)x$ is a locally constant function on $\mathrm{GL}_2(F_\mathfrak{p})$, as is f. So we can choose an open covering $\{U_i \mid i \in I\}$ of $\mathrm{GL}_2(F_\mathfrak{p})$ such that $g \mapsto f(g)\pi(g)x$ is constant on each U_i. Since the function f is compactly supported, the integral in Definition 3.1.3 then reduces to a finite sum. □

Proposition 3.1.5. *If $\pi : \mathrm{GL}_2(F_\mathfrak{p}) \to \mathrm{GL}(E)$ is an admissible representation in the sense of Definition 2.1.13 then the induced representation $\tilde{\pi} : \mathfrak{H}_\mathfrak{p} \to \mathrm{GL}(E)$ is also admissible. Conversely, if $\tilde{\pi} : \mathfrak{H}_\mathfrak{p} \to \mathrm{GL}(E)$ is admissible then it is induced by $\pi : \mathrm{GL}_2(F_\mathfrak{p}) \to \mathrm{GL}(E)$ defined by*

$$\pi(g)x := \tilde{\pi}\big(\lambda(g)f\big) \qquad \textit{where } f \in \mathfrak{H}_\mathfrak{p} \textit{ is such that } x = \tilde{\pi}(f)x\ ,$$

which is again admissible.

Proof. For $x \in E$, the stabilizer $\mathrm{Stab}_{\mathrm{GL}_2(F_\mathfrak{p})}(x)$ is open by Lemma 2.1.12. Hence we can find a compact open neighbourhood U of x contained in $\mathrm{Stab}_{\mathrm{GL}_2(F_\mathfrak{p})}(x)$. Then $f := \mathrm{vol}(U)^{-1} \cdot \mathbb{1}_U$, where $\mathbb{1}_U$ is the characteristic function of U, satisfies condition (i) of Definition 3.1.2.

Now let $f \in \mathfrak{H}_\mathfrak{p}$ and let us consider *elementary idempotents*, i. e. functions on $\mathrm{GL}_2(F_\mathfrak{p})$ that vanish outside of K and have the form

$$\xi(k) = \sum_{i=1}^{r} \dim \sigma_i \operatorname{Tr} \sigma_i(k^{-1}) \qquad \text{with } \sigma_i \in \widehat{K} \text{ pairwise inequivalent}$$

on K. It is known that we can find such an elementary idempotent satisfying $f * \xi = \xi * f = f$ (see [33, Ch. I, § 8]). Hence $\pi(f) = \pi(f) \circ \pi(\xi)$. Now, for every $x \in E$ we have

$$\pi(\xi)x = \sum_{i=1}^{r} \int_K \dim \sigma_i \operatorname{Tr}\sigma_i(k^{-1})\, \pi(k)x\, dk = \sum_{i=1}^{r} \operatorname{pr}_{\sigma_i}(x)$$

where $\operatorname{pr}_{\sigma_i} : E \to E(\sigma_i)$ denotes the projection onto the σ_i-isotypic component of π. By assumption, π is admissible and hence $\dim E(\sigma_i) < \infty$ for all i. So we see that $\pi(\xi)$ maps E into a finite-dimensional space, and it follows that $\pi(f)E = \pi(f)\big(\pi(\xi)E\big)$ is also of finite dimension.

For the converse see [41, § 2]. □

Remark 3.1.6. Instead of (ii) in Definition 3.1.2, many authors, among them [41], choose a slightly different condition. They only require that (ii) be satisfied if f is an elementary idempotent. But in light of the proof just given, we see that both definitions are equivalent. □

Although our main interest lies in the admissible representations of $\mathfrak{H}_\mathfrak{p}$, Proposition 3.1.5 allows us to study admissible representations of $\mathrm{GL}_2(F_\mathfrak{p})$ instead. In the remainder of this section we will therefore give the complete classification of the latter.

Let μ_1, μ_2 be quasi-characters of $F_\mathfrak{p}^*$ and

$$\mathcal{B}(\mu_1,\mu_2) := \left\{ f : \mathrm{GL}_2(F_\mathfrak{p}) \to \mathbb{C} \;\middle|\; \text{locally constant, } f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \mu_1(a)\mu_2(d)\left|\frac{a}{d}\right|_\mathfrak{p}^{\frac{1}{2}} f(g) \right\}.$$

Theorem 3.1.7. *The right regular representation*

$$\rho(\mu_1,\mu_2) : \mathrm{GL}_2(F_\mathfrak{p}) \to \mathrm{GL}\big(\mathcal{B}(\mu_1,\mu_2)\big)$$

is admissible.

(i) If $\mu_1\mu_2^{-1} \neq |\cdot|_\mathfrak{p}$ and $\mu_1\mu_2^{-1} \neq |\cdot|_\mathfrak{p}^{-1}$ then $\rho(\mu_1,\mu_2)$ is irreducible. It will be denoted by $\pi(\mu_1,\mu_2)$.

(ii) If $\mu_1\mu_2^{-1} = |\cdot|_\mathfrak{p}$ then the space $\mathcal{B}(\mu_1,\mu_2)$ contains exactly one proper invariant subspace denoted by $\mathcal{B}_s(\mu_1,\mu_2)$, which is infinite-dimensional. The corresponding factor space $\mathcal{B}_f(\mu_1,\mu_2) = \mathcal{B}(\mu_1,\mu_2)/\mathcal{B}_s(\mu_1,\mu_2)$ is one-dimensional.

(iii) If $\mu_1\mu_2^{-1} = |\cdot|_\mathfrak{p}^{-1}$ then $\mathcal{B}(\mu_1,\mu_2)$ contains exactly one proper invariant subspace $\mathcal{B}_f(\mu_1,\mu_2)$, which is one-dimensional and generated by $\chi \circ \det$ for some quasi-character χ of $F_\mathfrak{p}^*$. The factor space $\mathcal{B}_s(\mu_1,\mu_2) = \mathcal{B}(\mu_1,\mu_2)/\mathcal{B}_f(\mu_1,\mu_2)$ is infinite-dimensional.

(iv) Any irreducible admissible representation of $\mathrm{GL}_2(F_\mathfrak{p})$ *not appearing in the above list is called* absolutely cuspidal.

In the cases (ii) and (iii), we denote the representation $\rho(\mu_1,\mu_2)\colon \mathrm{GL}_2(F_\mathfrak{p})\to\mathrm{GL}\big(\mathcal{B}_s(\mu_1,\mu_2)\big)$ *by* $\sigma(\mu_1,\mu_2)$ *and call it a* special representation. *The only equivalences between these representations are*

$$\pi(\mu_1,\mu_2)\sim\pi(\mu_2,\mu_1) \qquad \textit{and} \qquad \sigma(\mu_1,\mu_2)\sim\sigma(\mu_2,\mu_1)\ .$$

Proof. See [41, Thm. 3.3 and Thm 2.7] and [32, § 1.10, Thm. 6 and § 1.11, Thm. 7]. □

Remark 3.1.8. The quasi-character χ mentioned in *(iii)* depends on μ_1,μ_2 in the following way: The condition $\mu_1\mu_2^{-1}=|\cdot|_\mathfrak{p}^{-1}$ implies that $\mu_1=\psi|\cdot|_\mathfrak{p}^r$ and $\mu_2=\psi|\cdot|_\mathfrak{p}^{r+1}$ for some unitary character ψ of $\mathfrak{o}_\mathfrak{p}^*$ and some $r\in\mathbb{C}$ (cf. Example 2.1.4). If we put $\chi:=\mu_1|\cdot|_\mathfrak{p}^{1/2}$ then a simple calculation shows that $\chi\circ\det$ is contained in $\mathcal{B}(\mu_1,\mu_2)$ and must therefore be a generator. Thus, any special representation is of the form

$$\sigma\big(\chi|\cdot|_\mathfrak{p}^{-1/2},\,\chi|\cdot|_\mathfrak{p}^{1/2}\big)\sim\sigma\big(\chi|\cdot|_\mathfrak{p}^{1/2},\,\chi|\cdot|_\mathfrak{p}^{-1/2}\big) \qquad \text{for some quasi-character } \chi \text{ of } F_\mathfrak{p}\ .$$

□

Lemma 3.1.9. *Suppose that* $\mathfrak{n}$ *is an integral square-free ideal of* F. *If* $\pi:\mathrm{GL}_2(F_\mathfrak{p})\to\mathrm{GL}(E)$ *is an absolutely cuspidal representation then* E *contains no element that is invariant under*

$$K_0(\mathfrak{d}_\mathfrak{p},\mathfrak{n}_\mathfrak{p})=\big(K_0(\mathfrak{d},\mathfrak{n})\big)_\mathfrak{p}=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\begin{pmatrix}\mathfrak{o}_\mathfrak{p} & \mathfrak{d}_\mathfrak{p}^{-1}\\ \mathfrak{n}_\mathfrak{p}\mathfrak{d}_\mathfrak{p} & \mathfrak{o}_\mathfrak{p}\end{pmatrix}\ \Big|\ ad-bc\in\mathfrak{o}_\mathfrak{p}^*\right\}\ .$$

Proof. Let $\delta\in F_\mathfrak{p}^*$ be a generator of $\mathfrak{d}_\mathfrak{p}$. Suppose $x\in E$ is $K_0(\mathfrak{d}_\mathfrak{p},\mathfrak{n}_\mathfrak{p})$-invariant. Then a simple calculation shows that

$$\pi\left(\begin{pmatrix}1 & 0\\ 0 & \delta^{-1}\end{pmatrix}\right)x\in E$$

is invariant under the group $K_0(1,\mathfrak{n}_\mathfrak{p})$. But if $\mathfrak{p}\nmid\mathfrak{n}$ then $K_0(1,\mathfrak{n}_\mathfrak{p})=\mathrm{GL}_2(\mathfrak{o}_\mathfrak{p})$, and in this case [41, Lemma 3.9] tells us that π cannot be absolutely cuspidal. If $\mathfrak{p}\mid\mathfrak{n}$ then we apply [65, Lemma 14] to obtain the same result. □

3.2 The local Hecke algebra (real case)

Although Hecke algebras can also be defined for complex places, there is no need for us to do so because we consider only totally real number fields. We will therefore omit the complex case and refer the reader to [41, § 6] for further information.

We choose the Haar measure dg on $\mathrm{GL}_2(\mathbb{R})$ such that its maximal compact subgroup $\mathrm{O}_2(\mathbb{R})$ has volume 1.

Definition 3.2.1 (Hecke algebra for $\mathrm{GL}_2(\mathbb{R})$). Let

$$\mathfrak{H}_1 := \{f \in C_c^\infty(\mathrm{GL}_2(\mathbb{R})) \mid f \text{ is } \mathrm{O}_2(\mathbb{R})\text{-finite on both sides}\}$$

and

$$\mathfrak{H}_2 := \big\langle \text{matrix coefficients of irreducible representations of } \mathrm{O}_2(\mathbb{R}) \big\rangle \ .$$

If we endow $\mathfrak{H}_1$ and $\mathfrak{H}_2$ with the convolution product and define in a natural way

$$(\xi * f)(g) := \int_{\mathrm{O}_2(\mathbb{R})} \xi(k) f(k^{-1}g)\, dk \qquad \text{and} \qquad (f * \xi)(g) := \int_{\mathrm{O}_2(\mathbb{R})} f(gk^{-1})\xi(k)\, dk$$

for $f \in \mathfrak{H}_1$ and $\xi \in \mathfrak{H}_2$ then $\mathfrak{H}_\mathbb{R} := \mathfrak{H}_1 + \mathfrak{H}_2$ becomes an algebra, which we call the *Hecke algebra* of $\mathrm{GL}_2(\mathbb{R})$.

Definition 3.2.2 (Admissible representation of $\mathfrak{H}_\mathbb{R}$). Let $E \subseteq C(\mathrm{GL}_2(\mathbb{R}))$ be some space consisting of continuous functions on $\mathrm{GL}_2(\mathbb{R})$. Let $\pi : \mathfrak{H}_\mathbb{R} \to \mathrm{GL}(E)$ be a representation. We say that π is *admissible* if the following conditions are satisfied:

(i) For any $x \in E$, there exist finitely many $f_i \in \mathfrak{H}_1$ and $x_i \in E$ such that $x = \sum_i \pi(f_i)x_i$.

(ii) For any $f \in \mathfrak{H}_1$, the space $\pi(f)E$ is finite-dimensional.

For $X \in \mathfrak{g}$ and $f \in C_c^\infty(\mathrm{GL}_2(\mathbb{R}))$ define

$$(X * f)(g) := \frac{d}{dt} f\big(\exp(-tX)g\big)\big|_{t=0} \ , \qquad (f * X)(g) := \frac{d}{dt} f\big(g\exp(-tX)\big)\big|_{t=0} \ .$$

If, in particular, $f \in \mathfrak{H}_1$ then also $f * X \in \mathfrak{H}_1$ and $X * f \in \mathfrak{H}_1$.

Proposition 3.2.3. *Let $\pi : \mathfrak{H}_\mathbb{R} \to \mathrm{GL}(E)$ be an admissible representation. We can associate to π representations $\tilde\pi$ on $U(\mathfrak{g}_\mathbb{C})$, $Z(\mathrm{GL}_2(\mathbb{R}))$ and $\mathrm{O}_2(\mathbb{R})$ by putting*

$$\tilde\pi : U(\mathfrak{g}_\mathbb{C}) \to \mathrm{GL}(E) \ , \qquad \tilde\pi(X)\big(\pi(f)v\big) := \pi(X * f)v \ ,$$

$$\tilde\pi : H \to \mathrm{GL}(E) \ , \qquad \tilde\pi(h)\big(\pi(f)v\big) := \pi\big(\lambda(h)f\big)v \qquad \text{for } H = \begin{cases} \mathrm{O}_2(\mathbb{R}) \\ Z\big(\mathrm{GL}_2(\mathbb{R})\big) \end{cases}$$

and extending this definition linearly to arbitrary elements in E. The following formulae hold for all $X \in U(\mathfrak{g}_\mathbb{C}), f \in \mathfrak{H}_1, h \in \mathrm{O}_2(\mathbb{R})$ or $h \in Z(\mathrm{GL}_2(\mathbb{R}))$

$$\tilde\pi(X)\pi(f) = \pi(X * f) \ , \qquad \pi(f)\pi(X) = \pi(f * X) \ ,$$
$$\tilde\pi\big(\mathrm{Ad}(h)X\big) = \tilde\pi(h)\tilde\pi(X)\tilde\pi(h^{-1}) \ .$$

Proof. Cf. [41, § 5]. □

Remark 3.2.4. Note that the definition we just gave for the representation $\tilde{\pi}$ on the subgroups $\mathrm{O}_2(\mathbb{R})$ and $Z(\mathrm{GL}_2(\mathbb{R}))$ cannot be extended to a representation on the whole group $\mathrm{GL}_2(\mathbb{R})$. The reason is that for $f \in \mathfrak{H}_1$ and an arbitrary $g \in \mathrm{GL}_2(\mathbb{R})$, the function $\lambda(g)f$ is not necessarily in $\mathfrak{H}_1$ because the left translation does not preserve left-$\mathrm{O}_2(\mathbb{R})$-finiteness. If, however, $g \in \mathrm{O}_2(\mathbb{R})$ or $g \in Z(\mathrm{GL}_2(\mathbb{R}))$ then $\lambda(g)f \in \mathfrak{H}_1$ and the above definition makes sense. □

As in the non-archimedean case, we want to give a complete list of admissible representations of the local Hecke algebra. It is again useful to consider the space $\mathcal{B}(\mu_1, \mu_2)$, whose definition resembles the non-archimedean situation and will be given now.

Let μ_1, μ_2 be characters of $\mathbb{R}^*$. According to Example 2.1.4, they are necessarily of the form

$$\mu_i(t) = |t|^{s_i}\mathrm{sgn}(t)^{m_i} \qquad \text{for some } s_i \in \mathbb{C} \text{ and } m_i \in \{0,1\} \ .$$

Let $s := s_1 - s_2$ and $m := |m_1 - m_2|$, so that

$$\mu_1\mu_2^{-1}(t) = |t|^s \mathrm{sgn}(t)^m \ .$$

Define

$$\mathcal{B}(\mu_1,\mu_2) \\ := \left\{ \phi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C} \mid \text{right-}\mathrm{SO}_2(\mathbb{R})\text{-finite, } \phi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \mu_1(a)\mu_2(d)\left|\frac{a}{d}\right|^{\frac{1}{2}}\phi(g) \right\} \ .$$

According to [41, § 5], the space $\mathcal{B}(\mu_1, \mu_2)$ is generated by the functions

$$\phi_l\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} r(\theta)\right) := \mu_1(a)\mu_2(d)\left|\frac{a}{d}\right|^{\frac{1}{2}} e^{-il\theta} \qquad \text{where } l \equiv m \bmod 2 \ .$$

Theorem 3.2.5. *The representation*

$$\rho(\mu_1,\mu_2) : \mathfrak{H}_{\mathbb{R}} \to \mathrm{GL}\big(\mathcal{B}(\mu_1,\mu_2)\big) \ , \qquad \big(\rho(\mu_1,\mu_2)f\big)(\phi)(g) := \int_{\mathrm{GL}_2(\mathbb{R})} \phi(gh)f(h)\,dh$$

is admissible.

(i) If $\mu_1\mu_2^{-1}(t)$ is not of the form $t^p\mathrm{sgn}(t)$ for some $0 \neq p \in \mathbb{Z}$ then $\rho(\mu_1,\mu_2)$ is irreducible. It will be denoted by $\pi(\mu_1,\mu_2)$ and will be called a representation of the principal series.

(ii) If $\mu_1\mu_2^{-1}(t) = t^p\mathrm{sgn}(t)$ for $p \in \mathbb{Z}$, $p > 0$ then $\mathcal{B}(\mu_1,\mu_2)$ contains exactly one proper invariant subspace $\mathcal{B}_s(\mu_1,\mu_2)$. This subspace is infinite-dimensional. The quotient space $\mathcal{B}_f(\mu_1,\mu_2) := \mathcal{B}(\mu_1,\mu_2)/\mathcal{B}_s(\mu_1,\mu_2)$ is of finite dimension.

(iii) *If* $\mu_1\mu_2^{-1}(t) = t^p \mathrm{sgn}(t)$ *for* $p \in \mathbb{Z}$, $p < 0$ *then* $\mathcal{B}(\mu_1, \mu_2)$ *contains exactly one proper invariant subspace* $\mathcal{B}_f(\mu_1, \mu_2)$. *This subspace is of finite dimension. The quotient space* $\mathcal{B}_s(\mu_1, \mu_2) := \mathcal{B}(\mu_1, \mu_2)/\mathcal{B}_f(\mu_1, \mu_2)$ *is infinite-dimensional.*

(iv) *The above is a complete list of the irreducible admissible representations of* $\mathfrak{H}_{\mathbb{R}}$.

In the cases (ii) and (iii), we denote any representation $\rho(\mu_1, \mu_2) : \mathfrak{H}_{\mathbb{R}} \to \mathrm{GL}\big(\mathcal{B}_s(\mu_1, \mu_2)\big)$ *by* $\sigma(\mu_1, \mu_2)$ *and call it a* special representation. *Similarly, any representation of the form* $\rho(\mu_1, \mu_2) : \mathfrak{H}_{\mathbb{R}} \to \mathrm{GL}\big(\mathcal{B}_f(\mu_1, \mu_2)\big)$ *will be denoted by* $\pi(\mu_1, \mu_2)$. *The only equivalences between these representations are*

$$\begin{aligned}
&\pi(\mu_1, \mu_2) \sim \pi(\mu_2, \mu_1) \ , \\
&\sigma(\mu_1, \mu_2) \sim \sigma(\mu_2, \mu_1) \sim \sigma(\mu_1 \,\mathrm{sgn},\ \mu_2 \,\mathrm{sgn}) \sim (\mu_2 \,\mathrm{sgn},\ \mu_1 \,\mathrm{sgn}) \ .
\end{aligned}$$

Proof. See [41, Thm. 5.11] and [32, § 2.3]. □

The proof of the previous Classification Theorem makes use of the representations $\tilde{\rho}(\mu_1, \mu_2)$ on $U(\mathfrak{g}_{\mathbb{C}})$, $Z(\mathrm{GL}_2(\mathbb{R}))$ and $\mathrm{O}_2(\mathbb{R})$ that are associated to $\rho(\mu_1, \mu_2)$ via the construction in Proposition 3.2.3. Although we will not explain the proof in detail it is worth noting in which way certain elements of $U(\mathfrak{g}_{\mathbb{C}})$, $Z(\mathrm{GL}_2(\mathbb{R}))$ and $\mathrm{O}_2(\mathbb{R})$ act.

Lemma 3.2.6. *As earlier, define elements* H, L, R, Δ *in* $U(\mathfrak{g}_{\mathbb{C}})$ *by*

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad L := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \qquad R := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and the Casimir (or Laplace) operator by

$$\Delta := -\frac{1}{4}H^2 - \frac{1}{2}RL - \frac{1}{2}LR \ .$$

Then

$$\big(\rho(\mu_1, \mu_2)\Delta\big)\phi = -\frac{s^2 - 1}{4}\phi \qquad \textit{for all } \phi \in \mathcal{B}(\mu_1, \mu_2) \ .$$

Proof. See [41, Lemma 5.6]. □

Lemma 3.2.7. *Let* $\tilde{\rho}(\mu_1, \mu_2)$ *be the representation of* $H = Z\big(\mathrm{GL}_2(\mathbb{R})\big)$ *or* $H = \mathrm{O}_2(\mathbb{R})$ *associated to the representation* $\rho(\mu_1, \mu_2)$ *of* $\mathfrak{H}_{\mathbb{R}}$. *Then*

$$\big(\tilde{\rho}(\mu_1, \mu_2)(h)\big)\phi = \rho(h)\phi \qquad \textit{for all } h \in H \textit{ and } \phi \in \mathcal{B}(\mu_1, \mu_2) \ ,$$

where ρ *denotes the usual right regular representation of* $\mathrm{GL}_2(\mathbb{R})$.

Proof. For the sake of clarity, we write ρ_μ and $\tilde{\rho}_\mu$ instead of $\rho(\mu_1, \mu_2)$ and $\tilde{\rho}(\mu_1, \mu_2)$. By part (i) of Definition 3.2.2 it is sufficient to verify the formula for an element of the form $\rho_\mu(f)\phi$ where $f \in \mathfrak{H}_1$, $\phi \in \mathcal{B}(\mu_1, \mu_2)$. For every $h \in H$,

$$\tilde{\rho}_\mu(h)\big(\rho_\mu(f)\phi\big)(g) = \big(\rho_\mu\big(\lambda(h)f\big)\phi\big)(g) = \int\limits_{\mathrm{GL}_2(\mathbb{R})} \phi(gk)\big(\lambda(h)f\big)(k)\,dk$$

$$= \int\limits_{\mathrm{GL}_2(\mathbb{R})} \phi(ghk)f(k)\,dk = \big(\rho_\mu(f)\phi\big)(gh)\ .$$

Hence $\tilde{\rho}_\mu(h)\phi = \rho(h)\phi$ for all $h \in H$ and $\phi \in \mathcal{B}(\mu_1, \mu_2)$. □

In the light of the previous lemma, we sometimes refer to $\rho(\mu_1, \mu_2)$ as the *right regular representation* of $\mathfrak{H}_\mathbb{R}$.

3.3 The global Hecke algebra

The aim of this chapter, which we will approach in the next section, is to examine the behaviour of the space of Hilbert automorphic forms under admissible representations of the global Hecke algebra $\mathfrak{H}$. However, we hardly need to refer to the global object $\mathfrak{H}$ itself because it is essentially the collection of the local data so that all proofs can and will be carried out locally. Mainly for the sake of completeness, this section provides the basic definitions of the global Hecke algebra and its representations.

Definition 3.3.1 (Global Hecke algebra for $\mathrm{GL}_2(\mathbb{A}_F)$). Let F be a totally real number field, v a place of F,

$$K_v := \begin{cases} \mathrm{O}_2(\mathbb{R}) & \text{if } v \text{ is real,} \\ \mathrm{GL}_2(\mathfrak{o}_v) & \text{if } v \text{ is non-archimedean,} \end{cases}$$

and $\mathbb{1}_{K_v}$ the characteristic function of K_v. The algebra

$$\mathfrak{H} := \big\langle \otimes_v f_v \mid f_v \in \mathfrak{H}_v\ , \text{ and } f_v = \mathbb{1}_{K_v} \text{ for almost all } v \big\rangle$$

is called the *global Hecke algebra for* $\mathrm{GL}_2(\mathbb{A}_F)$.

Definition 3.3.2 (Admissible representations of $\mathfrak{H}$). A representation $\pi : \mathfrak{H} \to \mathrm{GL}(E)$ is called *admissible* if the following conditions are satisfied:

(i) For any $x \in E$, there exist finitely many $f_i \in \mathfrak{H}'$ and $x_i \in E$ such that $x = \sum_i \pi(f_i)x_i$. Here $\mathfrak{H}'$ is a certain subalgebra of $\mathfrak{H}$ the precise definition of which can be found in [41, § 9].

(ii) For any elementary idempotent ξ, the space $\pi(\xi)E$ is finite-dimensional.

(iii) For all $x \in E$, the map

$$\xi_{v_0}\mathfrak{H}_{v_0}\xi_{v_0} \to \pi(\xi)E \ , \qquad f_{v_0} \mapsto \pi\Big(f_{v_0} \otimes \bigotimes_{v \neq v_0} \xi_v\Big)x \qquad \text{(where } v_0 \mid \infty \text{ is fixed)}$$

is continuous. Here, $\xi = \bigotimes_v \xi_v$ where each ξ_v is an elementary idempotent, almost all of which are equal to $\mathbb{1}_{K_v}$.

The reader is referred to [41, § 9] for an extensive discussion of admissible representations of $\mathfrak{H}$. For our purposes it is not relevant to know the exact wording of the definition. It suffices to be aware of the connection between the global and the local admissible representations, which is the content of the following proposition.

Proposition 3.3.3. *For all places v of F, let there be given an admissible representation $\pi_v : \mathfrak{H}_v \to \mathrm{GL}(E_v)$ such that for almost all v, the restriction of π_v to K_v contains the identity representation exactly once. For every v, choose an element $e_v \in E_v$ that is fixed under $\pi_v(K_v)$. Let*

$$E := \langle \otimes_v x_v \mid x_v \in E_v \ \textit{where}\ x_v = e_v \ \textit{for almost every}\ v \rangle \ .$$

Then

$$\pi := \otimes_v \pi_v : \mathfrak{H} \to \mathrm{GL}(E) \ , \qquad \big(\pi(\otimes_v f_v)\big)(\otimes_v x_v) := \otimes_v \big(\pi_v(f_v)x_v\big)$$

defines an admissible representation of $\mathfrak{H}$. Conversely, every irreducible admissible representation of $\mathfrak{H}$ can be obtained in this way.

Proof. See [41, Prop. 9.1]. □

3.4 Representation of the Hecke algebra on the space of Hilbert automorphic forms

To simplify matters we assume from now on that the level $\mathfrak{n}$ is square-free. Further, we consider only weight vectors $\mathbf{k}$ satisfying $k_v \geq 2$ for all $v \mid \infty$.

In this section we would like to use our knowledge of the Hecke algebra $\mathfrak{H}$, in particular the Classification Theorems 3.1.7 and 3.2.5 for the irreducible admissible representations of the local Hecke algebras, to examine the space $\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big)$. Unfortunately, this space is not invariant under the action of $\mathfrak{H}$, so that we cannot immediately apply the results of the previous sections. Instead, we recall from Corollary 2.2.14 the decomposition

$$\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big) = \bigoplus_{\omega} \mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1,\omega\big)$$

where all $\omega_\infty = \mathrm{sgn}^{\mathbf{k}}$ and every $\omega_{\mathfrak{p}}$ is unramified. Each $\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1,\omega\big)$ is a subspace of the larger space $\mathcal{A}_0(\omega)$ of all cusp forms with central character ω (cf. Definition 2.2.3). The Hecke algebra $\mathfrak{H}$ now operates on each $\mathcal{A}_0(\omega)$, which is stated in the following proposition.

Proposition 3.4.1. *The Hecke algebra $\mathfrak{H}$ acts on $\mathcal{A}_0(\omega)$. Under this action, $\mathcal{A}_0(\omega)$ decomposes into a direct sum of irreducible subspaces each occurring with multiplicity at most one. More precisely, we have*

$$\mathcal{A}_0(\omega) = \bigoplus_{\pi} V_{\omega,\pi}^{e_{\omega,\pi}} \qquad \text{where } e_{\omega,\pi} \in \{0,1\}$$

and where the sum runs over all irreducible admissible representations π of $\mathfrak{H}$.

Proof. Cf. [41, Prop. 10.9 and Prop. 11.1.1]. □

Consequently,

$$\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big) \subseteq \bigoplus_{\omega}\bigoplus_{\pi} V_{\omega,\pi}^{e_{\omega,\pi}} \ . \tag{3.1}$$

It will turn out that the properties **(H.1)**–**(H.6)**, which we introduced in Definition 2.2.9 in order to define Hilbert automorphic forms, are restrictive enough to imply that the projection of $\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big)$ onto certain of the $V_{\omega,\pi}^{e_{\omega,\pi}}$ is 0, so that these spaces do not contribute to the direct sum in (3.1). The aim of this section is to investigate which of the $V_{\omega,\pi}^{e_{\omega,\pi}}$ can be omitted so that equation (3.1) still holds true.

To this end, take a function $\mathbf{f} \in \mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big)$ and write it in the form

$$\mathbf{f} = \sum f_{\omega,\pi} \qquad \text{where} \qquad 0 \neq f_{\omega,\pi} \in V_{\omega,\pi}$$

according to the decomposition in equation (3.1).

First, we fix an archimedean place v and consider the v-th component of $\mathfrak{H}$, which is isomorphic to $\mathfrak{H}_{\mathbb{R}}$. By Theorem 3.2.5, any irreducible admissible representation of $\mathfrak{H}_{\mathbb{R}}$ is a quotient or a subrepresentation of $\rho(\mu_1,\mu_2) : \mathfrak{H}_{\mathbb{R}} \to \mathrm{GL}\big(\mathcal{B}(\mu_1,\mu_2)\big)$. Recall that the space $\mathcal{B}(\mu_1,\mu_2)$ is the span of the functions

$$\phi_l\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} r(\theta)\right) := \mu_1(a)\mu_2(d)\left|\frac{a}{d}\right|^{\frac{1}{2}} e^{-il\theta} \qquad \text{where } l \equiv m \bmod 2 \ ,$$

where the characters μ_1,μ_2 are given by

$$\mu_i(t) = |t|^{s_i}\,\mathrm{sgn}(t)^{m_i} \qquad \text{for some } s_i \in \mathbb{C}, m_i \in \{0,1\}$$

and m, s are defined by

$$m := |m_1 - m_2| \ , \qquad s := s_1 - s_2 \ .$$

Let us investigate the action of the Laplace operator on the space $\mathcal{B}(\mu_1, \mu_2)$. According to Lemma 3.2.6, it is given by

$$\Delta_v \phi = -\frac{s^2-1}{4}\phi \qquad \text{for all } \phi \in \mathcal{B}(\mu_1, \mu_2) \ .$$

Thus

$$\Delta_v f_{\omega,\pi} = -\frac{s^2-1}{4} f_{\omega,\pi}$$

if $V_{\omega,\pi}$ is a quotient or subrepresentation of $\mathcal{B}(\mu_1, \mu_2)$ with $\mu_1\mu_2^{-1} = |\cdot|^s \mathrm{sgn}^m$. On the other hand, we have by definition of $\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d}, \mathfrak{n}), 1\big)$

$$\Delta_v \mathbf{f} = -\frac{k_v}{2}\left(\frac{k_v}{2}-1\right)\mathbf{f} \ .$$

These two actions of Δ_v are compatible if and only if the parameter s, which depends on the pair (ω, π), satisfies

$$\frac{k_v}{2}\left(\frac{k_v}{2}-1\right) = \frac{s^2-1}{4} \ , \qquad \text{hence} \qquad s^2 = (k_v-1)^2 \tag{3.2}$$

for all (ω, π) for which $f_{\omega,\pi} \neq 0$. Property **(H.6)** thus restricts the set of $V_{\omega,\pi}$ that we need to consider.

Next, we examine the consequences of condition **(H.1)**. Consider a scalar matrix

$$z_v := \begin{pmatrix} z_v & 0 \\ 0 & z_v \end{pmatrix} \in Z(\mathrm{GL}_2(\mathbb{R})) \ .$$

Each basis element ϕ_l and hence every $\phi \in \mathcal{B}(\mu_1, \mu_2)$ satisfies

$$\phi\,(z_v g) = \mu_1(z_v)\mu_2(z_v)\phi(g) = |z_v|^{s_1+s_2}\mathrm{sgn}(z)^{m_1+m_2}\phi(g) \qquad \text{for all } g \in \mathrm{GL}_2(\mathbb{R}) \ .$$

On $\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d}, \mathfrak{n}), 1\big)$, however, the action of a scalar matrix at an archimedean place is given by multiplication with sgn^{k_v} because all relevant größencharacters ω satisfy $\omega^\infty = \mathrm{sgn}^{\mathbf{k}}$. As before, we are only interested in those representations for which both actions agree. If we consider the case $z_v > 0$ we immediately obtain the condition

$$s_1 + s_2 = 0 \ . \tag{3.3}$$

Similarly, if $z_v < 0$ we get $m_1 + m_2 \equiv k_v \bmod 2$, and hence

$$m \equiv k_v \bmod 2 \ . \tag{3.4}$$

To finish the discussion of the archimedean places, we note that no further restrictions on π arise from condition **(H.3)**. It tells us that the rotation matrices $r(\theta_v)$ act on $\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d}, \mathfrak{n}), 1, \omega\big)$ by multiplication by $e^{-ik_v\theta_v}$. On $\mathcal{B}(\mu_1, \mu_2)$, however,

$$\phi_l(gr(\theta_v)) = e^{-il\theta_v}\phi_l(g) \ .$$

So the action of $r(\theta_v)$ on $\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big)$ is compatible with $\rho(\mu_1,\mu_2)$ if $\phi_{k_v} \in \mathcal{B}(\mu_1,\mu_2)$. This is the case if $k_v \equiv m \bmod 2$, which results in the same condition as (3.4).

At the non-archimedean places we do not know that $\pi_{\mathfrak{p}}$ is equivalent to a quotient or subrepresentation of $\rho(\mu_1,\mu_2)$. A priori there is also the possibility that it is absolutely cuspidal (see Theorem 3.1.7). We will however show that this case never occurs if $f_{\omega,\pi} \neq 0$.

To this end, recall that every function in $\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big)$ is invariant under right translation with elements in the group $K_0(\mathfrak{d},\mathfrak{n})$. Therefore, we only need to consider such representations π for which $\pi_{\mathfrak{p}}$ contains $\big(K_0(\mathfrak{d},\mathfrak{n})\big)_{\mathfrak{p}}$-invariant vectors. But by Lemma 3.1.9 this implies that $\pi_{\mathfrak{p}}$ cannot be absolutely cuspidal.

Now that we have proved that $\pi_{\mathfrak{p}}$ is indeed equivalent to a quotient or subrepresentation of $\rho(\mu_1,\mu_2)$, we can continue in a similar fashion as in the archimedean situation.

Let $\phi \in \mathcal{B}(\mu_1,\mu_2)$ and consider

$$k := \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \big(K_0(\mathfrak{d},\mathfrak{n})\big)_{\mathfrak{p}} \qquad \text{where } a,d \in \mathfrak{o}_{\mathfrak{p}}^* .$$

Let $g \in Z\big(\mathrm{GL}_2(F_{\mathfrak{p}})\big)$. Then

$$\phi(gk) = \phi\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} g\right) = \mu_1(a)\mu_2(d)\left|\frac{a}{d}\right|_{\mathfrak{p}}^{\frac{1}{2}} \phi(g) = \mu_1(a)\mu_2(d)\phi(g)$$

since $|a|_{\mathfrak{p}} = |d|_{\mathfrak{p}} = 1$. On the other hand we know by **(H.3)** that $\mathbf{f}(gk) = \mathbf{f}(g)$ for all $\mathbf{f} \in \mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big)$. Hence, μ_1 and μ_2 must both be trivial on $\mathfrak{o}_{\mathfrak{p}}^*$.

We summarize this discussion in the following lemma, the statement of which can also be found in [65, Eq. (6.1) and (6.2)].

Lemma 3.4.2. *Let $0 \neq \mathbf{f} \in \mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big)$. In a decomposition $\mathbf{f} = \sum f_{\omega,\pi}$ where each $f_{\omega,\pi} \in V_{\omega,\pi}$ as before, we have $f_{\omega,\pi} = 0$ unless π satisfies the following conditions:*

(i) Each archimedean component π_v is equivalent to a special representation $\sigma(\mu_1,\mu_2)$ with characters $\mu_1 = |\cdot|^{\frac{k_v-1}{2}}$ and $\mu_2 = |\cdot|^{-\frac{k_v-1}{2}}\mathrm{sgn}^{k_v}$.

(ii) Each non-archimedean component $\pi_{\mathfrak{p}}$ is of the form $\pi_{\mathfrak{p}} = \pi(\mu_1,\mu_2)$ or $\pi_{\mathfrak{p}} = \sigma(\mu_1,\mu_2)$ for some unramified characters μ_1,μ_2.

Proof. Let π be an admissible irreducible representation for which $f_{\omega,\pi} \neq 0$.

(i) By the Classification Theorem 3.2.5 we know that each archimedean component π_v is equivalent to either a special representation $\sigma(\mu_1,\mu_2)$ or a $\pi(\mu_1,\mu_2)$ where the characters μ_1,μ_2 satisfy $\mu_1\mu_2^{-1}(t) = |t|^s\mathrm{sgn}^m(t) = t^s\mathrm{sgn}^{s+m}(t)$. But the conditions (3.2), (3.3) and (3.4), that we have just established, imply

$$s_1 = \pm\frac{k_v - 1}{2}\ , \qquad s_2 = \mp\frac{k_v - 1}{2}\ , \qquad s - m \equiv (k_v - 1) - k_v \equiv 1 \bmod 2\ .$$

Hence $\mu_1\mu_2^{-1}(t) = t^s\mathrm{sgn}$. All that is left to show is that π_v is infinite-dimensional. Suppose not. Then $\pi_v = \pi(\mu_1, \mu_2)$. According to [31, Ch. VII, § 5.2] (cf. also [29, Remark 4.7]) such a π_v cannot correspond to a unitary representation of $\mathrm{GL}_2(\mathbb{R})$ because $|s_1 - s_2| = k_v - 1$ is an integer ≥ 1. But this is a contradiction because we know that the right regular representation of $\mathrm{GL}_2(\mathbb{R})$ on the space of square-integrable functions, in which $\mathcal{A}_0(\omega)$ is contained, is unitary (cf. also Proposition 2.2.4).

(ii) We saw that $\pi_{\mathfrak{p}}$ is not absolutely cuspidal, so it is either one-dimensional or equivalent to a $\pi(\mu_1, \mu_2)$ or to a $\sigma(\mu_1, \mu_2)$. We also saw that in these cases the characters must be trivial on $\mathfrak{o}_{\mathfrak{p}}^*$, i. e. unramified. So again it remains to rule out the one-dimensional case. Let us suppose $\dim \pi_{\mathfrak{p}} = 1$. Then $\mathrm{GL}_2(F_{\mathfrak{p}})$ acts on $\mathbf{f}$ by

$$\rho(g)\mathbf{f} = (\chi \circ \det)(g)\mathbf{f}$$

(cf. Theorem 3.1.7). In particular, $\mathbf{f}$ is $\mathrm{SL}_2(F_{\mathfrak{p}})$-invariant on the right. Consider the function

$$\phi_g : \mathbb{A}_F \to \mathbb{C}\ , \quad x \mapsto \mathbf{f}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\ .$$

For all $g \in \mathrm{GL}_2(F_{\mathfrak{p}}) \hookrightarrow \mathrm{GL}_2(\mathbb{A}_F)$ and $x \in F_{\mathfrak{p}}$,

$$\phi_g(x) = \big(\rho(g)\mathbf{f}\big)\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \chi(\det(g))\mathbf{f}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \chi(\det(g))\mathbf{f}(I)$$

(I the (2×2)-identity matrix). So ϕ_g is constant on $F_{\mathfrak{p}}$ and hence equals its 0-th Fourier coefficient, which is 0 for almost all g since $\mathbf{f}$ is a cusp form (cf. Remark 2.2.7). Hence $\mathbf{f}$ must also be 0, which is a contradiction to our general assumption.

□

Definition 3.4.3. Suppose that $\mathfrak{n}$ is a square-free integral ideal in F. Then

$$U(\mathfrak{n}) \subseteq \mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d}, \mathfrak{n}), 1\big)$$

is defined to be the subspace of all $\mathbf{f} = \sum f_{\omega,\pi}$ such that $f_{\omega,\pi} = 0$ unless π satisfies the following conditions:

(i) Each archimedean component π_v is equivalent to a special representation $\sigma(\mu_1, \mu_2)$ with characters $\mu_1 = |\cdot|^{\frac{k_v-1}{2}}$ and $\mu_2 = |\cdot|^{-\frac{k_v-1}{2}}\mathrm{sgn}^{k_v}$.

(ii) Each non-archimedean component $\pi_{\mathfrak{p}}$ is of the form $\pi_{\mathfrak{p}} = \pi(\mu_1, \mu_2)$ or $\pi_{\mathfrak{p}} = \sigma(\mu_1, \mu_2)$ for some unramified characters μ_1, μ_2.

(iii) If $\mathfrak{p} \mid \mathfrak{n}$ then $\pi_\mathfrak{p}$ is a special representation.

Remark 3.4.4. What really distinguishes the elements lying in $U(\mathfrak{n})$ from those lying in $\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big) \smallsetminus U(\mathfrak{n})$ is condition (iii). Conditions (i) and (ii) are only mentioned for the sake of completeness. They are always fulfilled as we have just proved.

Let us briefly explain the role that condition (iii) plays so that it may become clearer why it has been introduced. Up to now we have examined the representations that are connected to an arbitrary cusp form $\mathbf{f} \in \mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big)$. We did not pay any attention to the question in which way any additional properties of $\mathbf{f}$ might affect the nature of these representations.

However, from classical theory we know that it often proves useful to divide the space of all cusp forms of level $\mathfrak{n}$ into two distinct classes: The first one contains the *oldforms*, which are either cusp forms with respect to some "lower" level $\mathfrak{m}$ properly dividing $\mathfrak{n}$ or certain translates thereof. It is well known (cf. for example [1]) that these oldforms form a subspace of the space of all cusp forms of level $\mathfrak{n}$. Its orthocomplement is spanned by the *newforms*, which—by construction—do not come from forms of "lower" level.

In terms of representation theory a newform in $\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big)$ is a function $\mathbf{f}$ which is invariant under right translation by elements in $K_0(\mathfrak{d},\mathfrak{n})$ but not by elements in any $K_0(\mathfrak{d},\mathfrak{m})$ where $\mathfrak{m} \mid \mathfrak{n}$ is a proper divisor of $\mathfrak{n}$.

Let us momentarily denote by $V_\mathfrak{p}$ the space of all $\mathbf{f} \in \mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big)$ viewed as functions on the $\mathfrak{p}$-th component alone. A theorem of Casselman (see [10, Thm. 1]) then states that for every $\mathfrak{p} < \infty$ there is a minimal $r_\mathfrak{p} \in \mathbb{Z}$ such that the fixed space

$$V_\mathfrak{p}^{K_0(\mathfrak{d}_\mathfrak{p},\mathfrak{p}^{r_\mathfrak{p}})} := \big\{\mathbf{f} \in V_\mathfrak{p} \mid \pi_\mathfrak{p}(k)\mathbf{f} = \mathbf{f} \text{ for all } k \in K_0(\mathfrak{d}_\mathfrak{p},\mathfrak{p}^{r_\mathfrak{p}})\big\}$$

is non-empty, and in this case $\dim_\mathbb{C} V_\mathfrak{p}^{K_0(\mathfrak{d}_\mathfrak{p},\mathfrak{p}^{r})} = 1$. Moreover, the minimal $r_\mathfrak{p}$ is equal to 0 if $\pi_\mathfrak{p} = \pi(\mu_1,\mu_2)$ for some unramified characters μ_1,μ_2, and $r_\mathfrak{p} = 1$ if $\pi_\mathfrak{p} = \sigma(\mu_1,\mu_2)$ with unramified μ_1,μ_2 (cf. [49, Thm. 3.5]).

So $\mathbf{f}$ is a newform if and only if it belongs to the minimal non-empty fixed space for every $\mathfrak{p} < \infty$. For $\mathfrak{p} \nmid \mathfrak{n}$ this is always true because in this case $\mathbf{f}$ is invariant under $K_0(\mathfrak{d}_\mathfrak{p},1)$, so that $\mathbf{f}$ lies in $V_\mathfrak{p}^{K_0(\mathfrak{d}_\mathfrak{p},\mathfrak{p}^0)}$, which then must necessarily be the minimal non-empty fixed space. For $\mathfrak{p} \mid \mathfrak{n}$ we know that $\mathbf{f}$ is right-invariant under $K_0(\mathfrak{d}_\mathfrak{p},\mathfrak{p})$. So it is a newform if and only if $r_\mathfrak{p} = 1$ for all $\mathfrak{p} \mid \mathfrak{n}$, which leads to the condition that $\pi_\mathfrak{p}$ be a special representation if $\mathfrak{p} \mid \mathfrak{n}$. So the purpose of condition (iii) of the previous definition is to single out the space of newforms in $\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big)$.

For a further discussion of this topic see [49, § 3 and § 4] and [29, Thm. 4.23 and § 5B]. □

In view of the previous remark the following proposition can be understood as a representation theoretic version of [1, Thm. 5]. (Cf. also [22, IV § 1]).

Proposition 3.4.5. *Assume that $\mathfrak{n}$ is square-free. For every prime ideal $\mathfrak{p} \mid \mathfrak{n}$ let $\varpi_{\mathfrak{p}} \in \mathfrak{o}_{\mathfrak{p}}$ be a uniformizing element. Then*

$$\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big) = \bigoplus_{\mathfrak{m}|\mathfrak{n}} \bigoplus_{\mathfrak{a}|\mathfrak{n}\mathfrak{m}^{-1}} \rho\left(\begin{pmatrix} 1 & 0 \\ 0 & \alpha_{\mathfrak{a}} \end{pmatrix}\right) U(\mathfrak{m})$$

where $\alpha_{\mathfrak{a}} = \prod_{\mathfrak{p}|\mathfrak{a}} \varpi_{\mathfrak{p}} \in \mathbb{A}_F^*$.

Proof. In [65, § 6.3] a similar result is shown for $\mathcal{H}^0_{\mathbf{k}}\big(K_0(1,\mathfrak{n}),1\big)$ and subspaces $\tilde{U}(\mathfrak{n})$ hereof that are define by the same set of conditions as we used in Definition 3.4.3. As $K_0(1,\mathfrak{n})$ and $K_0(\mathfrak{d},\mathfrak{n})$ are conjugate in the following way

$$K_0(\mathfrak{d},\mathfrak{n}) = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} K_0(1,\mathfrak{n}) \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}^{-1},$$

where δ is an adelic generator of $\mathfrak{d}$, we see that

$$\mathbf{f} \in \mathcal{H}^0_{\mathbf{k}}\big(K_0(1,\mathfrak{n}),1\big) \iff \rho\left(\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}\right)\mathbf{f} \in \mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big)\ .$$

So the claimed decomposition follows directly from the above-mentioned result in [65, § 6.3]. □

Chapter 4

Harmonic homogeneous polynomials

We saw in Section 1.3 that homogeneous harmonic polynomials come naturally into play when we construct theta series that are modular forms of weight ≥ 2, and these polynomials are non-constant if the weight is > 2. Unfortunately many authors prefer to restrict themselves to the technically easier case of weight 2, so that literature on the general case is rather limited.

We find it therefore advisable to explain the role of the harmonic polynomials in more detail and devote the entire chapter to this subject. For the sake of completeness we also choose to include Section 4.1, although it is not essential for the rest of this thesis. It discusses different inner products on the space of harmonic polynomials and the role of the Gegenbauer polynomials. Detailed explanations of these topics are somewhat neglected in the literature on modular forms, although they are used by a number of authors (for example [25], [4], [39]) when treating polynomial-valued automorphic forms and related subjects.

The action of the Hamiltonians on the space of homogeneous polynomials is explained in Section 4.2, and finally we discuss the connection between harmonic polynomials and adelic automorphic forms.

4.1 Gegenbauer polynomials

Denote by $\mathrm{Hom}_\nu[X_1, \ldots, X_m]$ the complex space of homogeneous polynomials in m variables of total degree ν. Recall that a polynomial $P \in \mathrm{Hom}_\nu[X_1, \ldots, X_m]$ is said to be *harmonic* if it is annihilated by the Laplace operator, i. e.

$$\Delta P = 0 \ .$$

The complex subspace of all harmonic polynomials in $\mathrm{Hom}_\nu[X_1,\ldots,X_m]$ will be denoted by $\mathrm{Harm}_\nu[X_1,\ldots,X_m]$.

Theorem 4.1.1. *A homogeneous polynomial $P \in \mathrm{Hom}_\nu[X_1,\ldots,X_m]$ is harmonic if and only if it is either*

(i) constant, if $\nu = 0$,

(ii) or linear, if $\nu = 1$,

(iii) or a linear combination of terms of the form $(z^t X)^\nu$ where each $z \in \mathbb{C}^m$ satisfies $z^t z = 0$.

Proof. Cf. [40, Theorem 9.1]. □

Theorem 4.1.2. *For $\nu \geq 2$, the space of homogeneous harmonic polynomials of degree ν in m indeterminates has dimension*

$$\dim_{\mathbb{C}}\left(\mathrm{Harm}_\nu[X_1,\ldots,X_m]\right) = \binom{\nu+m-1}{m-1} - \binom{\nu+m-3}{m-1} = \frac{(\nu+m-3)!}{\nu!(m-2)!}(m+2\nu-2)$$

over the complex field. In particular, for $m = 4$,

$$\dim_{\mathbb{C}}\left(\mathrm{Harm}_\nu[X_1,\ldots,X_4]\right) = (\nu+1)^2 \ .$$

Proof. This dimension formula can be proved by a simple counting argument (see for example [35, Satz 32] or [40, Corollary 9.2]). □

Lemma 4.1.3. *Let V be a subspace of $\mathrm{Hom}_\nu[X_1,\ldots,X_m]$, and let $\{P_1,\ldots,P_d\}$ be an orthonormal basis of V with respect to some inner product $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle$ on V. Then the polynomial $K_\nu(x,y)$ defined by*

$$K_\nu(x,y) := \sum_{i=1}^{d} P_i(x)\overline{P_i(y)} \ ,$$

is in fact independent of the choice of the orthonormal basis. Moreover, when considered as a polynomial in x alone, $K_\nu(x,y)$ is the unique element in V that satisfies the reproducing kernel condition

$$P(y) = \left\langle\!\left\langle P,\ K_\nu(\,\cdot\,,y)\right\rangle\!\right\rangle$$

for all $P \in V$ and all $y = (y_1,\ldots,y_m) \in \mathbb{C}^m$.

Proof. Most of the assertions are evident, and we will only show the independence of the choice of basis. Let $\{Q_1,\ldots,Q_d\}$ be another orthonormal basis of V, then we can find a unitary matrix $T = (t_{ij}) \in \mathrm{SU}_d(\mathbb{C})$ such that $P_i = \sum t_{ij} Q_j$ for all $i = 1, \ldots, d$. Using this and the fact that $\overline{T}^t T = I_d$, a simple calculation shows that $\sum P_i(x)\overline{P_i(y)} = \sum Q_i(x)\overline{Q_i(y)}$. For the uniqueness of the kernel function see for example [58, Ch. II, § 1, Thm. 1.1]. □

We will now make a specific choice for the inner product $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle$. For $\mathbf{j} = (j_1, \ldots, j_m) \in \mathbb{N}_0^m$, we use the multi-index notation

$$|\mathbf{j}| := j_1 + \ldots + j_m \ , \qquad \mathbf{j}! := j_1! \cdot \ldots \cdot j_m! \ , \qquad c(\mathbf{j}) := \frac{|\mathbf{j}|!}{\mathbf{j}!} \ , \qquad X^{\mathbf{j}} := X_1^{j_1} \cdot \ldots \cdot X_m^{j_m} \ ,$$

so that any homogeneous polynomial $P \in \mathrm{Hom}_\nu[X_1, \ldots, X_m]$ can uniquely be expressed as

$$P = \sum_{|\mathbf{j}|=\nu} c(\mathbf{j}) a_{\mathbf{j}} X^{\mathbf{j}} \qquad \text{for suitable coefficients } a_{\mathbf{j}} \in \mathbb{C} \ .$$

We endow the space $\mathrm{Hom}_\nu[X_1, \ldots, X_m]$ with the inner product $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle$ defined by

$$\langle\!\langle P, Q \rangle\!\rangle := \sum_{|\mathbf{j}|=\nu} c(\mathbf{j}) a_{\mathbf{j}} \bar{b}_{\mathbf{j}} \qquad \text{for} \qquad P = \sum_{|\mathbf{j}|=\nu} c(\mathbf{j}) a_{\mathbf{j}} X^{\mathbf{j}} \ , \qquad Q = \sum_{|\mathbf{j}|=\nu} c(\mathbf{j}) b_{\mathbf{j}} X^{\mathbf{j}} \ .$$

Our next aim is to find a more explicit description of the reproducing kernel $K_\nu(x, y)$ on the space $\mathrm{Harm}_\nu[X_1, \ldots, X_m]$ of harmonic polynomials. But before we tackle this problem, let us add a few comments on the inner product $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle$ that we are using. A possibly more natural inner product to consider would be

$$\langle P, Q \rangle := \int_{S^{m-1}} P(x)\overline{Q(x)}\, dx \ ,$$

where S^{m-1} is the unit sphere in $\mathbb{R}^m$. It has the advantage of being obviously rotation invariant, i. e.

$$\langle P \circ \rho,\, Q \circ \rho \rangle = \langle P, Q \rangle \qquad \text{for all } P, Q \in \mathrm{Harm}_\nu[X_1, \ldots, X_m] \text{ and } \rho \in \mathrm{SO}(m) \ ,$$

but the disadvantage that it is somewhat inconvenient to use when it comes to explicit computations. Therefore we prefer to use the inner product $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle$, which is defined solely in terms of the coefficients of P and Q. To justify our choice, we point out that $\langle \cdot\,,\,\cdot \rangle$ and $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle$ in fact differ only by a constant factor, as will be shown in the next Proposition.

Proposition 4.1.4. *For $P, Q \in \mathrm{Harm}_\nu[X_1, \ldots, X_m]$,*

$$\langle P, Q \rangle = \frac{\nu!\, \Gamma\left(\frac{m}{2} + 1\right)}{2^\nu\, \Gamma\left(\frac{m}{2} + \nu + 1\right)} \mathrm{vol}(S^{m-1})\, \langle\!\langle P, Q \rangle\!\rangle \ .$$

Proof. The proof makes use of the identities (cf. [40, § 9.2])

$$\int_{S^{m-1}} \Delta P(x)\, dx = \nu(\nu + m) \int_{S^{m-1}} P(x)\, dx \qquad \text{for all } P \in \mathrm{Hom}_\nu[X_1, \ldots, X_m]$$

and

$$\Delta(PQ) = 2\sum_{i=1}^{m} \frac{\partial P}{\partial X_i}\frac{\partial Q}{\partial X_i} \qquad \text{for all } P, Q \in \mathrm{Harm}_\nu[X_1, \dots, X_m] \ .$$

Applying these identities to $P\overline{Q} \in \mathrm{Hom}_{2\nu}[X_1, \dots, X_m]$, we get

$$\nu(2\nu+m)\int_{S^{m-1}} P(x)\overline{Q(x)}\,dx = \int_{S^{m-1}} \sum_{i=1}^{m} \frac{\partial P(x)}{\partial X_i}\frac{\partial \overline{Q}(x)}{\partial X_i}\,dx = \sum_{i=1}^{m}\int_{S^{m-1}} \frac{\partial P(x)}{\partial X_i}\frac{\partial \overline{Q}(x)}{\partial X_i}\,dx \ ,$$

where both $\frac{\partial P}{\partial X_i}$ and $\frac{\partial \overline{Q}}{\partial X_i}$ are in $\mathrm{Harm}_{\nu-1}[X_1, \dots, X_m]$. We repeat this process ν times and get

$$\prod_{q=1}^{\nu} q(2q+m)\int_{S^{m-1}} P(x)\overline{Q(x)}\,dx = \int_{S^{m-1}} \sum_{i_1,\dots,i_\nu=1}^{m} \frac{\partial^\nu P(x)}{\partial X_{i_1}\dots\partial X_{i_\nu}}\frac{\partial^\nu \overline{Q}(x)}{\partial X_{i_1}\dots\partial X_{i_\nu}}\,dx \ .$$

If we write P and Q in the form $P = \sum_{|\mathbf{j}|=\nu} c(\mathbf{j})a_{\mathbf{j}}X^{\mathbf{j}}$ and $Q = \sum_{|\mathbf{j}|=\nu} c(\mathbf{j})b_{\mathbf{j}}X^{\mathbf{j}}$ then the derivatives are

$$\frac{\partial^\nu P}{\partial X_{i_1}\dots\partial X_{i_\nu}} = j_1! \cdot \ldots \cdot j_m!\, c(\mathbf{j})a_{\mathbf{j}} = \nu!\, a_{\mathbf{j}} \qquad \text{where } j_k := \#\{l \mid i_l = k\}$$

and similarly for $\overline{Q}$. As there are exactly $c(\mathbf{j})$ tuples $(i_1, \dots, i_\nu) \in \{1, \dots, m\}^\nu$ leading to the same derivative, we finally obtain

$$\begin{aligned}
\int_{S^{m-1}} P(x)\overline{Q(x)}\,dx &= \left(\prod_{q=1}^{\nu} q(2q+m)\right)^{-1}\int_{S^{m-1}} \sum_{|\mathbf{j}|=\nu} c(\mathbf{j})\,(\nu!)^2\, a_{\mathbf{j}}\bar{b}_{\mathbf{j}}\,dx \\
&= \left(\nu!2^\nu \prod_{q=1}^{\nu}\left(\frac{m}{2}+q\right)\right)^{-1} \cdot \mathrm{vol}(S^{m-1})\,(\nu!)^2 \sum_{|\mathbf{j}|=\nu} c(\mathbf{j})\,a_{\mathbf{j}}\bar{b}_{\mathbf{j}} \\
&= \frac{\nu!\,\Gamma\left(\frac{m}{2}+1\right)}{2^\nu\,\Gamma\left(\frac{m}{2}+\nu+1\right)}\mathrm{vol}(S^{m-1}) \sum_{|\mathbf{j}|=\nu} c(\mathbf{j})\,a_{\mathbf{j}}\bar{b}_{\mathbf{j}} \ .
\end{aligned}$$

□

Now we are ready to find an explicit description of the reproducing kernel $K_\nu(x, y)$ on the space $\mathrm{Harm}_\nu[X_1, \dots, X_m]$ with respect to $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$. We follow the argumentation in [35, § 5] and [62, Proof of Theorem 3].

Lemma 4.1.5. *Let $x, y \in \mathbb{R}^m$. Then $K_\nu(x, y)$ is $\mathrm{SO}(m)$-invariant, i. e.*

$$K_\nu(\rho(x), \rho(y)) = K_\nu(x, y) \qquad \textit{for all } \rho \in \mathrm{SO}(m) \ ,$$

and must therefore be of the form

$$K_\nu(x, y) = (|x||y|)^\nu C\left(\frac{\langle x, y\rangle}{|x||y|}\right) \qquad \textit{for some } C \in \mathbb{C}[X] \ , \tag{4.1}$$

where $\langle\cdot,\cdot\rangle$ and $|\cdot|$ are the usual euclidian inner product and norm on $\mathbb{R}^m$.

Proof. Let $\{P_1, \dots, P_d\}$ be an orthonormal basis of $\mathrm{Harm}_\nu[X_1, \dots, X_m]$ with respect to $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$, and let $\rho \in \mathrm{SO}(m)$. Note that $\mathrm{Harm}_\nu[X_1, \dots, X_m]$ is invariant under ρ so that $\{P_1 \circ \rho, \dots, P_d \circ \rho\}$ is again a basis of this space. As a consequence of Proposition 4.1.4, we know that $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$ is rotation invariant. Therefore we have

$$\langle\!\langle P_i \circ \rho,\ P_j \circ \rho \rangle\!\rangle = \langle\!\langle P_i, P_j \rangle\!\rangle = \delta_{ij} \qquad \text{for all } 1 \le i, j \le d \ ,$$

so $\{P_1 \circ \rho, \dots, P_d \circ \rho\}$ is in fact an orthonormal basis. But $K_\nu(x, y)$ is independent of the choice of basis, as we have seen in Lemma 4.1.3. So it follows that

$$K_\nu(\rho(x), \rho(y)) = \sum_{i=1}^{d} (P_i \circ \rho)(x)\overline{(P_i \circ \rho)(y)} = \sum_{i=1}^{d} P_i(x)\overline{P_i(y)} = K_\nu(x, y) \ .$$

Now let $x, y, z, w \in S^{m-1}$ be elements on the unit sphere. If $\langle x, y \rangle = \langle z, w \rangle$ then there exists a rotation ρ such that $\rho(x) = z$ and $\rho(y) = w$, and consequently $K_\nu(x, y) = K_\nu(z, w)$. In other words, $K_\nu(x, y)$, when restricted to the unit sphere, depends only on $\langle x, y \rangle$. But since $K_\nu(x, y)$ is of homogeneous degree ν in x and y, respectively, we can write

$$K_\nu(x, y) = |x|^\nu |y|^\nu K_\nu\left(\frac{x}{|x|}, \frac{y}{|y|}\right) \qquad \text{for arbitrary } x, y \in \mathbb{R}^m \ ,$$

and we see that on the whole, $K_\nu(x, y)$ depends only on $|x|$, $|y|$ and $\langle x, y \rangle$. But as it must be a homogeneous polynomial of degree ν, that only leaves the possibility

$$K_\nu(x, y) = \sum_{i=0}^{\lfloor \frac{\nu}{2} \rfloor} a_i \langle x, y \rangle^{\nu - 2i} |x|^{2i} |y|^{2i} = (|x||y|)^\nu \sum_{i=0}^{\lfloor \frac{\nu}{2} \rfloor} a_i \left(\frac{\langle x, y \rangle}{|x||y|}\right)^{\nu - 2i}$$

for suitable coefficients $a_i \in \mathbb{C}$. $\square$

As mentioned in Lemma 4.1.3, $K_\nu(x, y)$ is itself a harmonic polynomial in x. As a matter of fact, the property $\Delta K_\nu(x, y) = 0$ is restrictive enough to make an explicit description of the polynomial C in equation (4.1) possible. Indeed, writing $K_\nu(x, y)$ as in equation (4.1) and taking derivatives with respect to x, leads after a lengthy but straightforward calculation to

$$\begin{aligned} \Delta K_\nu(x, y) = |x|^{\nu-2} |y|^\nu \Biggl(&\left(1 - \frac{\langle x, y \rangle^2}{|x|^2 |y|^2}\right) C''\left(\frac{\langle x, y \rangle}{|x||y|}\right) \\ &+ (1 - m) \frac{\langle x, y \rangle}{|x||y|} C'\left(\frac{\langle x, y \rangle}{|x||y|}\right) + \nu(\nu + m - 2) C\left(\frac{\langle x, y \rangle}{|x||y|}\right) \Biggr) \ . \end{aligned}$$

The polynomial C must therefore satisfy

$$(1 - u^2) C''(u) + (1 - m) u C'(u) + \nu(\nu + m - 2) C(u) = 0 \ . \tag{4.2}$$

This differential equation is sometimes called *Gegenbauer differential equation* and its polynomial solutions are well-known.

Theorem 4.1.6. *For $m > 1$ and an integer $\nu \ge 0$, the Gegenbauer differential equation* (4.2) *has a polynomial solution, which is unique up to a constant.*

Proof. See [69, Theorem 4.2.2]. □

Definition 4.1.7 (Gegenbauer polynomial). We call

$$C_\nu^{(\lambda)}(x) = \sum_{q=0}^{\lfloor \frac{\nu}{2} \rfloor} (-1)^q \frac{\Gamma(\nu - q + \lambda)}{\Gamma(\lambda)\Gamma(q+1)\Gamma(\nu - 2q + 1)} (2x)^{\nu - 2q}$$

Gegenbauer (or *ultraspherical*) *polynomial* of degree ν and parameter λ.

Example 4.1.8. The first Gegenbauer polynomials are

$$\begin{aligned}
C_0^{(\lambda)}(x) &= 1 \ , \\
C_1^{(\lambda)}(x) &= 2\lambda x \ , \\
C_2^{(\lambda)}(x) &= 2\lambda(\lambda+1)x^2 - \lambda \ , \\
C_3^{(\lambda)}(x) &= \frac{4}{3}\lambda(\lambda+1)(\lambda+2)x^3 - 2\lambda(\lambda+1)x \ .
\end{aligned}$$

Proposition 4.1.9. *Let $m > 1$ and $\lambda = \frac{m}{2} - 1$.*

(i) The Gegenbauer polynomial $C_\nu^{(\lambda)}$ is a solution of the differential equation (4.2).

(ii) The reproducing kernel $K_\nu(\,\cdot\,, y)$ on $\mathrm{Harm}_\nu[X_1, \ldots, X_m]$ with respect to $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$ is given by

$$\begin{aligned}
K_\nu(x,y) &= \frac{1}{d_0}(|x||y|)^\nu C_\nu^{(\lambda)}\left(\frac{\langle x, y\rangle}{|x||y|}\right) \\
&= \frac{1}{d_0}\sum_{|\mathbf{j}|=\nu}\left(\sum_{q=0}^{\lfloor \frac{\nu}{2} \rfloor} d_q |y|^{2q} \sum_{|\mathbf{k}|=q} c(\mathbf{k})c(\mathbf{j}-2\mathbf{k})y^{\mathbf{j}-2\mathbf{k}}\right)x^{\mathbf{j}}
\end{aligned}$$

where

$$d_q = (-1)^q 2^{\nu-2q} \frac{\Gamma\left(\nu - q + \frac{m}{2} - 1\right)}{\Gamma\left(\frac{m}{2} - 1\right)\Gamma(q+1)\Gamma(\nu - 2q + 1)} \in \mathbb{Q} \ .$$

Proof. The first part can be found in any book on Gegenbauer polynomials, for example [69, Equation (4.7.5)]. For the second part observe that

$$(|x||y|)^\nu C_\nu^{(\lambda)}\left(\frac{\langle x, y\rangle}{|x||y|}\right) = \sum_{q=0}^{\lfloor \frac{\nu}{2} \rfloor} d_q \langle x, y\rangle^{\nu-2q}(|x||y|)^{2q}$$

and

$$\langle x, y\rangle^{\nu-2q} = \sum_{|\mathbf{j}|=\nu-2q} c(\mathbf{j})x^{\mathbf{j}}y^{\mathbf{j}} \qquad \text{and} \qquad (|x||y|)^{2q} = |y|^{2q}\sum_{|\mathbf{k}|=q} c(\mathbf{k})x^{2\mathbf{k}} \ ,$$

which proves the second equality in *(ii)*. By the uniqueness of the reproducing kernel, by Lemma 4.1.5 and Theorem 4.1.6, it is then clear that the reproducing kernel for the space

$\mathrm{Harm}_\nu[X_1,\ldots,X_m]$ with respect to $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$ must be a scalar multiple of $(|x||y|)^\nu C_\nu^{(\lambda)}\left(\frac{\langle x,y\rangle}{|x||y|}\right)$. In order to show that the scalar is d_0^{-1}, it suffices to check the reproducing kernel property for an arbitrarily chosen $0\neq P\in\mathrm{Harm}_\nu[X_1,\ldots,X_m]$. In the light of Theorem 4.1.1 we take

$$P(x)=\Big(\sum_{i=1}^m u_ix_i\Big)^\nu=\sum_{[\mathbf{j}]=\nu}c(\mathbf{j})w^{\mathbf{j}}x^{\mathbf{j}}\qquad\text{where}\quad\sum_{i=1}^m u_i^2=0\quad\text{if }\nu\geq 2\ .$$

So we have for all $y\in\mathbb{R}^m$

$$\begin{aligned}\langle\!\langle P,\,K_\nu(\,\cdot\,,y)\rangle\!\rangle&=\frac{1}{d_0}\sum_{|\mathbf{j}|=\nu}u^{\mathbf{j}}\left(\sum_{q=0}^{\lfloor\frac{\nu}{2}\rfloor}d_q|y|^{2q}\sum_{|\mathbf{k}|=q}c(\mathbf{k})c(\mathbf{j}-2\mathbf{k})y^{\mathbf{j}-2\mathbf{k}}\right)\\&=\frac{1}{d_0}\sum_{q=0}^{\lfloor\frac{\nu}{2}\rfloor}d_q|y|^{2q}\sum_{|\mathbf{j}|=\nu}\sum_{|\mathbf{k}|=q}u^{2\mathbf{k}}c(\mathbf{k})c(\mathbf{j}-2\mathbf{k})u^{\mathbf{j}-2\mathbf{k}}y^{\mathbf{j}-2\mathbf{k}}\\&=\frac{1}{d_0}\sum_{q=0}^{\lfloor\frac{\nu}{2}\rfloor}d_q|y|^{2q}\sum_{|\mathbf{k}|=q}u^{2\mathbf{k}}c(\mathbf{k})\sum_{|\mathbf{i}|=\nu-2q}c(\mathbf{i})u^{\mathbf{i}}y^{\mathbf{i}}\\&=\frac{1}{d_0}\sum_{q=0}^{\lfloor\frac{\nu}{2}\rfloor}d_q|y|^{2q}\Big(\sum_{i=1}^m u_i^2\Big)^q\Big(\sum_{i=1}^m u_iy_i\Big)^{\nu-2q}\ .\end{aligned}$$

If $\nu=0,1$ then there is no term for $q>0$. If $\nu\geq 2$ then we have $\sum_{i=1}^m u_i^2=0$ by assumption, so again, all terms for $q>0$ vanish, and we are left with

$$\langle\!\langle P,\,K_\nu(\,\cdot\,,y)\rangle\!\rangle=\frac{1}{d_0}d_0\Big(\sum_{i=1}^m u_iy_i\Big)^\nu=P(y)\ .$$

Finally, it is easily verified that $d_q\in\mathbb{Q}$. $\square$

4.2 The action of the Hamiltonians on homogeneous polynomials

In this section we restrict our attention to the case $m=2$. The inner product on $\mathrm{Hom}_\nu[X,Y]$ then simplifies to

$$\langle\!\langle P,Q\rangle\!\rangle:=\sum_{j=0}^\nu\binom{\nu}{j}a_j\bar{b}_j\qquad\text{for}\qquad P=\sum_{j=0}^\nu\binom{\nu}{j}a_jX^{\nu-j}Y^j\ ,\quad Q=\sum_{j=0}^\nu\binom{\nu}{j}b_jX^{\nu-j}Y^j\ .$$

Let $\mathbf{H}$ denote the Hamilton quaternion algebra, in which we fix the canonical basis $\{1,\mathfrak{i},\mathfrak{j},\mathfrak{k}\}$ where $\mathfrak{i}^2=\mathfrak{j}^2=-1$, $\mathfrak{i}\mathfrak{j}=-\mathfrak{j}\mathfrak{i}=\mathfrak{k}$. Then $\mathbf{H}$ can be identified with

$$\left\{\begin{pmatrix}z&w\\-\overline{w}&\overline{z}\end{pmatrix}\;\middle|\;z,w\in\mathbb{C}\right\}\qquad\text{via}\qquad x\mapsto X_1(x)$$

where for $x = x_0 1 + x_1 \mathfrak{i} + x_2 \mathfrak{j} + x_3 \mathfrak{k}$,

$$z := x_0 + ix_1, \quad w := x_2 + ix_3 \in \mathbb{C} \qquad \text{and} \qquad X_1(x) := \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} .$$

The group $\mathbf{H}^1$ of norm-1 quaternions then corresponds to $\mathrm{SU}_2(\mathbb{C})$, i. e. the subgroup of $\mathrm{GL}_2(\mathbb{C})$ consisting of those matrices $X_1(x)$ that satisfy

$$\det\big(X_1(x)\big) = z\overline{z} + w\overline{w} = x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 .$$

Definition 4.2.1 (Symmetric power representation). Define the *ν-th symmetric power representation* $\sigma_\nu : \mathbf{H}^* \to \mathrm{GL}(\mathrm{Hom}_\nu[X,Y])$ by

$$\big(\sigma_\nu(x)P\big)(X,Y) := P\big((X,Y)X_1(x)\big) = P\big(zX - \overline{w}Y,\, wX + \overline{z}Y\big) .$$

Define another representation $\Lambda_\nu : \mathbf{H}^* \to \mathrm{GL}(\mathrm{Hom}_\nu[X,Y])$ by

$$\Lambda_\nu(x)P := \mathrm{nrd}(x)^{-\frac{\nu}{2}} \sigma_\nu(x)P .$$

Lemma 4.2.2. *Denote by χ_{σ_ν} the character of the representation σ_ν. Then for all $x \in \mathbf{H}^*$ and $\lambda \in \mathbb{R}^*$,*

$$\sigma_\nu(\lambda x) = \lambda^\nu \sigma_\nu(x) \qquad \text{and} \qquad \chi_{\sigma_\nu}(\lambda x) = \lambda^\nu \chi_{\sigma_\nu}(x) .$$

Proof. This is due to the fact that the polynomials on which $\sigma_\nu(x)$ acts are homogeneous. □

Lemma 4.2.3. *The irreducible finite-dimensional representations of $\mathbf{H}^*$ are precisely the representations of the form*

$$\mathbf{H}^* \to \mathrm{GL}\big(\mathrm{Hom}_\nu[X,Y]\big) , \qquad x \mapsto \Big(\mathrm{nrd}(x)^r \cdot \sigma_\nu(x)\Big)$$

for some $\nu \in \mathbb{N}_0$, $r \in \mathbb{C}$. In particular, σ_ν and Λ_ν are irreducible.

Proof. Via $x \mapsto \big(\mathrm{nrd}(x)^{\frac{1}{2}}, \mathrm{nrd}(x)^{-\frac{1}{2}} x\big)$, we get an isomorphism $\mathbf{H}^* \cong \mathbb{R}_{>0} \times \mathrm{SU}_2(\mathbb{C})$. By Example 2.1.4 we know that the action of $t \in \mathbb{R}_{>0}$ is multiplication by t^r for some $r \in \mathbb{C}$. In particular, any subspace is invariant under $\mathbb{R}_{>0}$. Therefore, the irreducible finite-dimensional representations of $\mathbf{H}^*$ remain irreducible when restricted to $\mathrm{SU}_2(\mathbb{C})$. But these are exactly the symmetric power representations σ_ν (see [43, Ch. II, §§ 1,2]). Hence, every irreducible finite-dimensional representation of $\mathbf{H}^*$ is of the form

$$x \mapsto \mathrm{nrd}(x)^{\frac{r}{2}} \cdot \sigma_\nu\left(\mathrm{nrd}(x)^{-\frac{1}{2}} x\right) = \mathrm{nrd}(x)^{\frac{r-\nu}{2}} \cdot \sigma_\nu(x)$$

as claimed. □

Proposition 4.2.4. *For all $P, Q \in \mathrm{Hom}_\nu[X,Y]$,*

$$\begin{aligned} \big\langle\!\big\langle \sigma_\nu(x)P,\ \sigma_\nu(x)Q \big\rangle\!\big\rangle &= \langle\!\langle P,Q\rangle\!\rangle && \textit{for all } x \in \mathbf{H}^1\ , \\ \big\langle\!\big\langle \Lambda_\nu(x)P,\ \Lambda_\nu(x)Q \big\rangle\!\big\rangle &= \langle\!\langle P,Q\rangle\!\rangle && \textit{for all } x \in \mathbf{H}^*\ . \end{aligned}$$

Proof. As mentioned above, $X_1(x) \in \mathrm{SU}_2(\mathbb{C})$ for any $x \in \mathbf{H}^1$. Therefore, $\sigma_\nu(x)$ acts by a unitary transformation of the variables X, Y. From the discussion in the previous section, we can derive that $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$ is invariant under such unitary transformations (or see for example [57, Thm. 2.12] for a direct proof). For arbitrary $x \in \mathbf{H}^*$, not necessarily of norm 1, we immediately obtain

$$\langle\!\langle P,Q\rangle\!\rangle = \big\langle\!\big\langle \sigma_\nu\big(\mathrm{nrd}(x)^{-\frac{1}{2}}x\big)P,\, \sigma_\nu\big(\mathrm{nrd}(x)^{-\frac{1}{2}}x\big)Q \big\rangle\!\big\rangle = \big\langle\!\big\langle \Lambda_\nu(x)P,\ \Lambda_\nu(x)Q \big\rangle\!\big\rangle$$

by Lemma 4.2.2. □

In $\mathrm{Hom}_\nu[X,Y]$, consider the canonical basis

$$P_i := X^{\nu-i}Y^i \qquad \text{for } i = 0,\dots,\nu\ .$$

Clearly, the image of $X^{\nu-i}Y^i$ under the action of $\sigma_\nu(x)$ for $x \in \mathbf{H}^1$ is

$$\big(\sigma_\nu(x)P_i\big)(X,Y) = (zX - \overline{w}Y)^{\nu-i}(wX + \overline{z}Y)^i\ .$$

The matrix of $\sigma_\nu(x)$ with respect to the basis $\{X^{\nu-i}Y^i\}$ will be denoted by $X_\nu(x)$. Note that for $\nu = 1$ this definition of $X_1(x)$ coincides with the definition given at the beginning of this section.

Example 4.2.5. Let $x = x_0 1 + x_1\mathfrak{i} + x_2\mathfrak{j} + x_3\mathfrak{k} \in \mathbf{H}^*$. If we put $z = x_0 + ix_1$ and $w = x_2 + ix_3$ as before then

$$\begin{aligned} X_1(x) &= \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}\ , \\ X_2(x) &= \begin{pmatrix} z^2 & zw & w^2 \\ -2z\overline{w} & z\overline{z} - w\overline{w} & 2\overline{z}w \\ \overline{w}^2 & -\overline{zw} & \overline{z}^2 \end{pmatrix}\ , \\ X_3(x) &= \begin{pmatrix} z^3 & z^2w & zw^2 & w^3 \\ -3z^2\overline{w} & z^2\overline{z} - 2zw\overline{w} & 2z\overline{z}w - w^2\overline{w} & 3\overline{z}w^2 \\ 3z\overline{w}^2 & w\overline{w}^2 - 2z\overline{zw} & z\overline{z}^2 - 2\overline{z}w\overline{w} & 3\overline{z}^2w \\ \overline{w}^3 & \overline{zw}^2 & -\overline{z}^2\overline{w} & \overline{z}^3 \end{pmatrix}\ . \end{aligned}$$

Lemma 4.2.6. *As polynomials in the four variables $x_0,\dots,x_3$, the entries of the matrices $X_\nu(x)$ are harmonic polynomials of degree ν.*

Proof. See [22, II § 6, Prop. 6]. □

Corollary 4.2.7. *Let* $\{Q_1, \ldots, Q_{\nu+1}\}$ *be an orthonormal basis of* $\mathrm{Hom}_\nu[X,Y]$ *with respect to* $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$. *Then the matrix coefficients*

$$Q_{ij}(x) := \big\langle\!\big\langle \sigma_\nu(x) Q_i,\, Q_j \big\rangle\!\big\rangle$$

constitute an orthogonal basis of $\mathrm{Harm}_\nu[X_0, \ldots, X_3]$ *with respect to* $\langle \cdot\,, \cdot \rangle$. *More precisely,*

$$\int_{\mathbf{H}^1} Q_{ij}(x)\overline{Q_{kl}(x)}\, dx = \frac{1}{\dim \sigma_\nu}\, \delta_{ik}\delta_{jl} \ .$$

Proof. The previous lemma shows that the Q_{ij} are elements of $\mathrm{Harm}_\nu[X_0, \ldots, X_3]$. By Proposition 4.2.4, $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$ is invariant under $\sigma_\nu|_{\mathbf{H}^1}$. We may therefore apply the Schur orthogonality (see Theorem 2.1.5) to obtain

$$\int_{\mathbf{H}^1} \big\langle\!\big\langle \sigma_\nu(x)Q_i, Q_j \big\rangle\!\big\rangle \overline{\big\langle\!\big\langle \sigma_\nu(x)Q_k, Q_l \big\rangle\!\big\rangle}\, dx = \frac{1}{\dim \sigma_\nu} \langle\!\langle Q_i, Q_k \rangle\!\rangle \overline{\langle\!\langle Q_j, Q_l \rangle\!\rangle} = \frac{1}{\dim \sigma_\nu} \cdot \delta_{ik}\delta_{jl} \ ,$$

which proves that the Q_{ij} are orthogonal and in particular linearly independent. Finally observe that the span of the polynomials Q_{ij} is indeed all of $\mathrm{Harm}_\nu[X_0, \ldots, X_3]$ as both spaces have dimension $(\nu+1)^2$ (see Theorem 4.1.2). □

Corollary 4.2.8. *The character* χ_{σ_ν} *of* σ_ν *can be expressed in terms of the polynomials* Q_{ij} *as*

$$\chi_{\sigma_\nu}(x) = \sum_{i=1}^{\nu+1} Q_{ii}(x) \qquad \textit{for all } x \in \mathbf{H}^* \ .$$

Proof. With respect to the orthonormal basis $\{Q_1, \ldots, Q_{\nu+1}\}$, the operator $\sigma_\nu(x)$ has the matrix $\big(\langle\!\langle \sigma_\nu(x)Q_j, Q_i \rangle\!\rangle\big)_{i,j=1}^{\nu+1}$. Its trace is

$$\chi_{\sigma_\nu}(x) = \sum_{i=1}^{\nu+1} \langle\!\langle \sigma_\nu(x)Q_i, Q_i \rangle\!\rangle = \sum_{i=1}^{\nu+1} Q_{ii}(x) \ .$$

□

4.3 Harmonic polynomials and automorphic forms

As earlier, let A be a definite quaternion algebra over F and $\mathfrak{O}$ an Eichler order in A.

For each archimedean place v of F, fix a non-negative integer ν_v and collect all of them in a vector $\boldsymbol{\nu} = (\nu_{v_1}, \ldots, \nu_{v_n})$. In order to deal with all archimedean places at the same time we use the following obvious notation for any $\mathbf{j} \in \mathbb{N}_0^n$:

$$\mathbf{0} \leq \mathbf{j} \leq \boldsymbol{\nu} \qquad :\Longleftrightarrow \qquad 0 \leq j_v \leq \nu_v \text{ for all } v \mid \infty \ .$$

Put

$$\mathrm{Hom}_{\boldsymbol{\nu}}[X,Y] := \bigotimes_{v\mid\infty} \mathrm{Hom}_{\nu_v}[X,Y] \quad \text{and} \quad \mathrm{Harm}_{\boldsymbol{\nu}}[X_0,\dots,X_3] := \bigotimes_{v\mid\infty} \mathrm{Harm}_{\nu_v}[X_0,\dots,X_3] \ .$$

Then

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\boldsymbol{\nu}}[X,Y] = \prod_{v\mid\infty} (\nu_v + 1) \ .$$

Since the local quaternion algebra A_v is isomorphic to the Hamiltonians $\mathbf{H}$, we may let A_v^* act on the polynomial space $\mathrm{Hom}_{\nu_v}[X,Y]$ by Λ_{ν_v}, which is the representation of $\mathbf{H}^*$ that we introduced in the previous section. By collecting these local representations we obtain a representation $\Lambda_{\boldsymbol{\nu}}$ of A_∞^* given by

$$\Lambda_{\boldsymbol{\nu}} := \bigotimes_{v\mid\infty} \Lambda_{\nu_v} = \bigotimes_{v\mid\infty} \mathrm{nrd}(\,\cdot\,)^{-\frac{\nu_v}{2}} \sigma_{\nu_v} =: \mathrm{nrd}(\,\cdot\,)^{-\frac{\boldsymbol{\nu}}{2}} \sigma_{\boldsymbol{\nu}} \ ,$$

which acts on $\mathrm{Hom}_{\boldsymbol{\nu}}[X,Y]$. Similarly we write

$$X_{\boldsymbol{\nu}}(\,\cdot\,) := \big(x_{\mathbf{ij}}(\,\cdot\,)\big)_{\mathbf{i},\mathbf{j}=\mathbf{0}}^{\boldsymbol{\nu}} := \otimes_{v\mid\infty} X_{\nu_v}(\,\cdot\,) = \otimes_{v\mid\infty} \big(x_{i_v j_v}(\,\cdot\,)\big)_{i_v,j_v=0}^{\nu_v}$$

where each $X_{\nu_v}(\,\cdot\,)$ is the matrix defined in Section 4.2.

Consider the space V of functions $\phi : A_{\mathbb{A}}^* \to \mathbb{C}$ satisfying

(i) $\phi(\gamma g k) = \phi(g)$ for all $\gamma \in A_F^*$, $g \in A_{\mathbb{A}}^*$, $k \in \prod_{\mathfrak{p}<\infty} \mathfrak{O}_{\mathfrak{p}}^*$,

(ii) the right regular representation ρ of A_∞^* on $E_\phi := \langle \rho(g)\phi \mid g \in A_\infty^* \rangle$ is $\Lambda_{\boldsymbol{\nu}}$-isotypic, i. e. it is equivalent to a direct sum $\bigoplus \Lambda_{\boldsymbol{\nu}}$.

Lemma 4.3.1. *Suppose that there is an* $\varepsilon \in \mathfrak{o}_F^*$ *such that* $\mathrm{sgn}(\varepsilon)^{\boldsymbol{\nu}} = \prod_{v\mid\infty} \mathrm{sgn}(\varepsilon_v)^{\nu_v} = -1$. *Then*

$$V = \{0\} \ .$$

Proof. Assume that $\varepsilon \in \mathfrak{o}_F^*$ satisfies $\mathrm{sgn}(\varepsilon)^{\boldsymbol{\nu}} = -1$. Let $\phi \in V$. Since $\varepsilon \in Z(A_F^*)$ and $\varepsilon_{\mathfrak{p}} \in \mathfrak{O}_{\mathfrak{p}}^*$ for all $\mathfrak{p} < \infty$, we can use property (i) to see that

$$\phi(g) = \phi(\varepsilon g) = \phi(g\varepsilon^\infty \varepsilon^f) = \phi(g\varepsilon^\infty) = \big(\rho(\varepsilon^\infty)\phi\big)(g) \qquad \text{for all } g \in A_{\mathbb{A}}^* \ .$$

Now apply property (ii) and continue

$$\phi(g) = \big(\Lambda_{\boldsymbol{\nu}}(\varepsilon^\infty)\phi\big)(g) = \mathrm{nrd}(\varepsilon^\infty)^{-\frac{\boldsymbol{\nu}}{2}} \big(\sigma_{\boldsymbol{\nu}}(\varepsilon^\infty)\phi\big)(g) = \Big(\prod_{v\mid\infty} (\varepsilon_v^2)^{-\frac{\nu_v}{2}} \varepsilon_v^{\nu_v}\Big)\phi(g) = \mathrm{sgn}(\varepsilon)^{\boldsymbol{\nu}} \phi(g) \ ,$$

where we made use of Lemma 4.2.2. From this equation and our assumption on ε it is clear that $\phi = 0$. □

In the light of this lemma we assume for the rest of this chapter that

$$\operatorname{sgn}(\varepsilon)^{\boldsymbol{\nu}} = 1 \qquad \text{for all} \qquad \varepsilon \in \mathfrak{o}_F^* \ .$$

By definition of the space V there is an isomorphism τ, depending on ϕ, such that the diagram

$$\begin{tikzcd} E_\phi \arrow[r, "\tau"] \arrow[d, "\rho(g)"'] & \bigoplus \mathrm{Hom}_{\boldsymbol{\nu}}[X,Y] \arrow[d, "\Lambda_\nu(g)"] \\ E_\phi \arrow[r, "\tau"] & \bigoplus \mathrm{Hom}_{\boldsymbol{\nu}}[X,Y] \end{tikzcd}$$

commutes for all $g \in A_\infty^*$. For $h \in A_{\mathbb{A}}^*$,

$$L_h : \bigoplus \mathrm{Hom}_{\boldsymbol{\nu}}[X,Y] \to \mathbb{C} \ , \qquad P \mapsto \big(\tau^{-1}(P)\big)(h)$$

is a linear map.

Lemma 4.3.2. *Fix $\phi \in V$ and let $P_0 = \tau(\phi)$. For $h \in A_{\mathbb{A}}^*$ define a map*

$$\tilde\phi(h) : A_\infty \to \mathbb{C} \ , \qquad g \mapsto \begin{cases} L_h\big(\Lambda_{\boldsymbol{\nu}}(g)P_0\big) & \text{if } g \neq 0 \ , \\ 0 & \text{if } g = 0 \ . \end{cases}$$

Then

(i) $\mathrm{nrd}(\,\cdot\,)^{\frac{\boldsymbol{\nu}}{2}}\tilde\phi(h)$ *is a harmonic polynomial in* $\mathrm{Harm}_{\boldsymbol{\nu}}[X_0,\dots,X_3]$.

(ii) $\phi(hg) = \tilde\phi(h)(g)$ *for all* $g \in A_\infty^*$.

(iii) $\phi(hz) = \phi(h)$ *for all* $z \in F_\infty^+ \subseteq Z(A_\infty^*)$.

Proof. (i) By definition, the restriction $\tilde\phi(h)|_{A_\infty^*}$ is a matrix coefficient of $\Lambda_{\boldsymbol{\nu}}$. Therefore $\mathrm{nrd}(\,\cdot\,)^{\frac{\boldsymbol{\nu}}{2}}\tilde\phi(h)$, when restricted to A_∞^*, is a matrix coefficient of $\sigma_{\boldsymbol{\nu}}$ and hence a harmonic polynomial by Lemma 4.2.6.

(ii) We use the commutativity of the diagram above to get

$$\phi(hg) = \big(\rho(g)\phi\big)(h) = \big((\tau^{-1}\circ\Lambda_{\boldsymbol{\nu}}(g)\circ\tau)(\phi)\big)(h) = \big(\tau^{-1}(\Lambda_{\boldsymbol{\nu}}(g)P_0)\big)(h) = L_h\big(\Lambda_{\boldsymbol{\nu}}(g)P_0\big) \ .$$

(iii) For $z \in F_\infty^+$, we have $z = \mathrm{nrd}(z)^{\frac12}$. Hence

$$\begin{aligned} \phi(hz) &= \mathrm{nrd}(z)^{-\frac{\boldsymbol{\nu}}{2}}\Big(\mathrm{nrd}(z)^{\frac{\boldsymbol{\nu}}{2}}\tilde\phi(h)(z)\Big) = \mathrm{nrd}\Big(\mathrm{nrd}(z)^{-\frac12}z\Big)^{\frac{\boldsymbol{\nu}}{2}}\tilde\phi(h)\Big(\mathrm{nrd}(z)^{-\frac12}z\Big) \\ &= \tilde\phi(h)(1) = \phi(h) \end{aligned}$$

where we used the homogenity of $\mathrm{nrd}(\,\cdot\,)^{\frac{\boldsymbol{\nu}}{2}}\tilde\phi(h)$, which was shown in *(i)*.

∎

Lemma 4.3.3. *Let $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$ be a $\Lambda_{\boldsymbol{\nu}}$-invariant inner product on $\bigoplus \operatorname{Hom}_{\boldsymbol{\nu}}[X, Y]$ and $\{Q_i\}$ an orthonormal basis with respect to $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$. For $h \in A_{\mathbb{A}}^*$ and $g \in A_\infty^*$,*

$$\begin{aligned}\phi(hg) &= \Big\langle\!\Big\langle \Lambda_{\boldsymbol{\nu}}(g)P_0,\, \sum_i \overline{L_h(Q_i)}\, Q_i \Big\rangle\!\Big\rangle \\ &= \operatorname{nrd}(g)^{-\frac{\boldsymbol{\nu}}{2}} \sum_i \sum_m \langle\!\langle P_0, Q_m \rangle\!\rangle L_h(Q_i) Q_{mi}(g)\end{aligned}$$

where $Q_{mi}(g) = \langle\!\langle \sigma_{\boldsymbol{\nu}}(g)Q_m, Q_i \rangle\!\rangle$ as in the previous section.

Proof. By the previous lemma, $\phi(hg) = \tilde{\phi}(h)(g) = L_h\big(\Lambda_{\boldsymbol{\nu}}(g)P_0\big)$. But for any polynomial $P \in \bigoplus \operatorname{Hom}_{\boldsymbol{\nu}}[X, Y]$ and in particular for $P = \Lambda_{\boldsymbol{\nu}}(g)P_0$, we can easily verify

$$L_h(P) = L_h\Big(\sum_i \langle\!\langle P, Q_i \rangle\!\rangle Q_i\Big) = \sum_i \langle\!\langle P, Q_i \rangle\!\rangle L_h(Q_i) = \Big\langle\!\Big\langle P, \sum_i \overline{L_h(Q_i)}\, Q_i \Big\rangle\!\Big\rangle\ ,$$

which proves the first equality. For the second equality observe that

$$\begin{aligned}\Big\langle\!\Big\langle \Lambda_{\boldsymbol{\nu}}(g)P_0,\, \sum_i \overline{L_h(Q_i)}\, Q_i \Big\rangle\!\Big\rangle &= \sum_i L_h(Q_i) \langle\!\langle \Lambda_{\boldsymbol{\nu}}(g)P_0, Q_i \rangle\!\rangle \\ &= \sum_i L_h(Q_i) \Big\langle\!\Big\langle \Lambda_{\boldsymbol{\nu}}(g)\Big(\sum_m \langle\!\langle P_0, Q_m \rangle\!\rangle Q_m\Big), Q_i \Big\rangle\!\Big\rangle \\ &= \sum_i \sum_m \langle\!\langle P_0, Q_m \rangle\!\rangle L_h(Q_i) \big\langle\!\big\langle \Lambda_{\boldsymbol{\nu}}(g)Q_m, Q_i \big\rangle\!\big\rangle\end{aligned}$$

and that $\Lambda_{\boldsymbol{\nu}}(g) = \operatorname{nrd}(g)^{-\frac{\boldsymbol{\nu}}{2}} \sigma_{\boldsymbol{\nu}}(g)$. □

We use the previous Lemmas to compute an integral that we will encounter in Section 5.3.

Lemma 4.3.4. *Let $h \in A_{\mathbb{A}}^*$ and $z \in A_\infty^*$. Then*

$$\int\limits_{A_\infty^1} \phi(hk)\chi_{\sigma_{\boldsymbol{\nu}}}(\overline{zk})\, dk = \frac{\operatorname{nrd}(z)^{\frac{\boldsymbol{\nu}}{2}}}{\dim \sigma_{\boldsymbol{\nu}}}\, \phi(h\overline{z}) = \frac{1}{\dim \sigma_{\boldsymbol{\nu}}} \Phi(h)(\overline{z})$$

where $\Phi(h) = \operatorname{nrd}(\,\cdot\,)^{\frac{\boldsymbol{\nu}}{2}} \big(\rho(\,\cdot\,)\phi\big)(h) \in \operatorname{Harm}_{\boldsymbol{\nu}}[X_0, \ldots, X_3]$.

Proof. As before, let $\{Q_i\}$ be an orthonormal basis of $\bigoplus \operatorname{Hom}_{\boldsymbol{\nu}}[X, Y]$. Use Corollary 4.2.8 to rewrite the character as

$$\chi_{\sigma_{\boldsymbol{\nu}}}(\overline{zk}) = \sum_p \langle\!\langle \sigma_{\boldsymbol{\nu}}(\overline{zk})Q_p, Q_p \rangle\!\rangle = \sum_p \big\langle\!\big\langle \sigma_{\boldsymbol{\nu}}(\overline{k})\big(\sigma_{\boldsymbol{\nu}}(\overline{z})Q_p\big), Q_p \big\rangle\!\big\rangle\ .$$

Since $\operatorname{nrd}(k) = 1$, the inner product is invariant under $\sigma_{\boldsymbol{\nu}}(\overline{k})$. Hence

$$\chi_{\sigma_{\boldsymbol{\nu}}}(\overline{zk}) = \sum_p \big\langle\!\big\langle \sigma_{\boldsymbol{\nu}}(\overline{z})Q_p,\, \sigma_{\boldsymbol{\nu}}(\overline{k}^{-1})Q_p \big\rangle\!\big\rangle = \sum_p \overline{\big\langle\!\big\langle \sigma_{\boldsymbol{\nu}}(k)Q_p,\, \sigma_{\boldsymbol{\nu}}(\overline{z})Q_p \big\rangle\!\big\rangle}\ .$$

Here $\sigma_{\boldsymbol{\nu}}(\overline{z})Q_p$ is again a homogeneous polynomial, which can be expressed in terms of the basis $\{Q_i\}$ as

$$\sigma_{\boldsymbol{\nu}}(\overline{z})Q_p = \sum_q \alpha_q^{(p)} Q_q \qquad \text{for some } \alpha_q^{(p)} \in \mathbb{C} .$$

Then

$$\chi_{\sigma_{\boldsymbol{\nu}}}(\overline{zk}) = \sum_p \overline{\langle\!\langle \sigma_{\boldsymbol{\nu}}(k)Q_p, \sum_q \alpha_q^{(p)} Q_q \rangle\!\rangle} = \sum_p \sum_q \alpha_q^{(p)} \overline{Q_{pq}(k)} .$$

Together with the results of Lemma 4.3.3 we get

$$\begin{aligned}
\int\limits_{A^1_\infty} \phi(hk)\chi_{\sigma_{\boldsymbol{\nu}}}(\overline{zk})\,dk &= \int\limits_{A^1_\infty} \sum_i \sum_m \langle\!\langle P_0, Q_m\rangle\!\rangle L_h(Q_i) Q_{mi}(k) \sum_p \sum_q \alpha_q^{(p)} \overline{Q_{pq}(k)}\,dk \\
&= \sum_i \sum_m \sum_p \sum_q \alpha_q^{(p)} \langle\!\langle P_0, Q_m\rangle\!\rangle L_h(Q_i) \int\limits_{A^1_\infty} Q_{mi}(k)\overline{Q_{pq}(k)}\,dk \\
&= \frac{1}{\dim \sigma_{\boldsymbol{\nu}}} \sum_i \sum_m \alpha_i^{(m)} \langle\!\langle P_0, Q_m\rangle\!\rangle L_h(Q_i)
\end{aligned}$$

by Corollary 4.2.7, in which the orthogonality of the $\{Q_{ij}\}$ was proven. We continue

$$\begin{aligned}
\int\limits_{A^1_\infty} \phi(hk)\chi_{\sigma_{\boldsymbol{\nu}}}(\overline{zk})\,dk &= \frac{1}{\dim \sigma_{\boldsymbol{\nu}}} \sum_m \langle\!\langle P_0, Q_m\rangle\!\rangle L_h\Big(\sum_i \alpha_i^{(m)} Q_i\Big) \\
&= \frac{1}{\dim \sigma_{\boldsymbol{\nu}}} \sum_m \langle\!\langle P_0, Q_m\rangle\!\rangle L_h\big(\sigma_{\boldsymbol{\nu}}(\overline{z})Q_m\big) .
\end{aligned}$$

Now use the $\Lambda_{\boldsymbol{\nu}}$-invariance of $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$ to obtain

$$\begin{aligned}
\int\limits_{A^1_\infty} \phi(hk)\chi_{\sigma_{\boldsymbol{\nu}}}(\overline{zk})\,dk &= \frac{\operatorname{nrd}(z)^{\frac{\boldsymbol{\nu}}{2}}}{\dim \sigma_{\boldsymbol{\nu}}} \sum_m \langle\!\langle \Lambda_{\boldsymbol{\nu}}(\overline{z})P_0,\, \Lambda_{\boldsymbol{\nu}}(\overline{z})Q_m\rangle\!\rangle L_h\big(\Lambda_{\boldsymbol{\nu}}(\overline{z})Q_m\big) \\
&= \frac{\operatorname{nrd}(z)^{\frac{\boldsymbol{\nu}}{2}}}{\dim \sigma_{\boldsymbol{\nu}}} \Big\langle\!\!\Big\langle \Lambda_{\boldsymbol{\nu}}(\overline{z})P_0,\, \sum_m \overline{L_h\big(\Lambda_{\boldsymbol{\nu}}(\overline{z})Q_m\big)}\,\Lambda_{\boldsymbol{\nu}}(\overline{z})Q_m \Big\rangle\!\!\Big\rangle \\
&= \frac{\operatorname{nrd}(z)^{\frac{\boldsymbol{\nu}}{2}}}{\dim \sigma_{\boldsymbol{\nu}}}\,\phi(h\overline{z}) .
\end{aligned}$$

The last equality can be shown with the same argument that was used in Lemma 4.3.3 because $\{\Lambda_{\boldsymbol{\nu}}(\overline{z})Q_m\}$ is again an orthonormal basis with respect to $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$. Finally, apply *(i)* and *(ii)* of Lemma 4.3.2 to get the desired result. $\square$

The rest of this chapter is devoted to the study of a certain subspace V_0 of V, which will be used in our discussion in Sections 5.2 and 5.3. To this end define the space

$$W := \Big\langle \langle\!\langle \Lambda_{\boldsymbol{\nu}}(\,\cdot\,)P,\, Q\rangle\!\rangle \;\Big|\; P, Q \in \operatorname{Hom}_{\boldsymbol{\nu}}[X,Y] \Big\rangle ,$$

which is generated by all matrix coefficients of $\Lambda_{\boldsymbol{\nu}}$. On W consider the left regular representation $\lambda : A^*_\infty \to \mathrm{GL}(W)$.

Lemma 4.3.5. *Let* $\{P_{\mathbf{i}} \mid \mathbf{i} = \mathbf{0}, \ldots, \boldsymbol{\nu}\}$ *be a basis of* $\mathrm{Hom}_{\boldsymbol{\nu}}[X,Y]$*. The space* W *decomposes into irreducible* λ*-invariant subspaces*

$$W_{\mathbf{j}} := \Big\langle \langle\!\langle \Lambda_{\boldsymbol{\nu}}(\,\cdot\,)P_{\mathbf{j}},\, Q \rangle\!\rangle \mid Q \in \mathrm{Hom}_{\boldsymbol{\nu}}[X,Y] \Big\rangle \qquad \textit{for} \quad \mathbf{j} = \mathbf{0}, \ldots, \boldsymbol{\nu} \ .$$

The action of λ *on each* $W_{\mathbf{j}}$ *is equivalent to* $\Lambda_{\boldsymbol{\nu}}$*. In particular, if*

$$P_{\mathbf{i}} := X^{\boldsymbol{\nu}-\mathbf{i}}Y^{\mathbf{i}} := \otimes_{v|\infty} X^{\nu_v - i_v} Y^{i_v} \qquad \textit{for} \qquad \mathbf{i} = \mathbf{0}, \ldots, \boldsymbol{\nu}$$

then the irreducible subspaces of W *are of the form*

$$W_{\mathbf{j}} = \big\langle \mathrm{nrd}(\,\cdot\,)^{-\frac{\boldsymbol{\nu}}{2}} x_{\mathbf{ij}}(\,\cdot\,) \mid \mathbf{i} = \mathbf{0}, \ldots, \boldsymbol{\nu} \big\rangle \qquad \textit{for} \quad \mathbf{j} = \mathbf{0}, \ldots, \boldsymbol{\nu}$$

where $X_{\boldsymbol{\nu}}(\,\cdot\,) = \big(x_{\mathbf{ij}}(\,\cdot\,)\big)$ *is the matrix defined above.*

Proof. On elements of the form $\langle\!\langle \Lambda_{\boldsymbol{\nu}}(\,\cdot\,)P, Q \rangle\!\rangle$ with homogeneous polynomials P, Q, the action of λ is given by

$$\big(\lambda(g)\langle\!\langle \Lambda_{\boldsymbol{\nu}}(\,\cdot\,)P, Q \rangle\!\rangle\big)(h) = \langle\!\langle \Lambda_{\boldsymbol{\nu}}(g^{-1}h)P, Q \rangle\!\rangle = \langle\!\langle \Lambda_{\boldsymbol{\nu}}(h)P,\, \Lambda_{\boldsymbol{\nu}}(g)Q \rangle\!\rangle \ .$$

Thus we see that each $W_{\mathbf{j}}$ is λ-invariant. Moreover, in Lemma 4.2.3 we saw that $\Lambda_{\boldsymbol{\nu}}$ is irreducible on $\mathrm{Hom}_{\boldsymbol{\nu}}[X,Y]$. Therefore, the spaces $W_{\mathbf{j}}$ are irreducible as well. Now consider the basis consisting of the polynomials $P_{\mathbf{i}} := X^{\boldsymbol{\nu}-\mathbf{i}}Y^{\mathbf{i}}$ for $\mathbf{i} = \mathbf{0}, \ldots, \boldsymbol{\nu}$. Then $\{P_{\mathbf{0}}, \ldots, P_{\boldsymbol{\nu}}\}$ is orthogonal with respect to $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$ and

$$\Lambda_{\boldsymbol{\nu}}(\,\cdot\,)P_{\mathbf{j}} = \mathrm{nrd}(\,\cdot\,)^{-\frac{\boldsymbol{\nu}}{2}} \sigma_{\boldsymbol{\nu}}(\,\cdot\,)P_{\mathbf{j}} = \mathrm{nrd}(\,\cdot\,)^{-\frac{\boldsymbol{\nu}}{2}} \sum_{\mathbf{i}=\mathbf{0}}^{\boldsymbol{\nu}} x_{\mathbf{ij}}(\,\cdot\,)P_{\mathbf{i}} \ .$$

Hence

$$\langle\!\langle \Lambda_{\boldsymbol{\nu}}(\,\cdot\,)P_{\mathbf{j}},\, P_{\mathbf{i}} \rangle\!\rangle = \kappa \cdot \mathrm{nrd}(\,\cdot\,)^{-\frac{\boldsymbol{\nu}}{2}} x_{\mathbf{ij}}(\,\cdot\,)$$

for some constant $\kappa \in \mathbb{C}$ which takes into account that the $P_{\mathbf{i}}$ are not normalized. In this particular situation

$$W_{\mathbf{j}} = \big\langle \mathrm{nrd}(\,\cdot\,)^{-\frac{\boldsymbol{\nu}}{2}} x_{\mathbf{ij}}(\,\cdot\,) \mid \mathbf{i} = \mathbf{0}, \ldots, \boldsymbol{\nu} \big\rangle$$

as claimed. □

Now recall from (1.1) the decomposition

$$A_{\mathbb{A}}^* = \coprod_{j=1}^{H} A_F^* y_j \mathfrak{O}_{\mathbb{A}}^* \qquad\qquad (\text{where } y_j^\infty = 1)$$

where H denotes the ideal class number of $\mathfrak{O}$-right ideals. For $i = 1, \ldots, H$ let $\mathfrak{O}_i$ be the Eichler order of A that is uniquely determined by the local data

$$(\mathfrak{O}_i)_{\mathfrak{p}} = \big(y_i \mathfrak{O} y_i^{-1}\big)_{\mathfrak{p}} \qquad \text{for all } \mathfrak{p} < \infty \ .$$

Lemma 4.3.6. *As before, assume that* $\operatorname{sgn}(\varepsilon)^{\boldsymbol{\nu}} = 1$ *for all* $\varepsilon \in \mathfrak{o}_F^*$ *and let* $P_{\mathbf{i}} := X^{\boldsymbol{\nu}-\mathbf{i}}Y^{\mathbf{i}}$ *for* $\mathbf{i} = \mathbf{0}, \ldots, \boldsymbol{\nu}$. *Fix an index* $\mathbf{j} \in \{\mathbf{0}, \ldots, \boldsymbol{\nu}\}$. *Define a subspace of* V *by*

$$V_0 := \left\{\phi \in V \mid \big(\rho(\,\cdot\,)\phi\big)(g) \in W_{\mathbf{j}} \text{ for all } g \in A_{\mathbb{A}}^*\right\} .$$

Then the following assertions hold:

(i) For $i \in \{1, \ldots, H\}$ *and* $\mathbf{m} \in \{\mathbf{0}, \ldots, \boldsymbol{\nu}\}$ *the elements*

$$Q_{i\mathbf{m}}(x) := \sum_{r \in \mathfrak{O}_i^*/\mathfrak{o}_F^*} \big\langle\!\big\langle \Lambda_{\boldsymbol{\nu}}(rx)P_{\mathbf{j}}, P_{\mathbf{m}} \big\rangle\!\big\rangle \qquad \text{for } x \in A_\infty^*$$

are $\mathfrak{O}_i^*$*-invariant on the left.*

(ii) For fixed i *the set* $\{Q_{i\mathbf{m}} \mid \mathbf{m} = \mathbf{0}, \ldots, \boldsymbol{\nu}\}$ *is a set of generators of*

$$W_{\mathbf{j}}^{(\mathfrak{O}_i^*)} := \{Q \in W_{\mathbf{j}} \mid Q \text{ is } \mathfrak{O}_i^*\text{-invariant on the left}\} .$$

(iii) The functions $\phi_{i\mathbf{m}}$ *for* $i = 1, \ldots, H$, $\mathbf{m} = \mathbf{0}, \ldots, \boldsymbol{\nu}$ *defined by*

$$\phi_{i\mathbf{m}}(g) := \begin{cases} Q_{i\mathbf{m}}(m^\infty) & \text{if } g = ay_im \in A_F^* y_i \mathfrak{O}_{\mathbb{A}}^* , \\ 0 & \text{else} \end{cases}$$

generate V_0.

Proof. (i) The quotient $\mathfrak{O}_i^*/\mathfrak{o}_F^*$ is finite (see [20, § 1, Satz 2]) and

$$\Lambda_{\boldsymbol{\nu}}(\varepsilon) = \operatorname{nrd}(\varepsilon)^{\frac{-\boldsymbol{\nu}}{2}} \sigma_{\boldsymbol{\nu}}(\varepsilon) = (\varepsilon^2)^{-\frac{\boldsymbol{\nu}}{2}} \varepsilon^{\nu} \sigma_{\boldsymbol{\nu}}(1) = \operatorname{sgn}(\varepsilon)^{\boldsymbol{\nu}} = 1 \qquad \text{for all} \qquad \varepsilon \in \mathfrak{o}_F^*$$

by assumption. So $Q_{i\mathbf{m}}$ is indeed well-defined. Moreover, left multiplication of the argument x by an element in $\mathfrak{O}_i^*$ only permutes the summands, so $Q_{i\mathbf{m}}$ is clearly left-$\mathfrak{O}_i^*$-invariant.

(ii) Let $Q \in W_{\mathbf{j}}^{(\mathfrak{O}_i^*)}$. Then $Q = \langle\!\langle \Lambda_{\boldsymbol{\nu}}(\,\cdot\,)P_{\mathbf{j}}, P \rangle\!\rangle$ for some $P \in \operatorname{Hom}_{\boldsymbol{\nu}}[X, Y]$, and because of the $\mathfrak{O}_i^*$-invariance we have $Q(x) = Q(rx)$ for all $r \in \mathfrak{O}_i^*$, i. e.

$$\langle\!\langle \Lambda_{\boldsymbol{\nu}}(\,\cdot\,)P_{\mathbf{j}},\, P \rangle\!\rangle = \big\langle\!\big\langle \Lambda_{\boldsymbol{\nu}}(\,\cdot\,)P_{\mathbf{j}},\, \Lambda_{\boldsymbol{\nu}}(r^{-1})P \big\rangle\!\big\rangle \qquad \text{for all } r \in \mathfrak{O}_i^* .$$

Summing both sides over a complete set of representatives for $\mathfrak{O}_i^*/\mathfrak{o}_F^*$ yields

$$\langle\!\langle \Lambda_{\boldsymbol{\nu}}(\,\cdot\,)P_{\mathbf{j}},\, P \rangle\!\rangle = \frac{1}{[\mathfrak{O}_i^* : \mathfrak{o}_F^*]} \sum_{r \in \mathfrak{O}_i^*/\mathfrak{o}_F^*} \big\langle\!\big\langle \Lambda_{\boldsymbol{\nu}}(\,\cdot\,)P_{\mathbf{j}},\, \Lambda_{\boldsymbol{\nu}}(r^{-1})P \big\rangle\!\big\rangle .$$

So if $P = \sum_{\mathbf{m}=\mathbf{0}}^{\boldsymbol{\nu}} \alpha_{\mathbf{m}} P_{\mathbf{m}}$ for some $\alpha_{\mathbf{m}} \in \mathbb{C}$ then

$$Q = \langle\!\langle \Lambda_{\boldsymbol{\nu}}(\,\cdot\,)P_{\mathbf{j}},\, P \rangle\!\rangle = \sum_{\mathbf{m}=\mathbf{0}}^{\boldsymbol{\nu}} \frac{\overline{\alpha}_{\mathbf{m}}}{[\mathfrak{O}_i^* : \mathfrak{o}_F^*]} Q_{i\mathbf{m}} ,$$

so Q lies indeed in the span of the $Q_{i\mathbf{m}}$.

(iii) First note that the $\phi_{i\mathbf{m}}$ are well-defined. Indeed, if $g = ay_i m = \tilde{a} y_i \tilde{m}$ then it is easily checked that $\tilde{m}^\infty = \tilde{a}^{-1} a m^\infty$ where $\tilde{a}^{-1} a \in A_F^* \cap y_i \mathfrak{O}_{\mathbb{A}}^* y_i^{-1} = \mathfrak{O}_i^*$. Since every $Q_{i\mathbf{m}}$ is $\mathfrak{O}_i^*$-invariant on the left we see that $\phi_{i\mathbf{m}}(g)$ does not depend on the particular factorization of g. Next we need to verify that $\phi_{i\mathbf{m}} \in V$. By construction we have $\phi_{i\mathbf{m}}(\gamma g k) = \phi_{i\mathbf{m}}(g)$ for all $\gamma \in A_F^*$, $g \in A_{\mathbb{A}}^*$ and $k \in \prod_{\mathfrak{p}<\infty} \mathfrak{O}_{\mathfrak{p}}^*$. Let $h \in A_\infty^*$. If $g = ay_i m$ as before then $gh = ay_i(mh)$ with $mh \in \mathfrak{O}_{\mathbb{A}}^*$. Hence

$$\big(\rho(h)\phi_{i\mathbf{m}}\big)(g) = Q_{i\mathbf{m}}\big((mh)^\infty\big) = \sum_{r \in \mathfrak{O}_i^*/\mathfrak{o}_F^*} \big\langle\!\big\langle \Lambda_{\boldsymbol{\nu}}(rm)\big(\Lambda_{\boldsymbol{\nu}}(h) P_{\mathbf{j}}\big),\, P_{\mathbf{m}} \big\rangle\!\big\rangle \ .$$

So $\rho(h)$ operates as $P_{\mathbf{j}} \mapsto \Lambda_{\boldsymbol{\nu}}(h) P_{\mathbf{j}}$, which means that the action of A_∞^* is $\Lambda_{\boldsymbol{\nu}}$-isotypic. Consequently, $\phi_{i\mathbf{m}} \in V$. Similarly,

$$\big(\rho(\,\cdot\,)\phi_{i\mathbf{m}}\big)(g) = \sum_{r \in \mathfrak{O}_i^*/\mathfrak{o}_F^*} \big\langle\!\big\langle \Lambda_{\boldsymbol{\nu}}(\,\cdot\,) P_{\mathbf{j}},\, \Lambda_{\boldsymbol{\nu}}\big((rm)^{-1}\big) P_{\mathbf{m}} \big\rangle\!\big\rangle \in W_{\mathbf{j}} \ ,$$

which shows that $\phi_{i\mathbf{m}} \in V_0$.

So all that is left to show is that the functions $\phi_{i\mathbf{m}}$ span all of V_0. To this end consider an arbitrary $\phi \in V_0$. Then $\big(\rho(\,\cdot\,)\phi\big)(h) \in W_{\mathbf{j}}$ for all $h \in A_{\mathbb{A}}^*$ by definition of V_0. More precisely, we have $\big(\rho(\,\cdot\,)\phi\big)(y_i) \in W_{\mathbf{j}}^{(\mathfrak{O}_i^*)}$, which can be seen by the following argument: If $\varepsilon \in \mathfrak{O}_i^* = A_F^* \cap y_i \mathfrak{O}_{\mathbb{A}}^* y_i^{-1}$, then there exists an $m \in \mathfrak{O}_{\mathbb{A}}^*$ such that $\varepsilon = y_i m y_i^{-1}$. In particular, $e^\infty = m^\infty$ since we always assume that $y_i^\infty = 1$. Now we use the A_F^*- and $\prod_{\mathfrak{p}} \mathfrak{O}_{\mathfrak{p}}^*$-invariance of ϕ to get for $x \in A_\infty^*$

$$\begin{aligned}\big(\rho(x)\phi\big)(y_i) &= \phi(y_i x) = \phi(\varepsilon y_i x) = \phi(y_i m x) = \phi(y_i m^\infty x m^f) \\ &= \phi\big(y_i m^\infty x m^f ((m^f)^{-1} \varepsilon^f)\big) = \phi(y_i \varepsilon^\infty x \varepsilon^f) = \phi(y_i \varepsilon x) = \big(\rho(\varepsilon x)\phi\big)(y_i) \ ,\end{aligned}$$

which shows that $\big(\rho(\,\cdot\,)\phi\big)(y_i)$ is indeed $\mathfrak{O}_i^*$-invariant on the left and hence lies in $W_{\mathbf{j}}^{(\mathfrak{O}_i^*)}$. It follows by part (ii) that

$$\big(\rho(\,\cdot\,)\phi\big)(y_i) = \sum_{\mathbf{m}=\mathbf{0}}^{\boldsymbol{\nu}} \alpha_{i\mathbf{m}} Q_{i\mathbf{m}} \qquad \text{for some } \alpha_{i\mathbf{m}} \in \mathbb{C} \ .$$

If $g = ay_i m$ as before then we use the A_F^*- and $\prod_{\mathfrak{p}<\infty} \mathfrak{O}_{\mathfrak{p}}^*$-invariance of ϕ again to obtain

$$\phi(g) = \phi(y_i m^\infty) = \sum_{\mathbf{m}=\mathbf{0}}^{\boldsymbol{\nu}} \alpha_{i\mathbf{m}} Q_{i\mathbf{m}}(m^\infty) = \sum_{\mathbf{m}=\mathbf{0}}^{\boldsymbol{\nu}} \alpha_{i\mathbf{m}} \phi_{i\mathbf{m}}(g) \ .$$

So ϕ is indeed a linear combination of the $\phi_{i\mathbf{m}}$ and the proof is complete.

□

Corollary 4.3.7. *Fix an index* $\mathbf{j} \in \{\mathbf{0}, \ldots, \boldsymbol{\nu}\}$. *To* $\phi_{i\mathbf{m}}$ *as defined above and to* $g \in A_{\mathbb{A}}^*$ *associate a harmonic polynomial* $\Phi_{i\mathbf{m}}(g) := \mathrm{nrd}(\,\cdot\,)^{\frac{\boldsymbol{\nu}}{2}} \big(\rho(\,\cdot\,)\phi_{i\mathbf{m}}\big)(g) \in \mathrm{Harm}_{\boldsymbol{\nu}}[X_0, \ldots, X_3]$ *as in Lemma 4.3.4. Then*

$$\Phi_{i\mathbf{m}}(g) = \begin{cases} \Big\langle\!\Big\langle \sigma_{\boldsymbol{\nu}}(\,\cdot\,) P_{\mathbf{j}},\, \displaystyle\sum_{r \in \mathfrak{O}_i^*/\mathfrak{o}_F^*} \Lambda_{\boldsymbol{\nu}}\big((rm)^{-1}\big) P_{\mathbf{m}} \Big\rangle\!\Big\rangle & \text{if } g = ay_i m \in A_F^* y_i \mathfrak{O}_{\mathbb{A}}^* \ , \\ 0 & \text{else} \ . \end{cases}$$

In particular, $\Phi_{i\mathbf{m}}(g)$ *is a linear combination of the entries in the* $\mathbf{j}$*-th column of* $X_{\boldsymbol{\nu}}(\,\cdot\,)$.

Proof. This is just a summary of the previous discussion. $\square$

Chapter 5

Theta series as generators of the space of Hilbert modular cusp forms

We will now turn to our main task of constructing explicit sets of generators for each of the spaces $S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_l\mathfrak{d},\mathfrak{n}),1\big)$, $l = 1,\ldots,h^+$ of Hilbert modular cusp forms. The key ingredient will be a result of Shimizu (see Theorem 5.2.3 and [65, § 6.5, Thm. 2]), which provides a set of generators of the space $U(\mathfrak{n})$ of adelic newforms (cf. Remark 3.4.4). Since we know by Theorem 2.3.7 that there is a complete correspondence between classical and adelic cusp forms, we may view Shimizu's result as an answer to our problem. It has, however, the drawback of being rather unexplicit.

Section 5.3 is therefore devoted to a detailed discussion and reformulation of Shimizu's Theorem by means of the correspondence 2.3.7. We will thus obtain an explicit set of generators of the spaces $S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_l\mathfrak{d},\mathfrak{n}),1\big)$ in terms of quaternionic theta series as were introduced in Section 1.3.

We begin with a short section on the Weil representation, which will be needed in order to state Shimizu's Theorem.

5.1 The Weil representation

In this section only, we denote by F a *local* field, which will be either the $\mathfrak{p}$-adic field $F_{\mathfrak{p}}$ or the real field $\mathbb{R}$.

Let A be a quaternion algebra over F and fix a non-trivial (additive) character of ψ of F.

We denote by $|\cdot|_F$ and $|\cdot|_A$ the *module* of F and A, respectively, which is defined by the equality $|a|_F\,dx = d(ax)$ for all $a \in F$ and any additive Haar measure dx on F, and similarly for A. For $F = \mathbb{R}$ or $F = F_{\mathfrak{p}}$, the module has the explicit form

$$|x|_{\mathbb{R}} = |x| \ , \qquad |x|_{F_{\mathfrak{p}}} = \mathcal{N}\mathfrak{p}^{-v_{\mathfrak{p}}(x)} \qquad \text{and} \qquad |a|_A = |\mathrm{nrd}(a)^2|_F \qquad \text{for } x \in F,\ a \in A$$

(cf. [78, I § 2, Cor. 3 of Thm. 3 and I § 4, Thm. 6]).

For a Schwartz-Bruhat function $f \in \mathcal{S}(A)$, we denote by $\hat{f}$ the *Fourier transform* with respect to the character ψ, that is

$$\hat{f}(x) = \int_A f(y)\psi\big(\mathrm{tr}(xy)\big)\,dy \ .$$

Definition 5.1.1 (Weil representation of $\mathbf{SL_2(F)}$). We define the local *Weil representation* $\Omega : \mathrm{SL}_2(F) \to \mathrm{GL}(\mathcal{S}(A))$ by

$$\Omega\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\right) f(x) = |a|_A^{1/2} f(ax) \ ,$$

$$\Omega\left(\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\right) f(x) = \psi\big(b\,\mathrm{nrd}(x)\big) f(x) \ ,$$

$$\Omega\left(\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right) f(x) = \gamma \hat{f}(\overline{x}) \qquad \text{where } \gamma = \begin{cases} 1 & \text{if } A \cong M_2(F) \ , \\ -1 & \text{if } A \text{ is a division algebra} \ . \end{cases}$$

The Weil representation was first introduced in [77] in a more general context. Since then it has proven to be a useful ingredient in the study of modular forms. However, we neither assume that the reader is familiar with the Weil representation nor will we go into details here. All that we need apart from the definition is the following lemma, which describes the interaction of Ω with the right and left regular representation. For further properties of Ω the reader is referred to [41, Prop. 1.3].

Lemma 5.1.2. *Let $s \in \mathrm{SL}_2(F)$ and $x \in A$. Then*

$$\rho(x)\Omega(s) = \Omega\left(\begin{pmatrix} \mathrm{nrd}(x) & 0 \\ 0 & 1\end{pmatrix} s \begin{pmatrix} \mathrm{nrd}(x)^{-1} & 0 \\ 0 & 1\end{pmatrix}\right)\rho(x) \ .$$

In particular, $\rho(x)\Omega(s) = \Omega(s)\rho(x)$ if $\mathrm{nrd}(x) = 1$. Similarly,

$$\lambda(x)\Omega(s) = \Omega(s)\lambda(x) \qquad \textit{if } \mathrm{nrd}(x) = 1 \ .$$

Proof. See [41, Lemma 1.4] or [65, Lemma 1], but note the misprint in the latter. $\square$

5.2 An explicit version of Shimizu's Theorem

Let F be a totally real number field of degree $n \geq 2$. As earlier we denote by $\mathfrak{d}$ the different of F. By [37, VIII § 63, Satz 176], there exists an element $\zeta \in F^*$ and an ideal $\mathfrak{b}$ of F such that

$$\zeta\mathfrak{d} = \mathfrak{b}^2 .$$

By $\mathfrak{c}_1, \ldots, \mathfrak{c}_{h^+}$ we denote a complete set of representatives of the narrow ideal classes of F. Further, let $\mathfrak{n}$ be a square-free integral ideal of F, and let $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ be a weight vector such that $k_i > 2$ for all i.

Our aim is to prove the following theorem:

Theorem 5.2.1.

(i) *Suppose that the degree $n = [F : \mathbb{Q}]$ is even. Then the space $S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_l\mathfrak{d}, \mathfrak{n}), 1\big)$ of Hilbert modular cusp forms of weight $\mathbf{k}$ for the group*

$$\Gamma_0(\mathfrak{c}_l\mathfrak{d}, \mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathfrak{o}_F & (\mathfrak{c}_l\mathfrak{d})^{-1} \\ \mathfrak{n}\mathfrak{c}_l\mathfrak{d} & \mathfrak{o}_F \end{pmatrix} \;\Big|\; ad - bc \in \mathfrak{o}_F^{*+} \right\}$$

with trivial character is generated by the set of all theta series of the form

$$\Theta(\mathbf{z}) = \sum_{\nu \in \mathfrak{o}_F^{*+}/\mathfrak{o}_F^2} \nu^{\frac{\mathbf{k}-\mathbf{2}}{2}} \sum_{a \in \eta^{-1}\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}} P(\overline{\eta a}) \exp\big(2\pi i \mathrm{Tr}(\nu \mathrm{nrd}(a)\mathbf{z})\big)$$

where for each divisor $\mathfrak{m}$ of $\mathfrak{n}$

- *$\mathfrak{a}$ runs through all divisors of $\mathfrak{n}\mathfrak{m}^{-1}$,*
- *we choose a quaternion algebra A together with an Eichler order $\mathfrak{O}$ of level (D_1, D_2) such that $\mathfrak{m} = D_1 D_2$,*
- *we put $I_{ij} := I_i I_j^{-1}$ where $I_1, \ldots, I_H$ is a complete set of representatives of the $\mathfrak{O}$-right ideal classes,*
- *of all pairs $(i, j) \in \{1, \ldots, H\}$ we consider only those for which there exists an $\eta \in A_F^*$ such that $\mathrm{nrd}(\eta^{-1}\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}) = \mathfrak{a}\mathfrak{c}_l$,*
- *and P runs through a set of at most $\prod_{i=1}^n (k_i - 1)^2$ harmonic polynomials of degree $\mathbf{k} - \mathbf{2}$ (a method of constructing suitable polynomials is described in Lemma 5.3.2 below).*

(ii) *If the degree n is odd then the construction in part (i) can be carried out except for the divisor $\mathfrak{m} = (1)$. The theta series that are obtained span a space U such that $S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_l\mathfrak{d}, \mathfrak{n}), 1\big) = U_1^{\mathrm{new}} \oplus U$ where U_1^{new} is generated by the new forms of level 1 and their right translates.*

Remark 5.2.2. In fact, we will only be able to prove this theorem by assuming the validity of a statement that has not yet been shown in detail (see Conjecture 5.2.5 below). However, we will defer any further comments on this gap until the end of this section. First it seems advisable to sketch the idea of our proof so that we will be able to see where the nonproven statement comes in. □

In order to state Shimizu's Theorem, which will be the main ingredient of the proof, we need to introduce some additional notation. Let us fix some adelic generators of the different $\mathfrak{d}$ and of the ideal class representatives $\mathfrak{c}_1, \ldots, \mathfrak{c}_{h^+}$. By this we mean elements

$$\delta \in \mathbb{A}_F^* \qquad \text{defined by} \qquad \delta^\infty = 1 \qquad \text{and} \qquad \delta_\mathfrak{p}\mathfrak{o}_\mathfrak{p} = \mathfrak{d}_\mathfrak{p} \qquad \text{for all } \mathfrak{p} < \infty$$

and

$$t_l \in \mathbb{A}_F^* \qquad \text{such that} \qquad t_l^\infty = 1 \qquad \text{and} \qquad t_{l,\mathfrak{p}}\mathfrak{o}_\mathfrak{p} = \mathfrak{c}_{l,\mathfrak{p}} \qquad \text{for all } \mathfrak{p} < \infty \ .$$

We use the element ζ, which was chosen above and satisfies $\zeta\mathfrak{d} = \mathfrak{b}^2$, to make a specific choice for the character ψ of $\mathbb{A}_F/F$, namely

$$\psi(x) = \tau(\zeta x) \ ,$$

where τ is the standard character on $\mathbb{A}_F/F$ as defined in Proposition 2.2.5. Then the archimedean components of ψ are of the form

$$\psi_v(x_v) = \exp(2\pi i \zeta_v x_v)$$

and

$$\prod_{v|\infty} \psi_v(x_v) = \prod_{v|\infty} \exp(2\pi i \zeta_v x_v) = \exp\big(2\pi i \mathrm{Tr}(\zeta x)\big) \ .$$

The numbers ζ_v will be collected in an element $u \in \mathbb{A}_F^*$ such that

$$u^\infty := (\zeta_v)_{v|\infty} \qquad \text{and} \qquad u^f := 1 \ .$$

The conductor of ψ at the non-archimedean places is

$$\mathrm{cond}(\psi_\mathfrak{p}) = \big\{x_\mathfrak{p} \in F_\mathfrak{p} \mid \mathrm{Tr}_{F_\mathfrak{p}/\mathbb{Q}_p}(\zeta x_\mathfrak{p}) \in \mathbb{Z}_p\big\} = \zeta^{-1}\mathfrak{d}_\mathfrak{p}^{-1} = \mathfrak{b}_\mathfrak{p}^{-2} \ .$$

For every place $\mathfrak{p} \nmid D_1$, we fix an Eichler order $\mathfrak{O}_\mathfrak{p}$ of $A_\mathfrak{p}$. Define an ideal J in A by the following local data

$$J_\mathfrak{p} = \mathfrak{b}_\mathfrak{p}^{-1}\mathfrak{O}_\mathfrak{p} \qquad \text{for all } \mathfrak{p} < \infty \ .$$

Then $J_\mathfrak{p}$ is a two-sided $\mathfrak{O}_\mathfrak{p}$-ideal of norm $\mathfrak{b}_\mathfrak{p}^{-2} = \mathrm{cond}(\psi_\mathfrak{p})$.

Denote by $\chi_{\mathbf{k-2}} := \chi_{\sigma_{\mathbf{k-2}}} = (\chi_{k_v-2})_{v|\infty}$ the character of the $(\mathbf{k}-\mathbf{2})$-th symmetric power representation $\sigma_{\mathbf{k-2}} = (\sigma_{k_v-2})_{v|\infty}$. Define a function $M(x) = \prod_v M_v(x_v)\prod_\mathfrak{p} M_\mathfrak{p}(x_\mathfrak{p})$ on $A_\mathbb{A}$ by

$$\begin{cases} M_v(x) := & \exp\big(-2\pi|u_v|\mathrm{nrd}(x)\big)\chi_{k_v-2}(\overline{x}) \quad \text{for all } v \mid \infty \ , \\ M_\mathfrak{p}(x) := & \mathbb{1}_{J_\mathfrak{p}}(x) \qquad\qquad\qquad\qquad\quad\ \text{for all } \mathfrak{p} < \infty \ . \end{cases}$$

In Lemma 4.3.6 we introduced the space V_0 of finite dimension d, say, consisting of all functions $\phi : A^*_{\mathbb{A}} \to \mathbb{C}$ satisfying

(i) $\phi(\gamma g k) = \phi(g)$ for all $\gamma \in A^*_F$, $g \in A^*_{\mathbb{A}}$, $k \in \prod_{\mathfrak{p}<\infty} \mathfrak{O}^*_{\mathfrak{p}}$,

(ii) the right regular representation ρ of A^*_∞ on $E_\phi := \langle \rho(g)\phi \mid g \in A^*_\infty \rangle$ is $\Lambda_{\mathbf{k-2}}$-isotypic,

(iii) $\big(\rho(\,\cdot\,)\phi\big)(h) \in \big\langle \mathrm{nrd}(\,\cdot\,)^{-\frac{\mathbf{k-2}}{2}} x_{\mathbf{ij}}(\,\cdot\,) \mid \mathbf{i} = \mathbf{0}, \dots, \mathbf{k-2} \big\rangle$ for all $h \in A^*_{\mathbb{A}}$ and an arbitrary but fixed index $\mathbf{j} \in \{\mathbf{0}, \dots, \mathbf{k-2}\}$,

where $X_{\mathbf{k-2}}(\,\cdot\,) = \big(x_{\mathbf{ij}}(\,\cdot\,)\big)$ is the matrix defined in Section 4.2.

Finally, recall from Definition 3.4.3 that we used the representations of the global Hecke algebra $\mathfrak{H}$ to specify a certain subspace $U(\mathfrak{n})$ of the space $\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d}, \mathfrak{n}), 1\big)$ of adelic Hilbert modular forms, which consists of the newforms of level $\mathfrak{n}$.

After these preparations, we are now ready to state Shimizu's Theorem, which will be the starting point for the proof of Theorem 5.2.1.

Theorem 5.2.3 (Shimizu, 1972). *Let $\mathbf{k} > 2$ and assume that A is a quaternion algebra of square-free discriminant D_1. Suppose that $\mathfrak{O}$ is a maximal order of A. Then the space $U(D_1)$ is generated by the functions*

$$\rho\left(\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}\right) \theta(\,\cdot\,; \phi_i, g_j) : \mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F) \longrightarrow \mathbb{C} \qquad \textit{for } i,j = 1,\dots,d$$

that are given by

$$\theta(s; \phi, g) := \dim(\sigma_{\mathbf{k}-2}) |\det(s)|_{\mathbb{A}_F} \int\limits_{A^1_F \backslash A^1_{\mathbb{A}}} \phi(xhg) \sum_{a \in A_F} \big(\Omega(s_1)M\big)(g^{-1}axhg)\, dx\,. \tag{5.1}$$

Here M is the function defined above, Ω is the Weil representation introduced in 5.1.1, $\sigma_{\mathbf{k-2}}$ is the $(\mathbf{k-2})$-th symmetric tensor representation introduced in 4.2.1, $h \in A^*_{\mathbb{A}}$ is such that $\mathrm{nrd}(h) = \det(s)\mathrm{sgn}(u)$*, $s_1 \in \mathrm{SL}_2(\mathbb{A}_F)$ is defined by*

$$s_1 := \begin{pmatrix} \mathrm{sgn}(u)\det(s)^{-1} & 0 \\ 0 & 1 \end{pmatrix} s \begin{pmatrix} \mathrm{sgn}(u) & 0 \\ 0 & 1 \end{pmatrix},$$

$\{\phi_1, \dots, \phi_d\}$ form a basis of the space V_0, and the elements $\{g_1, \dots, g_d\}$ in $A^*_{\mathbb{A}}$ may be arbitrarily chosen such that they satisfy $\det\big(\phi_i(g_j)\big) \neq 0$.

Proof. In [65, § 6, Thm. 2 and No. 11], Shimizu proves that the spaces

$$\tilde{U}(\mathfrak{n}) := \rho\left(\begin{pmatrix} 1 & 0 \\ 0 & \delta^{-1} \end{pmatrix}\right) U(\mathfrak{n}) \subseteq \mathcal{H}^0_{\mathbf{k}}\big(K_0(1, \mathfrak{n}), 1\big)$$

are generated by the functions $\theta(\,\cdot\,; \phi_i, g_j)$. As we use $K_0(\mathfrak{d}, \mathfrak{n})$ instead of $K_0(1, \mathfrak{n})$ we need to consider the right translates of the $\theta(\,\cdot\,; \phi_i, g_j)$ instead. □

Remark 5.2.4. Recall that an element in $\mathbb{A}_F^*$ is the reduced norm of some quaternion if and only if it is totally positive. Hence we can only find an element $h \in A_{\mathbb{A}}^*$ as claimed if $\det(s_v)\mathrm{sgn}(u_v) > 0$ for all archimedean places v. Without loss of generality we may assume that this is indeed the case. For by the Strong Approximation Theorem 2.3.1

$$\det(s)\mathrm{sgn}(u) \in \mathbb{A}_F^* = \coprod_{j=1}^{h^+} t_j F^* \big(F_\infty^+ \times \widehat{\mathfrak{o}_F^*}\big) \qquad \text{where } t_j^\infty = 1 \ .$$

So if $\det(s)\mathrm{sgn}(u) = t_j \mu m_\infty m^f$ for some $\mu \in F^*$, $m_\infty m^f \in F_\infty^+ \times \widehat{\mathfrak{o}_F^*}$ then we replace s by $\begin{pmatrix} \mu^{-1} & 0 \\ 0 & 1 \end{pmatrix} s$. This does not change the value of the function $\theta(\,\cdot\,;\phi,g)$, which is $\mathrm{GL}_2(F)$-invariant on the left. For the so constructed element the positivity condition

$$\det\left(\begin{pmatrix} \mu^{-1} & 0 \\ 0 & 1 \end{pmatrix} s\right)_v \mathrm{sgn}(u_v) > 0 \qquad \text{for all } v \mid \infty$$

is now satisfied. □

The idea of the proof of Theorem 5.2.1 is now evident: A set of generators of the space $S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_l\mathfrak{d},\mathfrak{n}),1\big)$ of classical Hilbert cusp forms can be derived from a set of (adelic) generators of the space $\mathcal{H}_{\mathbf{k}}^0\big(K_0(\mathfrak{d},\mathfrak{n}),1\big)$ using the correspondence

$$\bigoplus_{l=1}^{h^+} S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_l\mathfrak{d},\mathfrak{n}),1\big) \cong \mathcal{H}_{\mathbf{k}}^0\big(K_0(\mathfrak{d},\mathfrak{n}),1\big) \ ,$$

which has been established in Theorem 2.3.7. These adelic generators may be constructed by means of the decomposition

$$\mathcal{H}_{\mathbf{k}}^0\big(K_0(\mathfrak{d},\mathfrak{n}),1\big) = \bigoplus_{\mathfrak{m}\mid\mathfrak{n}} \bigoplus_{\mathfrak{a}\mid\mathfrak{n}\mathfrak{m}^{-1}} \rho\left(\begin{pmatrix} 1 & 0 \\ 0 & \alpha_{\mathfrak{a}} \end{pmatrix}\right) U(\mathfrak{m})$$

(Proposition 3.4.5) and Shimizu's Theorem 5.2.3, which tells us how to obtain generators of the spaces $U(\mathfrak{m})$.

Unfortunately, there is one crucial problem in this construction: Shimizu's Theorem is only applicable to $U(\mathfrak{m})$ if there exists a definite quaternion algebra over F of discriminant $\mathfrak{m}$. However, it is a fundamental fact that the number of prime divisors of a square-free discriminant of a definite quaternion algebra over F must have the same parity as the degree of F, i. e.

$$\#\{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal and } \mathfrak{p} \mid \mathrm{disc}(A)\} \equiv [F:\mathbb{Q}] \bmod 2 \qquad \text{if } A \text{ is definite} \ .$$

We saw that it is essential to be able to compute $U(\mathfrak{m})$ for *all* $\mathfrak{m}$ dividing the level $\mathfrak{n}$. Therefore we need a Shimizu-like theorem which allows us to construct $U(\mathfrak{m})$ in the cases where the number of primes dividing $\mathfrak{m}$ is *not* of the same parity as $[F:\mathbb{Q}]$.

Conjecture 5.2.5. *Shimizu's Theorem does not only hold if $\mathfrak{O}$ is a maximal order of A but also if it is an Eichler order of level (D_1, D_2). In this case, the theorem yields a set of generators of the space $U(D_1 D_2)$.*

Remark 5.2.6. The choice of the order has influence on the function M, on the class number H and on the space V_0 and its dimension d. □

Remark 5.2.7. We have good reason to believe that Conjecture 5.2.5 is true. Let us briefly explain why:

It is possible to define the Hecke algebra $\mathfrak{H}(A_{\mathbb{A}}^*)$ of a quaternion algebra A in a similar manner as we did in Section 3.3 when we introduced the Hecke algebra $\mathfrak{H}$ of $\mathrm{GL}_2(\mathbb{A}_F)$ (see for example [65, § 1, no.8]). In order to do so, an order of A has to be fixed, which plays the role of the compact subgroup. In Shimizu's proof of Theorem 5.2.3 this order is always assumed to be maximal.

One of the most essential steps in Shimizu's proof is to show that to every irreducible admissible representation of $\mathfrak{H}(A_{\mathbb{A}}^*)$ one can define an irreducible admissible representation of $\mathfrak{H}$ such that each constituent of the action of $\mathfrak{H}(A_{\mathbb{A}}^*)$ on the space of automorphic forms for $A_{\mathbb{A}}^*$ which is infinite-dimensional at all places $v \nmid D_1$ corresponds to a constituent of the representation of $\mathfrak{H}$ on the space $\mathcal{A}_0(\omega)$ of automorphic cusp forms (cf. also [41, Thm. 14.4]). Now, in order to examine the irreducible constituents of a representation π of a group G, say, the *Selberg trace formula* (see for example [30, Ch. 1, § 2, no.4]) often proves to be a useful tool. This formula contains information about the characters of irreducible subrepresentations of π, its specific form depending on the group G. Although the formula can become rather complicated, one may hope that by a close and careful analysis one may derive from it a classification of all irreducible constituents of π.

So what would be needed here in order to prove Conjecture 5.2.5 is a trace formula for $\mathfrak{H}(A_{\mathbb{A}}^*)$ that works even in the case of non-maximal Eichler orders. But such a formula exists, it has been shown by Shimizu himself in an earlier work [64]. With the help of this trace formula it should be possible to imitate the arguments in Shimizu's proof and thus verify Conjecture 5.2.5. However, since applications of the trace formula tend to become rather technical and tedious, this task is beyond the scope of this thesis. □

Remark 5.2.8. Note that even if the conjecture is true we cannot find generators for the space $U(1)$ if the degree $n = [F : \mathbb{Q}]$ is odd. Indeed, in this case there exists no definite quaternion algebra of discriminant $D_1 = (1)$. So no matter which Eichler order we choose, its level $D_1 D_2$ will always be divisible by the non-trivial factor D_1. With the methods explained above, we will therefore only be able do compute the subspace

$$\bigoplus_{\substack{\mathfrak{m}|\mathfrak{n} \\ \mathfrak{m}\neq(1)}} \bigoplus_{\mathfrak{a}|\mathfrak{n}\mathfrak{m}^{-1}} \rho\left(\begin{pmatrix} 1 & 0 \\ 0 & \alpha_{\mathfrak{a}} \end{pmatrix}\right) U(\mathfrak{m}) \subseteq \mathcal{H}_{\mathbf{k}}^0\big(K_0(\mathfrak{d},\mathfrak{n}),1\big) \ .$$

This is the content of part *(ii)* of Theorem 5.2.1. □

Remark 5.2.9. But what if the conjecture turns out to be wrong? In this case we will not obtain a set of generators of the whole space $S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_l\mathfrak{d},\mathfrak{n}),1\big)$ because we can only construct generators of those subspaces $U(\mathfrak{m})$ for which the number of prime divisors of $\mathfrak{m}$ has the same parity as $n=[F:\mathbb{Q}]$. However, there are at least two facts that remain valid even if Conjecture 5.2.5 is wrong:

- The theta series that we construct are cusp forms lying in the space $S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_l\mathfrak{d},\mathfrak{n}),1\big)$, although they may only span a subspace.
- If $\mathfrak{n}=(1)$ and $n=[F:\mathbb{Q}]$ is even then Theorem 5.2.1 is true. Indeed, in this case we only need to consider the divisor $\mathfrak{m}=(1)$ and hence only the space $U(1)$, which corresponds to a quaternion algebra over F of discriminant 1. Such an algebra exists if (and only if) n is even.

□

In what follows we will assume that Conjecture 5.2.5 is true. The proof of Theorem 5.2.1 will then follow the outline that we sketched above. Even though the idea of the proof is clear it will turn out that many tedious computations are needed before we arrive at a description of the generators of $S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_l\mathfrak{d},\mathfrak{n}),1\big)$, which is explicit enough for computational purposes. We will therefore devote the entire following section to the proof.

5.3 Proof of Theorem 5.2.1

Assume that Conjecture 5.2.5 is true.

As earlier let $\mathfrak{d}$ be the different of F, δ an adelic generator of $\mathfrak{d}$ and

$$\alpha := \alpha_{\mathfrak{a}} = \prod_{\mathfrak{p}|\mathfrak{a}} \varpi_{\mathfrak{p}} \in \mathbb{A}_F^* \qquad \text{where} \qquad \mathfrak{p}=\varpi_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}}$$

for any square-free integral ideal $\mathfrak{a}$ of F. By Proposition 3.4.5, Shimizu's Theorem 5.2.3 and Conjecture 5.2.5, there is a set of generators of $\mathcal{H}^0_{\mathbf{k}}\big(K_0(\mathfrak{d},\mathfrak{n}),1\big)$ consisting of functions of the form

$$\rho\left(\begin{pmatrix} 1 & 0 \\ 0 & \delta\alpha \end{pmatrix}\right)\theta(\,\cdot\,;\phi,g)$$

for certain ideals $\mathfrak{a}$, quaternion algebras of varying discriminant and Eichler orders of suitable level. The correspondence between classical and adelic automorphic forms, which we stated in Proposition 2.3.4 and Theorem 2.3.7, then implies that for $l=1,\dots,h^+$ the space

$S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_l\mathfrak{d},\mathfrak{n}),1\big)$ is generated by the corresponding functions

$$\mathbf{z} \mapsto j(\gamma_\infty,\mathbf{i})^{\mathbf{k}}\,\theta\left(\begin{pmatrix}1 & 0\\ 0 & \delta\alpha t_l\end{pmatrix}\gamma_\infty;\,\phi,g\right) \qquad \text{where } \gamma_\infty \in G_\infty^+ \text{ is such that } \gamma_\infty\mathbf{i} = \mathbf{z}\ .$$

In the remainder of this section we will derive an explicit description of these functions and thus prove Theorem 5.2.1.

As a first case let us consider the situation in which there exists a unit $\varepsilon \in \mathfrak{o}_F^*$ such that $\operatorname{sgn}(\varepsilon)^{\mathbf{k}} = -1$. Then Lemma 4.3.1 tells us that $V_0 \subseteq V = \{0\}$. The basis $\{\phi_1,\ldots,\phi_d\}$ of V_0, which is used to define the theta series in Shimizu's Theorem, is therefore empty. Consequently, $U(\mathfrak{m}) = \{0\}$ for all ideals $\mathfrak{m}$ and hence $S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_l\mathfrak{d},\mathfrak{n}),1\big) = \{0\}$.

We may therefore assume from now on that

$$\operatorname{sgn}(\varepsilon)^{\mathbf{k}} = 1 \qquad \text{for all} \qquad \varepsilon \in \mathfrak{o}_F^*\ .$$

We fix the quaternion algebra A of discriminant D_1, an Eichler order $\mathfrak{O}$ in A of level (D_1,D_2) and the ideals $\mathfrak{a}$ and $\mathfrak{c}_l$ under consideration.

Let us make a specific choice for γ_∞ in order to facilitate the subsequent computations. Let $\mathbf{z} \in \mathbb{H}^n$. If $z_v = a_v + ib_v$, let $\gamma_\infty \in G_\infty^+$ be the matrix whose v-th component is

$$\gamma_v = \begin{pmatrix}|\zeta_v|^{1/2} & 0\\ 0 & |\zeta_v|^{1/2}\end{pmatrix}\begin{pmatrix}b_v^{1/2} & a_v b_v^{-1/2}\\ 0 & b_v^{-1/2}\end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})\ ,$$

so that $\det\gamma_v = |\zeta_v|$ and $\gamma_\infty\mathbf{i} = \mathbf{z}$. Then

$$j(\gamma_\infty,\mathbf{i})^{\mathbf{k}} = \prod_{v|\infty}|\zeta_v|^{-\frac{k_v}{2}}\left(|\zeta_v|^{\frac12}b_v^{-\frac12}\right)^{k_v} = \operatorname{Im}(\mathbf{z})^{-\frac{\mathbf{k}}{2}}\ ,$$

and

$$\tilde{s} := \begin{pmatrix}1 & 0\\ 0 & \delta\alpha t_l\end{pmatrix}\gamma_\infty \in \mathrm{GL}_2(\mathbb{A}_F)$$

is the argument at which we need to evaluate $\theta(\,\cdot\,;\phi,g)$. First, we have to find an element $h \in A_{\mathbb{A}}^*$ whose norm equals $\det(\tilde{s})\operatorname{sgn}(u)$. As explained in Remark 5.2.4, such an element does not necessarily exist. But we saw that instead of $\tilde{s}$ we can take

$$s := \begin{pmatrix}\zeta^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \delta\alpha t_l\end{pmatrix}\gamma_\infty \in \mathrm{GL}_2(\mathbb{A}_F)$$

without altering the value of the theta function. Clearly,

$$\big(\det(s)\operatorname{sgn}(u)\big)_v = \zeta_v^{-1}|\zeta_v|\operatorname{sgn}(\zeta_v) = 1 > 0 \qquad \text{for all } v \mid \infty\ ,$$

so that we can find an $h \in A_{\mathbb{A}}^*$ such that $\operatorname{nrd}(h) = \det(s)\operatorname{sgn}(u)$. For this choice of s the factor $|\det(s)|_{\mathbb{A}_F}$ appearing in equation (5.1) becomes

$$\begin{aligned}|\det(s)|_{\mathbb{A}_F} &= |\zeta^{-1}|_{\mathbb{A}_F}\prod_{v|\infty}|\zeta|_v\prod_{\mathfrak{p}<\infty}|\delta\alpha t_l|_{\mathfrak{p}}\\ &= 1\cdot\left|N_{\mathbb{Q}}^F(\zeta)\right|\prod_{\mathfrak{p}<\infty}\mathcal{N}(\mathfrak{p})^{-v_{\mathfrak{p}}(\mathfrak{d}\mathfrak{a}\mathfrak{c}_l)} = \left|N_{\mathbb{Q}}^F(\zeta)\right|\mathcal{N}(\mathfrak{d}\mathfrak{a}\mathfrak{c}_l)^{-1}\ ,\end{aligned}$$

where $\mathcal{N}(\mathfrak{q}) = [\mathfrak{o}_F : \mathfrak{q}]$ denotes the absolute norm of an integral ideal $\mathfrak{q}$. Let $s_1 \in \mathrm{SL}_2(\mathbb{A}_F)$ be defined as in Theorem 5.2.3. Locally, s_1 has the form

$$\begin{cases} s_{1,v} = \begin{pmatrix} (|\zeta_v|^{-1}b_v)^{1/2} & 0 \\ 0 & (|\zeta_v|^{-1}b_v)^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & \mathrm{sgn}(u_v)a_v b_v^{-1} \\ 0 & 1 \end{pmatrix} & \text{for all } v \mid \infty \ , \\ s_{1,\mathfrak{p}} = \begin{pmatrix} (\alpha_\mathfrak{p}\delta_\mathfrak{p}t_{l,\mathfrak{p}})^{-1} & 0 \\ 0 & \alpha_\mathfrak{p}\delta_\mathfrak{p}t_{l,\mathfrak{p}} \end{pmatrix} & \text{for all } \mathfrak{p} < \infty \ . \end{cases}$$

Lemma 5.3.1. *Let the notation be as above and define the abbreviation*

$$e(x) := \exp\big(2\pi i \mathrm{Tr}(x)\big) \qquad \textit{for } x \in \mathbb{A}_F \ .$$

Then

$$\Omega(s_1)M = \big(\Omega(s_1)M\big)^\infty \big(\Omega(s_1)M\big)^f$$

where

$$\big(\Omega(s_1)M\big)^\infty(x) := \prod_{v\mid\infty} \big(\Omega(s_{1,v})M_v\big)(x_v) = \big(|\zeta|^{-1}\mathrm{Im}(\mathbf{z})\big)^{\frac{\mathbf{k}}{2}} \chi_{\mathbf{k-2}}(\overline{x}^\infty)\, e\big(\mathrm{nrd}(x)\mathbf{z}\big)$$

and

$$\big(\Omega(s_1)M\big)^f(x) := \prod_{\mathfrak{p}<\infty} \big(\Omega(s_{1,\mathfrak{p}})M_\mathfrak{p}\big)(x_\mathfrak{p}) = \begin{cases} \mathcal{N}(\mathfrak{a}\mathfrak{d}\mathfrak{c}_l)^2 & \textit{if } x_\mathfrak{p} \in (\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}\mathfrak{O})_\mathfrak{p} \textit{ for all } \mathfrak{p} < \infty \ , \\ 0 & \textit{else} \ . \end{cases}$$

Proof. Because of the particular shape of $s_{1,v}$ and $s_{1,\mathfrak{p}}$ we can compute the action of Ω directly by the formulae given in Definition 5.1.1. For each archimedean place $v \mid \infty$ we obtain

$$\begin{aligned} \big(\Omega(s_{1,v})M_v\big)(x_v) &= \big|\zeta_v^{-1}b_v\big|\Omega\left(\begin{pmatrix} 1 & \mathrm{sgn}(u_v)a_v b_v^{-1} \\ 0 & 1 \end{pmatrix}\right) M_v\big((|\zeta_v|^{-1}b_v)^{1/2}x_v\big) \\ &= \big|\zeta_v^{-1}b_v\big|\, \psi\big(\mathrm{sgn}(u_v)a_v b_v^{-1}\mathrm{nrd}((|\zeta_v|^{-1}b_v)^{1/2}x_v)\big)\, M_v\big((|\zeta_v|^{-1}b_v)^{1/2}x_v\big) \\ &= \big|\zeta_v^{-1}b_v\big|\, \chi_{k_v-2}\big(\overline{(|\zeta_v|^{-1}b_v)^{1/2}x_v}\big) \exp\big(2\pi i a_v \mathrm{nrd}(x_v) - 2\pi b_v \mathrm{nrd}(x_v)\big) \\ &= \big|\zeta_v^{-1}b_v\big|\, \big|\zeta_v^{-1}b_v\big|^{\frac{k_v-2}{2}} \chi_{k_v-2}(\overline{x_v}) \exp\big(2\pi i\, \mathrm{nrd}(x_v)(a_v+ib_v)\big) \\ &= \big|\zeta_v^{-1}b_v\big|^{\frac{k_v}{2}} \chi_{k_v-2}(\overline{x_v}) \exp\big(2\pi i\, \mathrm{nrd}(x_v)z_v\big) \ , \end{aligned}$$

from which the assertion for the archimedean part follows. In the non-archimedean case, apply Definition 5.1.1 again to obtain

$$\big(\Omega(s_1)M\big)^f(x) = \prod_{\mathfrak{p}<\infty} \big|\mathrm{nrd}(\alpha_\mathfrak{p}\delta_\mathfrak{p}t_{l,\mathfrak{p}})\big|_\mathfrak{p}^{-1} M_\mathfrak{p}\big((\alpha_\mathfrak{p}\delta_\mathfrak{p}t_{l,\mathfrak{p}})^{-1}x_\mathfrak{p}\big) \ .$$

Here

$$\prod_{\mathfrak{p}<\infty} \big|\mathrm{nrd}(\alpha_\mathfrak{p}\delta_\mathfrak{p}t_{l,\mathfrak{p}})\big|_\mathfrak{p}^{-1} = \prod_{\mathfrak{p}<\infty} \big|\alpha_\mathfrak{p}\delta_\mathfrak{p}t_{l,\mathfrak{p}}\big|_\mathfrak{p}^{-2} = \prod_{\mathfrak{p}<\infty} \mathcal{N}(\mathfrak{p})^{2v_\mathfrak{p}(\alpha_\mathfrak{p}\delta_\mathfrak{p}t_{l,\mathfrak{p}})} = \mathcal{N}(\mathfrak{a}\mathfrak{d}\mathfrak{c}_l)^2$$

and

$$\prod_{\mathfrak{p}<\infty} M_{\mathfrak{p}}\big((\alpha_{\mathfrak{p}}\delta_{\mathfrak{p}}t_{l,\mathfrak{p}})^{-1}x_{\mathfrak{p}}\big) = \begin{cases} 1 & (\alpha_{\mathfrak{p}}\delta_{\mathfrak{p}}t_{l,\mathfrak{p}})^{-1}x_{\mathfrak{p}} \in J_{\mathfrak{p}} \text{ for all } \mathfrak{p}<\infty\ , \\ 0 & \text{else}\ . \end{cases}$$

Since $J_{\mathfrak{p}} = \mathfrak{b}_{\mathfrak{p}}^{-1}\mathfrak{O}_{\mathfrak{p}}$, the assertion follows. $\square$

Now recall from (1.1) the decomposition

$$A_{\mathbb{A}}^* = \coprod_{j=1}^{H} A_F^* y_j \mathfrak{O}_{\mathbb{A}}^* \qquad\qquad (\text{where } y_j^{\infty} = 1)$$

where H denotes the ideal class number of $\mathfrak{O}$-right ideals. As earlier let $\mathfrak{O}_i$ be the order of A that is uniquely determined by the local data

$$(\mathfrak{O}_i)_{\mathfrak{p}} = y_i \mathfrak{O}_{\mathfrak{p}} y_i^{-1} \qquad \text{for all } \mathfrak{p} < \infty\ .$$

It is again an Eichler order. We saw in Lemma 4.3.6 that the space V_0 is generated by the functions

$$\phi_{i\mathbf{m}}(g) = \begin{cases} \sum\limits_{r\in\mathfrak{O}_i^*/\mathfrak{o}_F^*} \big\langle\!\big\langle \Lambda_{\mathbf{k-2}}(rm^{\infty})P_{\mathbf{j}},\, P_{\mathbf{m}}\big\rangle\!\big\rangle & \text{if } g = ay_i m \in A_F^* y_i \mathfrak{O}_{\mathbb{A}}^*\ , \\ 0 & \text{else} \end{cases}$$

for $i = 1,\ldots,H$, $\mathbf{m} = \mathbf{0},\ldots,\mathbf{k-2}$, for some fixed index $\mathbf{j}$ and $P_{\mathbf{i}} := X^{\mathbf{k-2-i}}Y^{\mathbf{i}}$.

Lemma 5.3.2. *The harmonic polynomial $\Phi_{i\mathbf{m}}(h) = \mathrm{nrd}(\,\cdot\,)^{\frac{\mathbf{k-2}}{2}}\big(\rho(\,\cdot\,)\phi_{i\mathbf{m}}\big)(h)$ associated to $\phi_{i\mathbf{m}}$ satisfies*

$$\Phi_{i\mathbf{m}}(y_k) = \begin{cases} \Big\langle\!\Big\langle \sigma_{\mathbf{k-2}}(\,\cdot\,)P_{\mathbf{j}},\, \sum\limits_{r\in\mathfrak{O}_i^*/\mathfrak{o}_F^*} \Lambda_{\mathbf{k-2}}(r)P_{\mathbf{m}} \Big\rangle\!\Big\rangle & \text{if } i = k\ , \\ 0 & \text{if } i \neq k\ . \end{cases}$$

For each $i = 1,\ldots,H$ choose among the polynomials $\{\Phi_{i\mathbf{m}}(y_i) \mid \mathbf{m} = \mathbf{0},\ldots,\mathbf{k-2}\}$ a set $\{\Phi_{i,1}(y_i),\ldots,\Phi_{i,d_i}(y_i)\}$ that is maximally linearly independent. Further choose elements $m_1^{(i)},\ldots,m_{d_i}^{(i)} \in A_{\infty}^1$ such that

$$\det\begin{pmatrix} \Phi_{i,1}(y_i)(m_1^{(i)}) & \ldots & \Phi_{i,1}(y_i)(m_{d_i}^{(i)}) \\ \vdots & & \vdots \\ \Phi_{i,d_i}(y_i)(m_1^{(i)}) & \ldots & \Phi_{i,d_i}(y_i)(m_{d_i}^{(i)}) \end{pmatrix} \neq 0\ . \tag{5.2}$$

Put

$$\{g_1,\ldots,g_d\} := \{y_1m_1^{(1)},\ldots,y_1m_{d_1}^{(1)},\quad \ldots,\quad y_Hm_1^{(H)},\ldots,y_Hm_{d_H}^{(H)}\}\ ,$$
$$\{\phi_1,\ldots,\phi_d\} := \{\phi_{1,1},\ldots,\phi_{1,d_1},\quad \ldots,\quad \phi_{H,1},\ldots,\phi_{H,d_H}\}\ ,$$

where $d = \sum_{i=1}^{H} d_i = \dim V_0$. *Then*

$$\det\big(\phi_i(g_j)\big) \neq 0 \ .$$

Further

$$g_\mathfrak{p}\mathfrak{O}_\mathfrak{p} = y_{i_\mathfrak{p}}\mathfrak{O}_\mathfrak{p} \qquad \textit{for all } \mathfrak{p} < \infty \ .$$

Proof. The specific form of the $\Phi_{i\mathbf{m}}(y_k)$ follows from Corollary 4.3.7. Now note that it is indeed possible to find $m_1^{(i)}, \dots, m_{d_i}^{(i)} \in A_\infty^1$ with the desired property because the polynomials $\Phi_{i,1}(y_i), \dots, \Phi_{i,d_i}(y_i)$ are homogeneous and linearly independent for fixed i. If the ϕ_i's and the g_j's are ordered in the above fashion then the matrix $\big(\phi_i(g_j)\big)_{i,j}$ becomes the $(d \times d)$-block matrix

$$\big(\phi_i(g_j)\big)_{i,j=1}^{d} = \begin{pmatrix} \boxed{B_1} & & 0 \\ & \ddots & \\ 0 & & \boxed{B_H} \end{pmatrix}$$

where the $(d_i \times d_i)$-blocks B_i are

$$B_i = \begin{pmatrix} \mathrm{nrd}(m_1^{(i)})^{-\frac{\mathbf{k}-2}{2}}\Phi_{i,1}(y_i)(m_1^{(i)}) & \dots & \mathrm{nrd}(m_{d_i}^{(i)})^{-\frac{\mathbf{k}-2}{2}}\Phi_{i,1}(y_i)(m_{d_i}^{(i)}) \\ \vdots & & \vdots \\ \mathrm{nrd}(m_1^{(i)})^{-\frac{\mathbf{k}-2}{2}}\Phi_{i,d_i}(y_i)(m_1^{(i)}) & \dots & \mathrm{nrd}(m_{d_i}^{(i)})^{-\frac{\mathbf{k}-2}{2}}\Phi_{i,d_i}(y_i)(m_{d_i}^{(i)}) \end{pmatrix} .$$

Since $\mathrm{nrd}(m_j^{(i)}) = 1$ for all i, j, this is exactly the matrix given in (5.2). In particular, $\det\big(\phi_i(g_j)\big) \neq 0$ as claimed. The last assertion is obvious. $\square$

Now that we have analyzed the term $\Omega(s_1)M$ and made a specific choice for the elements g_j, let us turn our attention to the integral occuring in the definition (5.1) of $\theta(s;\phi,g)$. Our next goal is to replace the integral by a finite sum that runs over the elements y_j. While we could easily obtain a description of $A_F^*\backslash A_\mathbb{A}^*$ in terms of the y_j from the decomposition

$$A_\mathbb{A}^* = \coprod_{j=1}^{H} A_F^* y_j \mathfrak{O}_\mathbb{A}^* \qquad\qquad (\text{where } y_j^\infty = 1) \ ,$$

it is more cumbersome to describe the quotient $A_F^1\backslash A_\mathbb{A}^1$ of norm-1-elements in terms of the y_j. The idea is to write an element $x \in A_\mathbb{A}^1$ as $x = cy_jk$ for some $c \in A_F^*$, $k \in \mathfrak{O}_\mathbb{A}^*$ and a unique $j \in \{1, \dots, H\}$. If we assume for a moment that $\mathrm{nrd}(y_j) = 1$ holds for every j then the norm-1-elements are characterized by the condition that $\mathrm{nrd}(ck) = 1$. Therefore we introduce the set

$$T := \{(c,k) \in A_F^* \times \mathfrak{O}_\mathbb{A}^* \mid \mathrm{nrd}(ck) = 1\}$$

and try to find a map from $A_F^1\backslash A_\mathbb{A}^1$ onto H copies of a suitable quotient of T, one for each y_j. Finally, these quotients of T can be described in terms of the quotients $y_j^{-1}\mathfrak{O}_j^1 y_j\backslash\mathfrak{O}_\mathbb{A}^1$ over which we can easily integrate.

Unfortunately, the situation is really a little more complicated because we cannot assume that $\mathrm{nrd}(y_j) = 1$ holds for all j. More precisely, if we define an equivalence relation $\sim$ on $A_{\mathbb{A}}^*$ by

$$x \sim y \qquad \Longleftrightarrow \qquad \mathrm{nrd}(xy^{-1}) \in F^+\big(F_\infty^+ \times \widehat{\mathfrak{o}_F^*}\big)$$

then $z \sim y_j$ for all $z \in A_F^* y_j \mathfrak{O}_{\mathbb{A}}^*$ (cf. Lemma 1.2.7). On the other hand

$$\mathrm{nrd}(A_{\mathbb{A}}^*) = \{x \in \mathbb{A}_F^* \mid x_v \geq 0 \text{ for all archimedean } v\} = \coprod_{l=1}^{h^+} t_l F^+\big(F_\infty^+ \times \widehat{\mathfrak{o}_F^*}\big)$$

by the Strong Approximation Theorem 2.3.1. Therefore, all narrow ideal classes of F must be represented in $\{(\mathrm{nrd}(y_1)), \ldots, (\mathrm{nrd}(y_H))\}$. It follows immediately that there must be h^+ inequivalent elements among the y_j's. In particular, if $h^+ > 1$ then not all y_j can be of norm 1.

Fortunately, all problems sketched above, in particular those arising from a non-trivial narrow class group, are only of a technical nature and can be mastered with the help of the following lemmas.

Lemma 5.3.3. *There is a group isomorphism*

$$\mathfrak{o}_F^{*+}/\mathrm{nrd}(\mathfrak{O}_j^*) \cong \big(\mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}\big)\big/\big(\mathrm{nrd}(\mathfrak{O}_j^*)/\mathfrak{o}_F^{*2}\big) \ ,$$

and the cardinality of the last factor is

$$[\mathrm{nrd}(\mathfrak{O}_j^*) : \mathfrak{o}_F^{*2}] = \frac{2e_j}{\#\mathfrak{O}_j^1} \qquad \textit{where } e_j := [\mathfrak{O}_j^* : \mathfrak{o}_F^*] \ .$$

Proof. The first part is an immediate consequence of the isomorphism theorem. For the second assertion, consider the map

$$\mathfrak{O}_j^* \to \mathrm{nrd}(\mathfrak{O}_j^*)/\mathfrak{o}_F^{*2} \qquad m \mapsto \mathrm{nrd}(m)\mathfrak{o}_F^{*2} \ ,$$

which is a surjective group homomorphism with kernel $\mathfrak{O}_j^1\mathfrak{o}_F^*$. Hence

$$\mathrm{nrd}(\mathfrak{O}_j^*)/\mathfrak{o}_F^{*2} \cong \mathfrak{O}_j^*/\mathfrak{O}_j^1\mathfrak{o}_F^* \cong \big(\mathfrak{O}_j^*/\mathfrak{o}_F^*\big)\big/\big(\mathfrak{O}_j^1\mathfrak{o}_F^*/\mathfrak{o}_F^*\big)$$

and

$$\mathfrak{O}_j^1\mathfrak{o}_F^*/\mathfrak{o}_F^* \cong \mathfrak{O}_j^1/(\mathfrak{O}_j^1 \cap \mathfrak{o}_F^*) = \mathfrak{O}_j^1/\{\pm 1\} \ .$$

Therefore, we obtain

$$[\mathrm{nrd}(\mathfrak{O}_j^*) : \mathfrak{o}_F^{*2}] = \frac{[\mathfrak{O}_j^* : \mathfrak{o}_F^*]}{\frac{1}{2}\#\mathfrak{O}_j^1} = \frac{2e_j}{\#\mathfrak{O}_j^1} \ .$$

∎

Lemma 5.3.4. *Let $z \in A_{\mathbb{A}}^*$.*

(i) If $z \sim y_j$ then there exist some elements $\eta := \eta_{z,j} \in A_F^*$ and $b := b_{z,j} \in \mathfrak{O}_{\mathbb{A}}^*$ such that $\mathrm{nrd}(zy_j^{-1}) = \mathrm{nrd}(\eta b)$.

(ii) In particular, let h be as before and $g = y_i m_u^{(i)}$ as in Lemma 5.3.2. Then $y_j \sim hg$ if and only if the ideal $\mathrm{nrd}(I_{ij})\zeta^{-1}\mathfrak{d}\mathfrak{a}\mathfrak{c}_l$ is trivial in the narrow class group, i. e. it has a totally positive generator.

Proof. (i) This is clear by definition of $\sim$ and Lemma 1.2.7. Note that η, b are not uniquely determined.

(ii) Suppose $y_j \sim hg$. Choose $\eta \in A_F^*$ and $b \in \mathfrak{O}_{\mathbb{A}}^*$ such that $\mathrm{nrd}(\eta b) = \mathrm{nrd}(hgy_j^{-1})$. By construction of h, g we have $\mathrm{nrd}(g) = \mathrm{nrd}(y_i)$ and

$$\mathrm{nrd}(h)_v = 1 \ , \qquad \mathrm{nrd}(h)_{\mathfrak{p}} = (\zeta^{-1}\delta\alpha t_l)_{\mathfrak{p}} \qquad \text{for all } v \mid \infty \text{ and } \mathfrak{p} < \infty \ .$$

Hence

$$\begin{aligned}\mathrm{nrd}(I_{ij})\zeta^{-1}\mathfrak{d}\mathfrak{a}\mathfrak{c}_l &= F \cap \bigcap_{\mathfrak{p}<\infty} \big(\mathrm{nrd}(y_i y_j^{-1})\zeta^{-1}\delta\alpha t_l\big)_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}} = F \cap \bigcap_{\mathfrak{p}<\infty} \big(\mathrm{nrd}(hgy_j^{-1})\big)_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}} \\ &= F \cap \bigcap_{\mathfrak{p}<\infty} \big(\mathrm{nrd}(\eta b)\big)_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}} = \mathrm{nrd}(\eta)\mathfrak{o}_F\end{aligned}$$

since $\mathrm{nrd}(b)_{\mathfrak{p}} \in \mathfrak{o}_{\mathfrak{p}}^*$ for all $\mathfrak{p} < \infty$. So $\mathrm{nrd}(\eta)$ is a totally positive generator of the ideal $\mathrm{nrd}(I_{ij})\zeta^{-1}\mathfrak{d}\mathfrak{a}\mathfrak{c}_l$. Conversely, suppose that $\mathrm{nrd}(I_{ij})\zeta^{-1}\mathfrak{d}\mathfrak{a}\mathfrak{c}_l = \mu\mathfrak{o}_F$ for some $\mu \in F^+$. Then

$$\mathrm{nrd}(hgy_j^{-1})\mathfrak{o}_{\mathfrak{p}} = \big(\mathrm{nrd}(I_{ij})\zeta^{-1}\mathfrak{d}\mathfrak{a}\mathfrak{c}_l\big)_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}} = \mu_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}} \qquad \text{for all } \mathfrak{p} < \infty \ ,$$

so for all $\mathfrak{p} < \infty$ we can find some $\varepsilon_{\mathfrak{p}} \in \mathfrak{o}_{\mathfrak{p}}^*$ such that $\mathrm{nrd}(hgy_j^{-1})_{\mathfrak{p}} = \mu_{\mathfrak{p}}\varepsilon_{\mathfrak{p}}$. Use Lemma 1.2.7 to find $\eta \in A_F^*$ and $b \in \mathfrak{O}_{\mathbb{A}}^*$ such that $\mathrm{nrd}(\eta) = \mu$ and

$$\mathrm{nrd}(b)_v = \mu_v^{-1} \ , \qquad \mathrm{nrd}(b)_{\mathfrak{p}} = \varepsilon_{\mathfrak{p}} \qquad \text{for all } v \mid \infty \text{ and } \mathfrak{p} < \infty \ .$$

For this choice of η and b it is easily verified that $\mathrm{nrd}(hgy_j^{-1}) = \mathrm{nrd}(\eta b)$, which proves that $y_j \sim hg$.

□

Lemma 5.3.5. *Consider the set*

$$T := \{(c,k) \in A_F^* \times \mathfrak{O}_{\mathbb{A}}^* \mid \mathrm{nrd}(ck) = 1\} \ .$$

For $j \in \{1, \dots, H\}$ define an equivalence relation $\sim_j$ on T by

$$(c,k) \sim_j (\tilde{c},\tilde{k}) \quad \iff \quad \tilde{c} = ecs^{-1} \text{ and } \tilde{k} = y_j^{-1}sy_jk \qquad \text{for some } s \in \mathfrak{O}_j^*,\ e \in A_F^1 \ .$$

Fix some $z \in A_{\mathbb{A}}^*$.

(i) The map

$$A_F^1 \backslash A_{\mathbb{A}}^1 \xrightarrow{\simeq} \coprod_{\substack{j=1 \\ y_j \sim z}}^{H} T/\sim_j \ , \qquad A_F^1 x \ \mapsto \ \big[(\eta^{-1}c,\ kb^{-1})\big]_j \qquad \textit{where } xz = cy_jk$$

and η, b *are chosen as in part (i) of the previous lemma is a bijection. (Here* $[\,\cdot\,]_j$ *denotes the equivalence class modulo* $\sim_j$*).*

(ii) Fix a complete set of representatives $R = \{\nu_1, \ldots, \nu_r\}$ *of* $\mathfrak{o}_F^{*+}/\mathrm{nrd}(\mathfrak{O}_j^*)$*. For each* $\nu_i \in R$*, fix some* $m_i \in \mathfrak{O}_{\mathbb{A}}^*$ *and* $c_i \in A_F^*$ *such that* $\nu_i = \mathrm{nrd}(m_i) = \mathrm{nrd}(c_i)$*. Put* $M = \{m_1, \ldots, m_r\}$ *and* $C = \{c_1, \ldots, c_r\}$*. Then there is a bijection*

$$\begin{aligned} \Psi_{R,M,C} : \ T/\sim_j \quad &\xrightarrow{\simeq} \quad R \times (y_j^{-1}\mathfrak{O}_j^1 y_j)\backslash \mathfrak{O}_{\mathbb{A}}^1 \\ [(c,k)]_j \quad &\mapsto \quad \big(\nu_i,\ [y_j^{-1}t^{-1}y_jkm_i^{-1}]\big) \end{aligned}$$

where $t \in \mathfrak{O}_j^*$ *is such that* $\mathrm{nrd}(k) = \nu_i \mathrm{nrd}(t)$*. (Again, the brackets indicate the cosets in the respective quotients.) The inverse map is*

$$\Psi_{R,M,C}^{-1}(\nu_i, [k]) = [(c_i^{-1},\ km_i)]_j \ .$$

Proof. (i) Take $A_F^1 x \in A_F^1 \backslash A_{\mathbb{A}}^1$. There is a unique $j \in \{1, \ldots, H\}$ for which we can find $c \in A_F^*$ and $k \in \mathfrak{O}_{\mathbb{A}}^*$ such that $xz = cy_jk$. Then $\mathrm{nrd}(zy_j^{-1}) = \mathrm{nrd}(ck) \in F^+\big(F_\infty^+ \times \widehat{\mathfrak{o}_F^*}\big)$, so $z \sim y_j$. It follows from the definition of η and b that $\big(\eta^{-1}c,\ kb^{-1}\big) \in T$. However, the coset $A_F^1 x$ does not determine the pair $\big(\eta^{-1}c,\ kb^{-1}\big)$ uniquely. But if $A_F^1 x = A_F^1 \tilde{x}$ and $xz = cy_jk$ and $\tilde{x}z = \tilde{c}y_j\tilde{k}$ then there exists an $e \in A_F^1$ such that

$$\tilde{c}y_j\tilde{k} = \tilde{x}z = exz = ecy_jk \ , \qquad \text{hence} \quad \tilde{c}^{-1}ec = y_j\tilde{k}k^{-1}y_j^{-1} \in A_F^* \cap y_j\mathfrak{O}_{\mathbb{A}}^*y_j^{-1} \ .$$

We have thus found an $e \in A_F^1$ and an $s := \tilde{c}^{-1}ec \in \mathfrak{O}_j^*$ such that

$$\eta^{-1}\tilde{c} = \eta^{-1}ecs^{-1} = \big(\eta^{-1}e\eta\big)\eta^{-1}cs^{-1} \qquad \text{and} \qquad \tilde{k}b^{-1} = y_j^{-1}sy_jkb^{-1} \ .$$

Consequently, $\big(\eta^{-1}c,\ kb^{-1}\big) \sim_j \big(\eta^{-1}\tilde{c},\ \tilde{k}b^{-1}\big)$, which shows that the map is well-defined. It is moreover bijective, the inverse image of a coset $[(c,k)]_j$ being

$$\begin{aligned} &\{A_F^1 x \in A_F^1\backslash A_{\mathbb{A}}^1 \mid xz = (\eta ecs^{-1})y_j(y_j^{-1}sy_jkb) \text{ for some } s \in \mathfrak{O}_j^*,\ e \in A_F^1\} \\ &\quad = \{A_F^1 x \in A_F^1\backslash A_{\mathbb{A}}^1 \mid x = \big(\eta e\eta^{-1}\big)\eta cy_jkbz^{-1} \text{ for some } e \in A_F^1\} \\ &\quad = \{A_F^1 \eta cy_jkbz^{-1}\} \ . \end{aligned}$$

(ii) Use Lemma 1.2.7 again to see that we may indeed find elements $m_i \in \mathfrak{O}_{\mathbb{A}}^*$ and $c_i \in A_F^*$ as required.

For $(c,k) \in T$, there is exactly one $\nu_i \in R$ such that $\mathrm{nrd}(k) \equiv \nu_i \bmod \mathrm{nrd}(\mathfrak{O}_j^*)$, and it is clear that all possible choices of $t \in \mathfrak{O}_j^*$ with $\mathrm{nrd}(k) = \nu_i\mathrm{nrd}(t)$ determine the same equivalence class $[y_j^{-1}t^{-1}y_jkm_i^{-1}]$. Now, if $(\tilde{c},\tilde{k}) \sim_j (c,k)$ then we can find some $e \in A_F^1$ and $s \in \mathfrak{O}_j^*$ such that $\tilde{c} = ecs^{-1}$ and $\tilde{k} = y_j^{-1}sy_jk$. But then the norms satisfy $\mathrm{nrd}(\tilde{k}) = \mathrm{nrd}(k)\mathrm{nrd}(s) = \nu_i\mathrm{nrd}(st)$. So the pair $(\tilde{c},\tilde{k})$ is mapped to $\big(\nu_i,\, [y_j^{-1}(st)^{-1}y_j\tilde{k}m_i^{-1}]\big) = \big(\nu_i,\, [y_j^{-1}t^{-1}y_jkm_i^{-1}]\big)$. Therefore, $\Psi_{R,M,C}$ is well-defined on equivalence classes modulo $\sim_j$. The inverse image of $\big(\nu_i,[m]\big)$ is

$$\begin{aligned}
&\Psi_{R,M,C}^{-1}\big(\nu_i,[m]\big)\\
&\quad= \big\{[(c,k)]_j \;\mid \mathrm{nrd}(k) = \mathrm{nrd}(t)\nu_i,\;\; [y_j^{-1}t^{-1}y_jkm_i^{-1}] = [m] \text{ for some } t \in \mathfrak{O}_j^*\big\}\\
&\quad= \big\{[(c,k)]_j \;\mid \mathrm{nrd}(c) = \mathrm{nrd}(ts)^{-1}\nu_i^{-1},\;\; k = y_j^{-1}tsy_jmm_i \text{ for some } t \in \mathfrak{O}_j^*,\, s \in \mathfrak{O}_j^1\big\}\\
&\quad= \big\{[(c,k)]_j \;\mid c = ec_i^{-1}s^{-1},\;\; k = y_j^{-1}sy_jmm_i \text{ for some } s \in \mathfrak{O}_j^*,\, e \in \mathfrak{O}_j^1\big\}\\
&\quad= \big\{[(c_i^{-1}, mm_i)]_j\big\}\ .
\end{aligned}$$

□

The algebra $A_{\mathbb{A}}^*$ and thus A_F^* and $\mathfrak{O}_{\mathbb{A}}^*$ are equipped with the usual Haar measure, and the set T in the previous lemma inherits their product measure $d(c,k) = dc\,dk$. On each quotient $T/\sim_j$ we may then define $d[c,k]_j$ as the quotient measure of $d(c,k)$ with respect to the canonical projection onto the equivalence classes modulo $\sim_j$. Likewise, the sets $A_F^1\backslash A_{\mathbb{A}}^1$ and $R \times (y_j^{-1}\mathfrak{O}_j^1y_j)\backslash\mathfrak{O}_{\mathbb{A}}^1$ are equipped with quotient Haar measures. With respect to these measures the map in part *(i)* of the lemma as well as Ψ^{-1} of part *(iii)* are measurable. Having at our disposal measurable maps $A_F^1\backslash A_{\mathbb{A}}^1 \to \coprod T/\sim_j$ and $T/\sim_j \to R\times(y_j^{-1}\mathfrak{O}_j^1y_j)\backslash\mathfrak{O}_{\mathbb{A}}^1$ allows us to replace the integral over $A_F^1\backslash A_{\mathbb{A}}^1$, which appears in the definition (5.1) of $\theta(s;\phi,g)$, by integrals over $(y_j^{-1}\mathfrak{O}_j^1y_j)\backslash\mathfrak{O}_{\mathbb{A}}^1$. These will then be simplified further.

Lemma 5.3.6. *Let $f : A_{\mathbb{A}}^* \to \mathbb{C}$ be a left-A_F^*-invariant integrable function. Let $W \subseteq \mathfrak{O}_{\mathbb{A}}^*$ be a set of $[\mathfrak{o}_F^{*+} : \mathfrak{o}_F^{*2}]$ elements such that* $\mathrm{nrd}(W)$ *is a complete set of representatives of $\mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}$. Then*

$$\int_{T/\sim_j} f(cy_jk)\, d[c,k]_j = \frac{\#\mathfrak{O}_j^1}{2e_j} \sum_{w\in W} \int_{(y_j^{-1}\mathfrak{O}_j^1y_j)\backslash\mathfrak{O}_{\mathbb{A}}^1} f(y_jkw)\, dk\ .$$

Proof. Let $R = \{\nu_1,\dots,\nu_r\}$, $M = \{m_1,\dots,m_r\}$ and $C = \{c_1,\dots,c_r\}$ be as in part *(iii)* of Lemma 5.3.5. Moreover, fix a set of representatives $S = \{\sigma_1,\dots,\sigma_s\}$ of $\mathrm{nrd}(\mathfrak{O}_j^*)/\mathfrak{o}_F^{*2}$, and for each $\sigma_l \in S$, choose some $s_l \in \mathfrak{O}_j^*$ such that $\mathrm{nrd}(s_l) = \sigma_l$. For each $l = 1,\dots,s$, put

$$\widetilde{R} := \{\nu_1\sigma_l,\dots,\nu_r\sigma_l\}\ ,\quad \widetilde{M} := \{m_iy_j^{-1}s_ly_j \mid i = 1,\dots,r\}\ ,\quad \widetilde{C} := \{c_is_l \mid i = 1,\dots,r\}\ .$$

These sets have the same properties as R,M,C. We may therefore construct $\Psi_{\widetilde{R},\widetilde{M},\widetilde{C}}$ as in Lemma 5.3.5. Then

$$\Psi_{\widetilde{R},\widetilde{M},\widetilde{C}}^{-1}\big(\nu_i\sigma_l,[k]\big) = \big\{[(c_is_l)^{-1},\, km_iy_j^{-1}s_ly_j)]_j\big\}\ ,$$

so that an application of the transformation rule for integrals yields

$$\int\limits_{T/\sim_j} f(cy_jk)\,d[c,k]_j = \sum_{i=1}^{r} \int\limits_{(y_j^{-1}\mathfrak{O}_j^1 y_j)\backslash\mathfrak{O}_{\mathbb{A}}^1} f\big((c_is_l)^{-1}y_j(km_iy_j^{-1}s_ly_j)\big)\,dk$$

for each $l = 1,\dots,s$. Taking into account that $s = [\mathrm{nrd}(\mathfrak{O}_j^*) : \mathfrak{o}_F^{*2}] = \frac{2e_j}{\#\mathfrak{O}_j^1}$ (see Lemma 5.3.3) and that f is left-A_F^*-invariant, we obtain by summing over all l

$$\begin{aligned}\frac{2e_j}{\#\mathfrak{O}_j^1}\int\limits_{T/\sim_j} f(cy_jk)\,d[c,k]_j &= \sum_{l=1}^{s}\sum_{i=1}^{r} \int\limits_{(y_j^{-1}\mathfrak{O}_j^1 y_j)\backslash\mathfrak{O}_{\mathbb{A}}^1} f\big((c_is_l)^{-1}y_j(km_iy_j^{-1}s_ly_j)\big)\,dk \\ &= \sum_{\tilde{w}\in\widetilde{W}} \int\limits_{(y_j^{-1}\mathfrak{O}_j^1 y_j)\backslash\mathfrak{O}_{\mathbb{A}}^1} f(y_jk\tilde{w})\,dk\ ,\end{aligned}$$

where $\widetilde{W} := \{m_iy_j^{-1}s_ly_j \mid i = 1,\dots,r,\ l = 1,\dots,s\} \subseteq \mathfrak{O}_{\mathbb{A}}^*$. Observe that the value of each of the integrals in the last formula depends only on the coset $\mathfrak{O}_{\mathbb{A}}^1\mathfrak{o}_F^*\tilde{w}$ but not on the particular choice of $\tilde{w}$. Instead of summing over $\widetilde{W}$, we may therefore sum over any set $W \subseteq \mathfrak{O}_{\mathbb{A}}^*$ containing exactly $rs = [\mathfrak{o}_F^{*+} : \mathfrak{o}_F^{*2}]$ elements such that $\mathrm{nrd}(W)$ is a set of representatives of $\mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}$. □

Now, consider the situation of Shimizu's Theorem 5.2.3 for a fixed $\phi = \phi_{j\mathbf{m}}$ and $g = y_im_u^{(i)}$, which we always assume to be chosen as in Lemma 5.3.2. For the sake of clarity we drop all indices and write only ϕ and g.

We use part *(ii)* of Lemma 5.3.5 (with $z = hg$) to rewrite $\big(\dim(\sigma_{\mathbf{k}-2})|\det(s)|_{\mathbb{A}_F}\big)^{-1}\theta(s;\phi,g)$ as follows

$$\begin{aligned}&\big(\dim(\sigma_{\mathbf{k}-2})|\det(s)|_{\mathbb{A}_F}\big)^{-1}\theta(s;\phi,g) \\ &\quad= \int\limits_{A_F^1\backslash A_{\mathbb{A}}^1} \phi(xhg)\sum_{a\in A_F}\big(\Omega(s_1)M\big)(g^{-1}axhg)\,dx \\ &\quad= \sum_{\substack{j=1\\ y_j\sim hg}}^{H}\ \int\limits_{T/\sim_j} \phi(cy_jkb)\sum_{a\in A_F}\big(\Omega(s_1)M\big)(g^{-1}acy_jkb)\,d[(c,k)]_j\ ,\end{aligned}$$

where $\eta \in A_F^*$, $b \in \mathfrak{O}_{\mathbb{A}}^*$ are such that

$$\mathrm{nrd}(hgy_j^{-1}) = \mathrm{nrd}(\eta b)\ . \tag{5.3}$$

Note that when passing from $A_F^1\backslash A_{\mathbb{A}}^1$ over to $T/\sim_j$, the argument xhg transforms into ηcy_jkb. But since ϕ is A_F^*-invariant on the left and $a\eta$ runs through all of A_F if a does, we can omit η. Now we apply Lemma 5.3.6 and obtain

$$\begin{aligned}&\big(\dim(\sigma_{\mathbf{k}-2})|\det(s)|_{\mathbb{A}_F}\big)^{-1}\theta(s;\phi,g) \\ &\quad= \sum_{\substack{j=1\\ y_j\sim hg}}^{H}\frac{\#\mathfrak{O}_j^1}{2e_j}\sum_{w\in W}\ \int\limits_{(y_j^{-1}\mathfrak{O}_j^1 y_j)\backslash\mathfrak{O}_{\mathbb{A}}^1} \phi(y_jkwb)\sum_{a\in A_F}\big(\Omega(s_1)M\big)(g^{-1}ay_jkwb)\,dk\ .\end{aligned} \tag{5.4}$$

Next, we will simplify the integral in (5.4). Consider a suitably integrable function $h(k)$, then

$$\int_{\mathfrak{O}_{\mathbb{A}}^1} h(k)\,dk = \int\limits_{(y_j^{-1}\mathfrak{O}_j^1 y_j)\backslash\mathfrak{O}_{\mathbb{A}}^1} \int\limits_{(y_j^{-1}\mathfrak{O}_j^1 y_j)} h(mk)\,dm\,dk = \int\limits_{(y_j^{-1}\mathfrak{O}_j^1 y_j)\backslash\mathfrak{O}_{\mathbb{A}}^1} \int\limits_{\mathfrak{O}_j^1} h(y_j^{-1} m y_j k)\,dm\,dk \;.$$

In particular, for $h(k) = \phi(y_j kwb)\sum_{a\in A_F}\big(\Omega(s_1)M\big)(g^{-1}ay_jkwb)$,

$$\begin{aligned}
&\int_{\mathfrak{O}_{\mathbb{A}}^1} \phi(y_jkwb)\sum_{a\in A_F}\big(\Omega(s_1)M\big)(g^{-1}ay_jkwb)\,dk\\
&\qquad = \int\limits_{(y_j^{-1}\mathfrak{O}_j^1 y_j)\backslash\mathfrak{O}_{\mathbb{A}}^1} \int\limits_{\mathfrak{O}_j^1} \phi(my_jkwb)\sum_{a\in A_F}\big(\Omega(s_1)M\big)(g^{-1}amy_jkwb)\,dm\,dk\\
&\qquad = \int\limits_{(y_j^{-1}\mathfrak{O}_j^1 y_j)\backslash\mathfrak{O}_{\mathbb{A}}^1} \int\limits_{\mathfrak{O}_j^1} \phi(y_jkwb)\sum_{a\in A_F}\big(\Omega(s_1)M\big)(g^{-1}ay_jkwb)\,dm\,dk
\end{aligned}$$

where, in the last equality, we make again use of the fact that the integrand is A_F^*-invariant on the left. Now we see that the integrand does not depend on m, and hence

$$\begin{aligned}
&\int_{\mathfrak{O}_{\mathbb{A}}^1} \phi(y_jkwb)\sum_{a\in A_F}\big(\Omega(s_1)M\big)(g^{-1}ay_jkwb)\,dk\\
&\qquad = \#\mathfrak{O}_j^1 \cdot \int\limits_{(y_j^{-1}\mathfrak{O}_j^1 y_j)\backslash\mathfrak{O}_{\mathbb{A}}^1} \phi(y_jkwb)\sum_{a\in A_F}\big(\Omega(s_1)M\big)(g^{-1}ay_jkwb)\,dk\;.
\end{aligned}$$

Inserting this expression into (5.4) leads to

$$\begin{aligned}
&\big(\dim(\sigma_{\mathbf{k}-2})|\det(s)|_{\mathbb{A}_F}\big)^{-1}\theta(s;\phi,g)\\
&\qquad = \sum_{\substack{j=1\\ y_j\sim hg}}^{H} \frac{1}{2e_j}\sum_{w\in W}\int_{\mathfrak{O}_{\mathbb{A}}^1} \phi(y_jkwb)\sum_{a\in A_F}\big(\Omega(s_1)M\big)(g^{-1}ay_jkwb)\,dk\;. \qquad (5.5)
\end{aligned}$$

Now let us turn our attention to the term $\big(\Omega(s_1)M\big)(g^{-1}ay_jwk)$. Recall from Lemma 5.3.2 that g is of the form $g = y_i m$ for some $i\in\{1,\dots,H\}$ and some $m\in\{m_1^{(i)},\dots,m_{d_i}^{(i)}\}\subseteq A_\infty^1$. Also $g_{\mathfrak{p}}\mathfrak{O}_{\mathfrak{p}} = y_{i\mathfrak{p}}\mathfrak{O}_{\mathfrak{p}}$ for all $\mathfrak{p}<\infty$. Using Lemma 5.3.1 we can deduce that the archimedean part of $\big(\Omega(s_1)M\big)(g^{-1}ay_jkwb)$ is

$$\big(\Omega(s_1)M\big)^{\infty}(g^{-1}ay_jkwb) = \big(|\zeta|^{-1}\mathrm{Im}(\mathbf{z})\big)^{\frac{\mathbf{k}}{2}}\,\chi_{\mathbf{k-2}}\big((\overline{g^{-1}ay_jkwb})^{\infty}\big)\,e\big(\mathrm{nrd}(g^{-1}ay_jkwb)\,\mathbf{z}\big)\;.$$

Recall that $\mathrm{nrd}(k)=1$, $\mathrm{nrd}(g^{\infty})=1$, $y_j^{\infty}=1$ and

$$\mathrm{nrd}(g^{-1}ay_jkwb) = \mathrm{nrd}\big((hg)^{-1}y_jb\big)\mathrm{nrd}(aw)\mathrm{nrd}(h) = \mathrm{nrd}(\eta^{-1})\mathrm{nrd}(aw)\mathrm{nrd}(h)$$

by equation (5.3). We have seen that $\mathrm{nrd}(h)_v = \big(\det(s)\mathrm{sgn}(u)\big)_v = 1$ at every archimedean place. Hence $\big(\Omega(s_1)M\big)(g^{-1}ay_jkwb)$ simplifies to

$$\big(\Omega(s_1)M\big)(g^{-1}ay_jkwb) = \big(|\zeta|^{-1}\mathrm{Im}(\mathbf{z})\big)^{\frac{\mathbf{k}}{2}}\,\chi_{\mathbf{k-2}}\big((\overline{g^{-1}akwb})^{\infty}\big)e\big(\mathrm{nrd}(\eta^{-1}aw)\mathbf{z}\big)\;.$$

The non-archimedean part is

$$\big(\Omega(s_1)M\big)^f(g^{-1}ay_jkwb) = \begin{cases} \mathcal{N}(\mathfrak{a}\mathfrak{d}\mathfrak{c}_l)^2 & \text{if } g^{-1}ay_jkwb \in (\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}\mathfrak{O})_\mathfrak{p} \text{ for all } \mathfrak{p} < \infty \ , \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \mathcal{N}(\mathfrak{a}\mathfrak{d}\mathfrak{c}_l)^2 & \text{if } a \in \mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij} \ , \\ 0 & \text{else} \end{cases}$$

since $kwb \in \mathfrak{O}_\mathbb{A}^*$ by construction. We use these descriptions of $\big(\Omega(s_1)M\big)^\infty$ and $\big(\Omega(s_1)M\big)^f$ to evaluate the integral in equation (5.5):

$$\int\limits_{\mathfrak{O}_\mathbb{A}^1} \phi(y_jkwb) \sum_{a\in A_F} \big(\Omega(s_1)M\big)(g^{-1}ay_jkwb)\, dk$$

$$= \big(|\zeta|^{-1}\mathrm{Im}(\mathbf{z})\big)^{\frac{\mathbf{k}}{2}}\mathcal{N}(\mathfrak{a}\mathfrak{d}\mathfrak{c}_l)^2 \int\limits_{\mathfrak{O}_\mathbb{A}^1} \phi(y_jkwb) \sum_{a\in A_F} e\big(\mathrm{nrd}(\eta^{-1}aw)\mathbf{z}\big)\chi_{\mathbf{k-2}}\big((\overline{g^{-1}akwb})^\infty\big)\mathbb{1}_{\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}}(a)dk$$

$$= \big(|\zeta|^{-1}\mathrm{Im}(\mathbf{z})\big)^{\frac{\mathbf{k}}{2}}\mathcal{N}(\mathfrak{a}\mathfrak{d}\mathfrak{c}_l)^2 \sum_{a\in\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}} e\big(\mathrm{nrd}(\eta^{-1}aw)\mathbf{z}\big) \int\limits_{\mathfrak{O}_\mathbb{A}^1} \phi(y_jkwb)\chi_{\mathbf{k-2}}\big((\overline{g^{-1}akwb})^\infty\big)\, dk \ .$$

We know that ϕ is $\prod_{\mathfrak{p}<\infty}\mathfrak{O}_\mathfrak{p}^*$-invariant on the right, so the last integrand depends only on the archimedean part $(kwb)^\infty$ of kwb. Hence

$$\int\limits_{\mathfrak{O}_\mathbb{A}^1} \phi(y_jkwb) \sum_{a\in A_F} \big(\Omega(s_1)M\big)(g^{-1}ay_jkwb)\, dk$$

$$= C\,\mathrm{Im}(\mathbf{z})^{\frac{\mathbf{k}}{2}} \cdot \sum_{a\in\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}} e\big(\mathrm{nrd}(\eta^{-1}aw)\mathbf{z}\big) \int\limits_{A_\infty^1} \phi\big(y_j(kwb)^\infty\big)\chi_{\mathbf{k-2}}\big((\overline{g^{-1}akwb})^\infty\big)\, dk^\infty \ , \quad (5.6)$$

where

$$C := \mathrm{vol}(A_\infty^1\backslash\mathfrak{O}_\mathbb{A}^1)\,|\zeta|^{-\frac{\mathbf{k}}{2}}\mathcal{N}(\mathfrak{a}\mathfrak{d}\mathfrak{c}_l)^2$$

is a constant depending neither on s nor on ϕ, g. From now on we assume that the archimedean part of w is

$$w^\infty = (\nu^{\frac{1}{2}}, \dots, \nu^{\frac{1}{2}}) \in Z(A_\infty^*) \qquad \text{where } \nu = \mathrm{nrd}(w) \ .$$

This assumption is indeed possible because by construction of the set W, the elements w can be chosen arbitrarily in $\mathfrak{O}_\mathbb{A}^*$ provided that their norms form a complete set of representatives of $\mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}$. On this assumption,

$$\chi_{\mathbf{k-2}}\big((\overline{g^{-1}akwb})^\infty\big) = \mathrm{nrd}(w)^{\frac{\mathbf{k-2}}{2}}\chi_{\mathbf{k-2}}\big((\overline{g^{-1}akb})^\infty\big)$$

by Lemma 4.2.2. Moreover,

$$\phi\big(y_j(kwb)^\infty\big) = \phi(y_j(kb)^\infty)$$

by part *(iii)* of Lemma 4.3.2. With these formulae together with Lemma 4.3.4 and a change of the integration variable $k^\infty \to (bkb^{-1})^\infty$, equation (5.6) becomes

$$\begin{aligned}
&\int_{\mathfrak{O}^1_{\mathbb{A}}} \phi(y_j kwb) \sum_{a\in A_F} \big(\Omega(s_1)M\big)(g^{-1}ay_jkwb)\,dk \\
&\quad = C\,\mathrm{Im}(\mathbf{z})^{\frac{\mathbf{k}}{2}} \sum_{a\in\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}} e\big(\mathrm{nrd}(\eta^{-1}aw)\mathbf{z}\big)\mathrm{nrd}(w)^{\frac{\mathbf{k-2}}{2}} \int_{A^1_\infty} \phi\big(y_j(kb)^\infty\big)\chi_{\mathbf{k-2}}\big((\overline{g^{-1}akb})^\infty\big)\,dk^\infty \\
&\quad = C\,\mathrm{Im}(\mathbf{z})^{\frac{\mathbf{k}}{2}} \sum_{a\in\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}} e\big(\mathrm{nrd}(\eta^{-1}aw)\mathbf{z}\big)\mathrm{nrd}(w)^{\frac{\mathbf{k-2}}{2}} \int_{A^1_\infty} \phi\big(y_j(bk)^\infty\big)\chi_{\mathbf{k-2}}\big((\overline{g^{-1}abk})^\infty\big)\,dk^\infty \\
&\quad = \frac{C\,\mathrm{Im}(\mathbf{z})^{\frac{\mathbf{k}}{2}}}{\dim\sigma_{\mathbf{k}-2}}\,\mathrm{nrd}(w)^{\frac{\mathbf{k-2}}{2}} \sum_{a\in\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}} e\big(\mathrm{nrd}(\eta^{-1}aw)\mathbf{z}\big)\Phi(y_jb^\infty)\big((\overline{g^{-1}ab})^\infty\big)
\end{aligned}$$

where $\Phi(y_j) = \mathrm{nrd}(\,\cdot\,)^{\frac{\mathbf{k-2}}{2}}\big(\rho(\,\cdot\,)\phi\big)(y_j) \in \mathrm{Harm}_{\mathbf{k}-2}[X_0,\dots,X_3]$ is the harmonic polynomial associated to ϕ. By Lemma 4.3.2, part *(iii)*, the polynomial expression can be rewritten as

$$\begin{aligned}
\Phi(y_jb^\infty)\big((\overline{g^{-1}ab})^\infty\big) &= \mathrm{nrd}(g^{-1}ab)^{\frac{\mathbf{k-2}}{2}}\phi\big(y_jb^\infty\overline{b}^\infty(\overline{g^{-1}a})^\infty\big) \\
&= \mathrm{nrd}(b)^{\frac{\mathbf{k-2}}{2}}\mathrm{nrd}(g^{-1}a)^{\frac{\mathbf{k-2}}{2}}\phi\big(y_j(\overline{g^{-1}a})^\infty\big) = \mathrm{nrd}(b)^{\frac{\mathbf{k-2}}{2}}\Phi(y_j)\big((\overline{g^{-1}a})^\infty\big)\ .
\end{aligned}$$

We insert this expression into equation (5.5) and use that $\mathrm{nrd}(w)=\nu$ to get

$$\begin{aligned}
&\theta(s;\phi,g) \\
&\quad= \sum_{\substack{j=1\\ y_j\sim hg}}^{H} \frac{C\,\mathrm{Im}(\mathbf{z})^{\frac{\mathbf{k}}{2}}\,|\det(s)|_{\mathbb{A}_F}}{2e_j} \sum_{w\in W} \mathrm{nrd}(wb)^{\frac{\mathbf{k-2}}{2}} \sum_{a\in\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}} \Phi(y_j)\big((\overline{g^{-1}a})^\infty\big)e\big(\mathrm{nrd}(\eta^{-1}aw)\mathbf{z}\big) \\
&\quad= \sum_{\substack{j=1\\ y_j\sim hg}}^{H} \frac{C\,\mathrm{Im}(\mathbf{z})^{\frac{\mathbf{k}}{2}}\,|\det(s)|_{\mathbb{A}_F}}{2e_j} \sum_{\nu\in\mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}} (\nu\mathrm{nrd}(b))^{\frac{\mathbf{k-2}}{2}} \sum_{a\in\eta^{-1}\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}} \Phi(y_j)\big((\overline{g^{-1}\eta a})^\infty\big)e\big(\nu\mathrm{nrd}(a)\mathbf{z}\big)\ .
\end{aligned}$$

Recall that $j(\gamma_\infty,\mathbf{i})^{\mathbf{k}} = \mathrm{Im}(\mathbf{z})^{-\frac{\mathbf{k}}{2}}$ and $|\det(s)|_{\mathbb{A}_F} = \big|N^F_{\mathbb{Q}}(\zeta)\big|\,\mathcal{N}(\mathfrak{d}\mathfrak{a}\mathfrak{c}_l)^{-1}$. Thus we have

$$\begin{aligned}
&j(\gamma_\infty,\mathbf{i})^{\mathbf{k}}\theta(s;\phi,g) \\
&\qquad = C_1 \sum_{\substack{j=1\\ y_j\sim hg}}^{H} \frac{\mathrm{nrd}(b)^{\frac{\mathbf{k-2}}{2}}}{e_j} \sum_{\nu\in\mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}} \nu^{\frac{\mathbf{k}-2}{2}} \sum_{a\in\eta^{-1}\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}} \Phi(y_j)\big((\overline{g^{-1}\eta a})^\infty\big)e\big(\nu\mathrm{nrd}(a)\mathbf{z}\big)\ ,
\end{aligned}$$

where

$$C_1 := \frac{\mathrm{vol}(A^1_\infty\backslash\mathfrak{O}^1_{\mathbb{A}})\big|N^F_{\mathbb{Q}}(\zeta)\big|\,\mathcal{N}(\mathfrak{d}\mathfrak{a}\mathfrak{c}_l)}{2|\zeta|^{\frac{\mathbf{k}}{2}}}$$

is again a constant that depends neither on s nor on ϕ, g.

Finally recall that ϕ and g, on which $\theta(\,\cdot\,;\phi,g)$ depends, are chosen as in Lemma 5.3.2. So if we write $\phi = \phi_{j\mathbf{m}}$ and $g = y_i m_u^{(i)}$ then

$$j(\gamma_\infty, \mathbf{i})^{\mathbf{k}} \theta(s; \phi_{j\mathbf{m}}, y_i m_u^{(i)})$$

$$= \begin{cases} C_2 \sum\limits_{\nu \in \mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}} \nu^{\frac{\mathbf{k}-2}{2}} \sum\limits_{a \in \eta^{-1}\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}} \Phi_{j\mathbf{m}}(y_j)\big((\overline{g^{-1}\eta a})^\infty\big) e\big(\nu \mathrm{nrd}(a)\mathbf{z}\big) & \text{if } y_j \sim hg, \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} C_2 \sum\limits_{\nu \in \mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}} \nu^{\frac{\mathbf{k}-2}{2}} \sum\limits_{a \in \eta^{-1}\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}} \big(\rho(m_u^{(i)})\Phi_{j\mathbf{m}}(y_j)\big)\big((\overline{\eta a})^\infty\big) e\big(\nu \mathrm{nrd}(a)\mathbf{z}\big) & \text{if } y_j \sim hg, \\ 0 & \text{else} \end{cases}$$

where

$$C_2 := \frac{\mathrm{nrd}(b)^{\frac{\mathbf{k}-\mathbf{2}}{2}} C_1}{e_j} = \frac{\mathrm{nrd}(b)^{\frac{\mathbf{k}-\mathbf{2}}{2}} \mathrm{vol}(A_\infty^1 \backslash \mathfrak{O}_{\mathbb{A}}^1) \big|N_{\mathbb{Q}}^F(\zeta)\big| \mathcal{N}(\mathfrak{d}\mathfrak{a}\mathfrak{c}_l)}{2e_j |\zeta|^{\frac{\mathbf{k}}{2}}}$$

depends only on $\phi_{j\mathbf{m}}$, on $y_i m_u^{(i)}$ and, via b, on the determinant of s. Recall that although s depends on the element $\mathbf{z} \in \mathbb{H}^n$, the determinant $\det(s)$ does not. So for each choice of $\phi_{j\mathbf{m}}$ and $y_i m_u^{(i)}$ we obtain a theta series

$$\mathbf{z} \mapsto \sum_{\nu \in \mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}} \nu^{\frac{\mathbf{k}-2}{2}} \sum_{a \in \eta^{-1}\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}} \big(\rho(m_u^{(i)})\Phi_{j\mathbf{m}}(y_j)\big)\big((\overline{\eta a})^\infty\big) e\big(\nu \mathrm{nrd}(a)\mathbf{z}\big) \tag{5.7}$$

which is a constant multiple of the above function. The polynomials $\big(\rho(m_u^{(i)})\Phi_{j\mathbf{m}}(y_j)\big)$ are clearly harmonic and for fixed i, j there are

$$d_i^2 \leq \big(\#\{\mathbf{m} \mid \mathbf{m} = \mathbf{0}, \ldots, \mathbf{k} - \mathbf{2}\}\big)^2 = \prod_{v=1}^{n} (k_v - 1)^2$$

of them (Lemma 5.3.2).

So we have finally brought the integrals in Shimizu's Theorem into the shape of quaternionic theta series. Our proof of Theorem 5.2.1 is now complete.

$\square$

Remark 5.3.7. Consider the theta series

$$\Theta(\mathbf{z}; Q) := \sum_{\nu \in \mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}} \nu^{\frac{\mathbf{k}-\mathbf{2}}{2}} \sum_{a \in \eta^{-1}\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}} Q(a^\infty) e\big(\nu \mathrm{nrd}(a)\mathbf{z}\big)$$

for some harmonic polynomial $Q \in \mathrm{Harm}_{\mathbf{k}-\mathbf{2}}[X_0, \ldots, X_3]$, which is the essential part of the series in equation (5.7). It is easily verified that $\mathrm{nrd}(\eta^{-1}\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}) = \mathfrak{a}^2\mathfrak{d}^2\mathfrak{c}_l^2\mathfrak{b}^{-2}\mathrm{nrd}(\eta^{-1}I_{ij})$ where

$$\begin{aligned} \mathrm{nrd}(\eta^{-1}I_{ij}) &= F \cap \bigcap_{\mathfrak{p}<\infty} \mathfrak{o}_{\mathfrak{p}} \mathrm{nrd}(\eta^{-1}y_i y_j^{-1}) = F \cap \bigcap_{\mathfrak{p}<\infty} \mathfrak{o}_{\mathfrak{p}} \mathrm{nrd}(h^{-1}b) \\ &= F \cap \bigcap_{\mathfrak{p}<\infty} \mathfrak{o}_{\mathfrak{p}} \zeta(\delta\alpha t_l)^{-1} = \zeta(\mathfrak{d}\mathfrak{a}\mathfrak{c}_l)^{-1} \end{aligned}$$

by equation (5.3) and the fact that $\mathrm{nrd}(b)_{\mathfrak{p}} \in \mathfrak{o}_{\mathfrak{p}}^{*}$. Hence

$$\mathrm{nrd}(\eta^{-1}\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}) = \mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-2}\zeta = \mathfrak{a}\mathfrak{c}_l$$

since $\mathfrak{d}\zeta = \mathfrak{b}^2$ by construction. By Corollary 1.3.3 we conclude that $\Theta(\mathbf{z}; Q)$ is a Hilbert modular form of weight $\mathbf{k}$ and character 1 for the group $\Gamma_0\big(\mathfrak{a}\mathfrak{c}_l\mathfrak{d}, D_1D_2\big)$. $\square$

Chapter 6

Explicit computations and results

Not only does Theorem 5.2.1 provide a set of generators of the spaces $S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_l\mathfrak{d},\mathfrak{n}),1\big)$ of Hilbert modular forms, these generators also have the advantage of being explicit enough for computational purposes. In this final chapter we would therefore like to present some examples and results of our computations, all of which have been carried out with the help of the computer algebra system MAGMA, V.2.13-1 [13].

We fix a constant $C > 0$, which we will call the *approximation level.* Then the results of our computations will be of the form

$$\Theta(\mathbf{z}) = \sum_a c(a)e\big(\mathrm{nrd}(a)\mathbf{z}\big)$$

where a runs through all elements in F such $\mathrm{nrd}(a)^{(i)} \leq C$ for all embeddings $i = 1, \ldots, n$ and such that the Fourier coefficient $c(a) \neq 0$.

6.1 Sketches of the algorithms

Most of the algorithms that we make use of are either straightforward or based on functions that are already included in the MAGMA-package. However, for the sake of completeness we will still sketch those parts of our program that seem to require a few more words of explanation.

6.1.1 Algorithms concerning the number field

Magma provides the functions `UnitGroup` and `RayClassGroup` with which we are able to compute the units and the narrow class group of the number field F.

Algorithm 6.1.1 (Computing $\mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}$). Given a totally real number field F, return a complete set of representatives of the totally positive units of F modulo squares.

1. Determine $\mathfrak{o}_F^*$ and find a set $U \subseteq \mathfrak{o}_F^*$ of representatives for the classes $\mathfrak{o}_F^*/\mathfrak{o}_F^{*2}$.
2. Delete from U all elements that are not totally positive. Return U.

□

Algorithm 6.1.2 (Computing ζ). Given a number field F, compute a number $\zeta \in F^*$ and an integral ideal $\mathfrak{b}$ of F such that the different $\mathfrak{d}$ of F can be written as $\zeta\mathfrak{d} = \mathfrak{b}^2$.

1. Compute the different $\mathfrak{d}$.
2. Compute representatives $\mathfrak{a}_i$, $i = 1, \ldots, h$ of all ideal classes of F.
3. For $i = 1, \ldots, h$ check whether $\mathfrak{a}_i^2\mathfrak{d}^{-1}$ is a principal ideal. If so, compute a generator ζ and put $\mathfrak{b} := \mathfrak{a}_i$. Return ζ and $\mathfrak{b}$ and terminate. Else continue with the next i.

□

6.1.2 Algorithms concerning the quaternion ideals

We make use of the numerous algorithms that Magma provides for computations with quaternion algebras. Among these there are algorithms for finding quaternion algebras of given discriminant as well as computing a maximal order therein. Most of the algorithms were implemented by J. Voight and D. Kohel, some are explained in [75].

Algorithm 6.1.3 (Finding a quaternion order). For a given squarefree integral ideal $\mathfrak{m} = \mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_r$ return an order $\mathfrak{O}$ in a suitable quaternion algebra A of level (D_1, D_2) such that $D_1 D_2 = \mathfrak{m}$.

1. If $r \equiv n \bmod 2$ then compute a quaternion algebra A of discriminant $\mathfrak{m}$ and return a maximal order $\mathfrak{O}$ of A.
2. If $r \not\equiv n \bmod 2$ and $\mathfrak{m} = (1)$ then terminate with an error message.
3. If $r \not\equiv n \bmod 2$ and $\mathfrak{m} \neq (1)$ then compute a quaternion algebra A of discriminant $\mathfrak{p}_2 \cdot \ldots \cdot \mathfrak{p}_r$. Compute a maximal order $\mathfrak{M}_1$ in A and put $I := \mathfrak{p}_1\mathfrak{M}_1$.

4. Choose a random element $x \in \mathfrak{M}_1$. If $X^2 - \mathrm{tr}(x)X + \mathrm{nrd}(x)$ has a root a modulo $\mathfrak{p}_1$ then continue. Else repeat this step.

5. Put $J := \mathfrak{M}_1(x-a)$, compute the right order $\mathfrak{M}_2$ of $I + J$. Use [12, Algorithm 1.5.1] to compute $\mathfrak{O} := \mathfrak{M}_1 \cap \mathfrak{M}_2$.

6. If the level of $\mathfrak{O}$ is (D_1, D_2) then return $\mathfrak{O}$. Else go back to step 4.

□

It is clear that Algorithm 6.1.3 returns the correct result if $r \equiv n \bmod 2$. It is also clear that no order with the desired property can be found if $\mathfrak{m} = (1)$ and $r \not\equiv n \bmod 2$ because in this case n is odd so that there exists no quaternion algebra of discriminant $D_1 = (1)$. In particular, $D_1 D_2 = (1)$ cannot be true.

We claim that the algorithm returns an Eichler order of level $D_2 = \mathfrak{p}_1$ in the remaining case. Let us briefly explain why this is indeed true. For any prime ideal $\mathfrak{p} \neq \mathfrak{p}_1$ the localizations of I and $\mathfrak{M}_1$ at $\mathfrak{p}$ coincide, i. e. $I_\mathfrak{p} = \mathfrak{M}_{1,\mathfrak{p}}$. It follows that $(I+J)_\mathfrak{p} = \mathfrak{M}_{1,\mathfrak{p}} = \mathfrak{M}_{2,\mathfrak{p}}$, so that $\mathfrak{O}$ is maximal at $\mathfrak{p}$. By construction

$$0 \equiv a^2 - \mathrm{tr}(x)a + \mathrm{nrd}(x) \equiv (x-a)(\overline{x-a}) \equiv \mathrm{nrd}(x-a) \bmod \mathfrak{p}_1 \ ,$$

so $v_{\mathfrak{p}_1}\big(\mathrm{nrd}(x-a)\big) \geq 1$. It can be shown that if $v_{\mathfrak{p}_1}\big(\mathrm{nrd}(x-a)\big) = 1$ then the right order of $I+J$ is $\mathfrak{M}_2 = (x-a)^{-1}\mathfrak{M}_1(x-a)$ and $\mathfrak{O} = \mathfrak{M}_1 \cap \mathfrak{M}_2$ is of level $D_2 = \mathfrak{p}_1$. In this case the algorithm terminates with the correct result, otherwise it will go back to find a different x.

Algorithm 6.1.4 (Computing the ideals I_{ij} and orders $\mathfrak{O}_i$). Given an order $\mathfrak{O}$ in a quaternion algebra A, compute the orders $\mathfrak{O}_i$ and ideals I_{ij} defined by the local data $(\mathfrak{O}_i)_\mathfrak{p} = y_i \mathfrak{O}_\mathfrak{p} y_i^{-1}$ and $(I_{ij})_\mathfrak{p} = y_i \mathfrak{O}_\mathfrak{p} y_j^{-1}$, respectively. Here $\{y_1, \dots, y_H\}$ denotes a complete set of representatives of the double cosets $A_F^* \backslash A_\mathbb{A}^* / \mathfrak{O}_\mathbb{A}$.

1. Use the MAGMA-function `RightIdealClasses` to obtain a complete set $\{I_1, \dots, I_H\}$ of representatives of the $\mathfrak{O}$-right ideal classes.

2. For each $i = 1, \dots, H$ and each $j = 1, \dots, H$ compute $I_{ij} := I_i I_j^{-1}$.

3. Return the ideals I_{ij} and the orders $\mathfrak{O}_i := I_{ii}$. (A type conversion is necessary to interpret $\mathfrak{O}_i$ correctly as quaternion *order*.)

□

A formula for the class number of an Eichler order in a definite quaternion algebra can be found in [72, Prop. 1.3]. Many examples of these class numbers have been computed and tabulated in [68, Anhang A]. We used these tables to double-check the return values of the function `RightIdealClasses`. Unfortunately, there were cases in which `RightIdealClasses` returned fewer ideal classes than predicted by [68]. Sometimes it was not even able to find any classes at all but stopped with an error message. Apparently, this problem has been fixed in the latest Magma-release. If not, one could also use a different approach of finding all ideal classes by using Kneser's method of neighbouring lattices, which is explained in [47, Kapitel IX]. An implementation of this method has been sketched in [59], following the ideas in [61].

However, in the examples that we present in the next two sections, no discrepancies occurred between the class numbers computed by Magma and [68].

Algorithm 6.1.5 (Finding short vectors in a quaternion ideal). Given a quaternion ideal I and $S, T \in \mathbb{R}^n$ determine all $x \in I$ such that $S_i \leq \mathrm{nrd}(x^{(i)}) \leq T_i$ for all embeddings $i = 1, \dots, n$.

Here we follow the enumeration algorithm described in [26, § 3]. For the step "Enumerate one coefficient" contained therein we use the Fourier-Motzkin elimination (see for example [70]). □

We also need to be able to compute $\mathfrak{O}^*/\mathfrak{o}_F^*$ for a quaternion order $\mathfrak{O}^*$. This factor group is finite and we need an explicit list of representatives. Although Magma provides an algorithm to compute the unit group, this algorithm leads to unexpected error messages for a number of orders, in particular for non-maximal orders. The error messages apparently result from Magma being unable to determine the exact group structure of $\mathfrak{O}^*/\mathfrak{o}_F^*$, but this bug might have been fixed in the current Magma-release. For our purposes, however, the group structure of $\mathfrak{O}^*/\mathfrak{o}_F^*$ is not relevant, we are only interested in the elements themselves. Therefore we decided to adopt the Magma-algorithm in the following way:

Algorithm 6.1.6 (Computing the units in a quaternion order). Given a quaternion order $\mathfrak{O}$ return all elements in $\mathfrak{O}^*$ modulo $\mathfrak{o}_F^*$.

1. Use Algorithm 6.1.1 to compute the set U of representatives of $\mathfrak{o}_F^{*+}/\mathfrak{o}_F^{*2}$.
2. For each $u \in U$ use Algorithm 6.1.5 with $S = T = \left(\mathrm{nrd}(u^{(1)}), \dots, \mathrm{nrd}(u^{(n)})\right)$ to compute all $x \in \mathfrak{O}$ such that $\mathrm{nrd}(x) = u$. Return the set of all such x.

□

Recall that for fixed $I_{ij}, \mathfrak{a}, \mathfrak{c}_l$ we also need an element $\eta \in A_F^*$ which is defined by a condition on the norm of the quaternion ideal $\eta^{-1}\mathfrak{a}\mathfrak{d}\mathfrak{b}^{-1}I_{ij}$. At a second glance we see that it is enough to know $\mathrm{nrd}(\eta)$, and this is easily obtained:

Algorithm 6.1.7 (Computing $\mathrm{nrd}(\eta_{\mathbf{ij}})$). Given a pair of indices $i, j \in \{1, \dots, H\}$ decide whether there exists an element $\eta_{ij} \in A_F^*$ that satisfies $\mathrm{nrd}(\eta^{-1}\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}) = \mathfrak{a}\mathfrak{c}_l$, and if so return $\mathrm{nrd}(\eta_{ij})$.

1. Compute $I := \mathrm{nrd}(I_{ij})\zeta^{-1}\mathfrak{d}\mathfrak{a}\mathfrak{c}_l$.
2. If I is trivial in Cl^+ then return `true` and a totally positive generator of I. Else return `false`.

□

6.1.3 Algorithms concerning the harmonic polynomials

The harmonic polynomials

$$\Phi_{i\mathbf{m}}(y_i) = \left\langle\!\!\left\langle \sigma_{\mathbf{k-2}}(\,\cdot\,)P_{\mathbf{j}}, \sum_{r \in \mathfrak{O}_i^*/\mathfrak{o}_F^*} \Lambda_{\mathbf{k-2}}(r)P_{\mathbf{m}} \right\rangle\!\!\right\rangle \qquad \text{for } i = 1, \dots, H,\ \mathbf{m} = \mathbf{0}, \dots, \mathbf{k-2}$$

from Lemma 5.3.2 can be constructed in a straightforward way. We only need to compute the unit group $\mathfrak{O}_i^*/\mathfrak{o}_F^*$ first, which can be achieved by Algorithm 6.1.6. In our examples we always used the index $\mathbf{j} = \mathbf{0}$, so $P_{\mathbf{j}} = X^{\mathbf{k-2}}$. The result will be a collection of

$$\#\{\mathbf{m} \mid \mathbf{m} = \mathbf{0}, \dots, \mathbf{k-2}\} = \prod_{v=1}^{n}(k_v - 1)$$

harmonic polynomials in $4n$ variables, which are not necessarily linearly independent. In fact, it may even happen that all of them are 0.

Now recall that these polynomials are not exactly what we need in order to evaluate the theta series

$$\sum_{a \in \mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}} \big(\rho(m_u^{(i)})\Phi_{j\mathbf{m}}(y_j)\big)(\overline{a}^\infty)\, e\big(\nu\mathrm{nrd}(\eta^{-1}a)\mathbf{z}\big)$$

in equation (5.7). What we are really interested in are the polynomials

$$\rho(m_u^{(i)})\big(\Phi_{j\mathbf{m}}(y_j)\big)$$

where the elements $m_u^{(i)}$ and the indices $u, \mathbf{m}$ are chosen as explained in Lemma 5.3.2.

Algorithm 6.1.8 (Finding elements $m_u^{(i)}$). Given a set $\{P_1, \dots, P_d\}$ of linearly independent complex homogeneous polynomials in $4n$ variables, compute elements $m_1, \dots, m_d \in A_\infty^1$ such that $\det\big(P_i(m_j)\big)_{i,j} \neq 0$.

1. For each P_i and each monomial $X_1^{e_1} \cdots X_{4n}^{e_{4n}}$ ($e_j \geq 0$) appearing in P_i construct the vector $e = (e_1, \dots, e_{4n})$. Collect all these vectors in a set E.

2. Pick d random vectors $e^{(1)}, \dots, e^{(d)} \in E$. For all $j = 1, \dots, d$ and $v = 1, \dots, n$ put
$$\widetilde{m}_v^{(j)} := e_{4v-3}^{(j)} 1 + e_{4v-2}^{(j)} \mathfrak{i} + e_{4v-1}^{(j)} \mathfrak{j} + e_{4v}^{(j)} \mathfrak{k} \qquad \text{and} \qquad m_v^{(j)} := \sqrt{\operatorname{nrd}(\widetilde{m}_v^{(j)})}^{-1} \widetilde{m}_v^{(j)} \; .$$
Put $m_j := (m_1^{(j)}, \dots, m_n^{(j)})$.
3. If $\det \big(P_i(m_j)\big)_{ij} \neq 0$ then return $m_1, \dots, m_d$. Else go back to step 2.

$\square$

The reason that we choose elements of that particular form is that we prefer elements whose coefficient vector with respect to the quaternion basis $1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ contains many zeros and small other coefficients. By restricting ourselves to those elements that correspond to the monomials appearing in the P_i's we avoid testing m_j's for which we trivially have $P_i(m_j) = 0$ for all i. Since the polynomials $P_1, \dots, P_d$ are linearly independent it is clear that by picking random elements $m_1, \dots, m_d$ we will eventually find some that satisfy the determinant condition $\det \big(P_i(m_j)\big)_{i,j} \neq 0$.

6.2 First example over $\mathbb{Q}(\sqrt{5})$ explained in detail

To illustrate how the different steps of the algorithm work we will explain an easy but non-trivial example in detail. Consider the situation
$$F = \mathbb{Q}(\sqrt{5}) \; , \qquad \mathfrak{n} = (2) \; , \qquad \mathbf{k} = (4, 4)$$
and choose the approximation level to be $C = 5$. In this setting
$$\mathfrak{o}_F = \mathbb{Z}[\omega] \; , \qquad \omega = \frac{1 + \sqrt{5}}{2} \; , \qquad h^+ = 1 \; , \qquad \mathfrak{d} = (1 - 2\omega) \; .$$
Algorithm 6.1.2 yields
$$\zeta = (1 - 2\omega)^{-1} = \frac{1}{5}(1 - 2\omega) \qquad \text{and} \qquad \mathfrak{b} = (1) \; .$$
For $\mathfrak{n} = (2)$ we need to consider the divisors $\mathfrak{m} = (1)$ and $\mathfrak{m} = (2)$ together with the ideals $\mathfrak{a} \in \{(1), (2)\}$ and $\mathfrak{a} = (1)$, respectively. This leads to quaternion orders of the levels
$$\begin{aligned} (D_1, D_2) &= (1, 1) \qquad \text{and} \qquad \mathfrak{a} = (1) \; , \\ (D_1, D_2) &= (1, 1) \qquad \text{and} \qquad \mathfrak{a} = (2) \; , \\ (D_1, D_2) &= (1, 2) \qquad \text{and} \qquad \mathfrak{a} = (1) \; . \end{aligned}$$
For each level we use Algorithm 6.1.3 to determine a suitable quaternion algebra A, namely
$$A = \langle 1, \mathfrak{i}, \mathfrak{j}, \mathfrak{k} \rangle \qquad \text{where} \qquad \mathfrak{i}^2 = \mathfrak{j}^2 = -1 \qquad \text{and} \qquad \mathfrak{i}\mathfrak{j} = -\mathfrak{j}\mathfrak{i} = \mathfrak{k} \; ,$$

and a maximal (or Eichler) order $\mathfrak{O}$. Then we have to construct all right $\mathfrak{O}$-ideals and all orders $\mathfrak{O}_i$. In this particular situation it turns out that for every level under consideration the class group is trivial, i. e.

$$H = H_{(D_1,D_2)} = 1 \ .$$

So there is only one order $\mathfrak{O}_i = \mathfrak{O}$, and it follows that $I_{ij} = \mathfrak{O}$ for each of the levels.

Next, we construct the harmonic polynomials

$$\Big\langle\!\Big\langle \sigma_{(\mathbf{2},\mathbf{2})}(\,\cdot\,)P_{\mathbf{0}}, \sum_{r\in\mathfrak{O}^*/\mathfrak{o}_F^*} \Lambda_{(\mathbf{2},\mathbf{2})}(r)P_{\mathbf{m}}\Big\rangle\!\Big\rangle \qquad \text{for} \qquad \mathbf{m} = \mathbf{0},\ldots,\mathbf{k}-\mathbf{2} = (\mathbf{2},\mathbf{2}) \ ,$$

where $P_{\mathbf{m}} = X^{\mathbf{k}-\mathbf{2}-\mathbf{m}}Y^{\mathbf{m}}$ as earlier. To this end we need to compute the unit group of $\mathfrak{O}$ using Algorithm 6.1.6. Note that since $\mathbf{k} = (4,4)$ we will obtain $(4-1)(4-1) = 9$ polynomials, but they are not necessarily linearly independent.

For $(D_1,D_2) = (1,1)$, we find $\#\mathfrak{O}^* = 120$. All 9 polynomials turn out to be 0. Consequently, there are no new forms of level $\mathfrak{m} = 1$, and we may terminate at this point.

For $(D_1,D_2) = (1,2)$ we have $\#\mathfrak{O}^* = 24$. The 9 polynomials span a 1-dimensional space generated by P, say. According to Algorithm 6.1.8 we choose an element $g \in A^1_\infty$ such that $P(g) \neq 0$. In our case the choice fell on

$$g = \big(1,\ \frac{1}{\sqrt{2}}(\mathfrak{i}+\mathfrak{k})\big) \in \prod_{v|\infty} A^1_v = A^1_\infty \ ,$$

so that

$$\begin{aligned}\rho(g)P = &-\tfrac{5}{8}\big((6-2\omega)-(3+\omega)i\big)X_1^2X_5^2 + \tfrac{5}{4}\big(1-(2+\omega)i\big)X_1^2X_5X_6\\ &-\tfrac{5}{4}\big((2+\omega)+i\big)X_1^2X_5X_7 - \tfrac{5}{4}\big((3+\omega)+(6-2\omega)i\big)X_1^2X_5X_8 + \tfrac{5}{8}(1+\omega)iX_1^2X_6^2\\ &+\tfrac{5}{4}(1+\omega)X_1^2X_6X_7 + \tfrac{5}{4}\big((2+\omega)+i\big)X_1^2X_6X_8 - \tfrac{5}{8}(1+\omega)iX_1^2X_7^2\\ &+\tfrac{5}{4}\big(1-(2+\omega)i\big)X_1^2X_7X_8 \quad + \quad \text{(91 further terms)} \ .\end{aligned}$$

This is the polynomial that we have to evaluate at $\overline{a}^\infty$ in order to compute the series

$$\sum_{a\in\mathfrak{a}\mathfrak{d}\mathfrak{c}_l\mathfrak{b}^{-1}I_{ij}} \big(\rho(m_u^{(i)})\Phi_{j\mathbf{m}}(y_j)\big)(\overline{a}^\infty)\, e\big(\mathrm{nrd}(\eta^{-1}a)\mathbf{z}\big) = \sum_{a\in\mathfrak{d}\mathfrak{O}} \big(\rho(g)P\big)(\overline{a}^\infty) e\big(\mathrm{nrd}(\eta^{-1}a)\mathbf{z}\big) \ .$$

Here

$$\mathrm{nrd}(\eta) = \mathrm{nrd}(\mathfrak{d}\mathfrak{O}) = \mathfrak{d}^2 = (5) \ .$$

As we want to compute the theta series up to an approximation level $C = 5$, we need to determine all $a \in \mathfrak{d}\mathfrak{O}$ such that

$$\mathrm{nrd}\big((\eta^{-1}a)^{(v)}\big) \leq 5 \ , \qquad \text{i. e.} \qquad \mathrm{nrd}\big(a^{(v)}\big) \leq 25 \qquad \text{for all embeddings } v \ .$$

To this end we apply Algorithm 6.1.5 with $I = \mathfrak{d}\mathfrak{O}$, $S = (0,0)$ and $T = (25,25)$.

We summarize these results in the following table:

(D_1, D_2)	$H_{(D_1,D_2)}$	$\#\mathfrak{O}^*$	lin. independent polynomials	$\mathfrak{a}$	#short vectors
$(1,1)$	1	120	0	(1)	—
$(1,1)$				(2)	—
$(1,2)$	1	24	1	(1)	3121

After evaluating the polynomial at each short vector we obtain the theta series

$$\begin{aligned}\Theta(\mathbf{z}) =& e(\mathbf{z}) + e\big((1+\omega)\mathbf{z}\big) + e\big((2-\omega)\mathbf{z}\big) - 4\,e(2\mathbf{z}) - 10\,e\big((2+\omega)\mathbf{z}\big) - 28\,e\big((3+\omega)\mathbf{z}\big) \\ & - 10\,e\big((3-\omega)\mathbf{z}\big) + 50\,e(3\mathbf{z}) + 16\,e(4\mathbf{z}) - 28\,e\big((4-\omega)\mathbf{z}\big) - 25\,e(5\mathbf{z}) + \dots\ .\end{aligned}$$

As we know by our main theorem 5.2.1 that this series must generate $S_{(\mathbf{4},\mathbf{4})}\big(\Gamma_0(\mathfrak{d},(2)),1\big)$ and as it is obviously non-zero, we can conclude that

$$\dim_{\mathbb{C}} S_{(\mathbf{4},\mathbf{4})}\big(\Gamma_0(\mathfrak{d},(2)),1\big) = 1\ .$$

In general, we will not be able to determine the precise dimension of the given space of cusp forms. As we can only compute a finite number of terms in each theta series we will not be able to detect any linear independencies that occur at a later point. In particular, if all computed coefficients in a series are 0 we cannot conclude that the series itself is 0, unless the number of computed terms is large enough. In [79] the question of how many terms need to be calculated to determine whether a given series is indeed 0 is addressed and solved. As the number of necessary terms turns out to be rather large we do not make use of this bound. Instead we contend ourselves with truncating the computed theta series after a certain term, which is determined by the chosen approximation level, and test those truncated series for linear dependencies. Thus we obtain a lower bound for the dimension of the respective space. However, we have to keep in mind that the true dimension might be larger.

But note that we do know an upper bound for the dimension of $S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{d},\mathfrak{n}),1\big)$: If d is the number of linearly independent polynomials for a certain level, as given in the fourth column of the table above, then d^2 is the maximal dimension of the space of newforms in this level. We may thus obtain an upper bound for the dimension of the whole space $S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{d},\mathfrak{n}),1\big)$, by adding all these d^2's up, here

$$\dim_{\mathbb{C}} S_{(\mathbf{4},\mathbf{4})}\big(\Gamma_0(\mathfrak{d},(2)),1\big) \le 0^2 + 0^2 + 1^2 = 1\ .$$

In all the examples that we present in the following section we will not only list the theta series that we found but also mention the maximal possible dimension of the respective space.

With the methods introduced in this thesis an exact determination of the true dimension is not possible.

6.3 Further results

After this detailed example we conclude this chapter by listing some results that were computed with the help of the algorithm sketched above. As always, we consider

$$\Gamma_0(\mathfrak{c}_l\mathfrak{d}, \mathfrak{n}) , \qquad l = 1, \ldots, h^+ .$$

The tables are organized in the following way:

- The number field, certain invariants and the weight $\mathbf{k}$ are given in the caption. Here we use the symbol ω for an element that generates the ring of integers in F, i. e. $\mathfrak{o}_F = \mathbb{Z}[\omega]$.
- In the first column we list all small elements $\mathrm{nrd}(a)$ that contribute to the theta series in the sense that the coefficient of $e\big(\mathrm{nrd}(a)\mathbf{z}\big)$ does not vanish. The size of these elements is restricted by the approximation level, which we always choose to be 5.
- The first row lists the different levels $\mathfrak{n}$ that were considered, the second row their respective norms $\mathcal{N}(\mathfrak{n})$.
- The third row shows an upper bound for the dimension of the space $S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{c}_l, \mathfrak{n}), 1\big)$. This bound is obtained by counting the linearly independent polynomials for each subspace of newforms and adding the squares of these numbers up, as was explained in the detailed example.
- Each column underneath an ideal $\mathfrak{n}$ belongs to a single theta series that was found for the level $\mathfrak{n}$. The number in the i-th row of each of these columns is the Fourier coefficient of the the term $e(\mathrm{nrd}(a)\mathbf{z})$ where $\mathrm{nrd}(a)$ is the element given in the i-th row of the leftmost column.

6.3.1 Theta series for $\mathbb{Q}(\sqrt{5})$

We saw in Remark 2.3.6 and in Lemma 4.3.1 that the space of cusp forms is $\{0\}$ if the weight $\mathbf{k}$ does not meet the condition

$$\mathrm{sgn}(\delta)^{\mathbf{k}} = 1 \qquad \text{for all} \qquad \delta \in \mathfrak{o}_F^* .$$

A fundamental unit of $\mathbb{Q}(\sqrt{5})$ is $\omega = \frac{1+\sqrt{5}}{2}$. Since the conjugates of ω have opposite sign, the above condition is satisfied if and only if

$$k_1 \equiv k_2 \equiv 0 \bmod 2 .$$

So we only have to compute theta series of even weight. We will restrict ourselves to the cases

$$\mathbf{k} \in \{(4,4),\ (4,6),\ (6,6)\} .$$

An approximation level of 5 means in this case that the first twelve terms of each series will be computed.

The third column of Table 6.1 summarizes the results of the previous section. Note that the coefficients in this table corroborate the results given in [15, Table 2] as far as they can be compared.

For weight $\mathbf{k} = (6, 6)$, we could only compute theta series of level $\mathfrak{n} = 1$. For all other levels up to norm 30, one of the errors occurred that we explained in Section 6.1.

Table 6.1: $F = \mathbb{Q}(\sqrt{5})$, $h^+ = 1$, $\omega = \frac{1+\sqrt{5}}{2}$, $\mathbf{k} = (4, 4)$

$\mathfrak{n}$	(1)	(2)	$(2+\omega)$	(3)		$(3+\omega)$
$\mathcal{N}(\mathfrak{n})$	1	4	5	9		11
max. dimension	0	1	1	4		1
0	—	0	0	0	0	0
1	—	1	1	1	0	1
$1+\omega$	—	1	1	1	0	1
$2-\omega$	—	1	1	1	0	1
2	—	-4	0	0	1	4
$2+\omega$	—	-10	-5	10	-2	4
$3+\omega$	—	-28	32	-28	2	-11
$3-\omega$	—	-10	-5	10	-2	4
3	—	50	-50	-9	0	-2
4	—	16	-64	16	7	-48
$4-\omega$	—	-28	32	-28	2	-10
5	—	-25	25	295	-12	-109

Table 6.2: $F = \mathbb{Q}(\sqrt{5})$, $h^+ = 1$, $\omega = \frac{1+\sqrt{5}}{2}$, $\mathbf{k} = (4, 6)$

$\mathfrak{n}$	(1)	(2)	$(2+\omega)$	(3)	
$\mathcal{N}(\mathfrak{n})$	1	4	5	9	
max. dimension	0	1	1	4	
0	—	0	0	0	0
1	—	1	1	1	0
$1+\omega$	—	$2-\omega$	$2-\omega$	$2-\omega$	0
$2-\omega$	—	$1+\omega$	$1+\omega$	$1+\omega$	0
2	—	8	10	0	2
$2+\omega$	—	$-18+6\omega$	$15-5\omega$	$3-\omega$	$3-\omega$
$3+\omega$	—	$48-132\omega$	$-192+108\omega$	$68-47\omega$	$-4+7\omega$
$3-\omega$	—	$12+6\omega$	$10+5\omega$	$2+\omega$	$2+\omega$
3	—	-90	30	12	3
4	—	64	-156	4	-32
$4-\omega$	—	$-84+132\omega$	$-84-108\omega$	$21+47\omega$	$3-7\omega$
5	—	-445	125	-295	-30

Table 6.3: $F = \mathbb{Q}(\sqrt{5})$, $h^+ = 1$, $\omega = \frac{1+\sqrt{5}}{2}$, $\mathbf{k} = (6, 6)$

$\mathfrak{n}$	(1)
$\mathcal{N}(\mathfrak{n})$	1
max. dimension	1
0	0
1	1
$1+\omega$	1
$2-\omega$	1
2	20
$2+\omega$	-90
$3+\omega$	252
$3-\omega$	-90
3	90
4	-624
$4-\omega$	252
5	4975

6.3.2 Theta series for $\mathbb{Q}(\sqrt{3})$

The field $\mathbb{Q}(\sqrt{3})$ has narrow class number $h^+ = 2$, the two ideal classes are represented by $\mathfrak{c}_1 = (1)$ and $\mathfrak{c}_2 = (\omega)$, where $\omega = \sqrt{3}$. For each weight $\mathbf{k}$ and level $\mathfrak{n}$ we will therefore obtain two spaces of cusp forms, namely $S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{d}, \mathfrak{n}), 1\big)$ and $S_{\mathbf{k}}\big(\Gamma_0(\omega\mathfrak{d}, \mathfrak{n}), 1\big)$.

A fundamental unit of $\mathfrak{o}^* = \mathbb{Z}[\omega]$ is $\varepsilon = 2 + \omega$, which is totally positive. The condition $\operatorname{sgn}(\delta)^{\mathbf{k}} = 1$ for all $\delta \in \mathfrak{o}_F^*$ then amounts to

$$k_1 + k_2 \equiv 0 \bmod 2 \ .$$

If it is not satisfied then the respective space of cusp forms will be $\{0\}$. Therefore we consider only

$$\mathbf{k} \in \{(3,3),\ (3,5),\ (4,4),\ (5,5)\} \ ,$$

levels of norm up to 3 and an approximation level of 5, which gives us the first ten terms in each theta series.

Some coefficients in Table 6.5 could not be computed correctly (indicated by "?"). It is very likely that this is due to some rounding errors that occurred when handling the complex harmonic polynomials, the coefficients of which are internally represented as floating point numbers. We would need a closer analysis of this problem in order to fill the remaining gaps in the table, but this is still in progress.

Table 6.4: $F = \mathbb{Q}(\sqrt{3})$, $h^+ = 2$, $\omega = \sqrt{3}$, $\mathbf{k} = (3,3)$

	$\mathfrak{n}$	(1)	$(1+\omega)$	$(3-2\omega)$
	$\mathcal{N}(\mathfrak{n})$	1	2	3
	max. dimension	0	0	1
	0	—	—	0
	1	—	—	1
	2	—	—	-8
	$2-\omega$	—	—	1
$\mathfrak{c}_1 = (1)$:	$2+\omega$	—	—	1
	3	—	—	-3
	$3-\omega$	—	—	6
	$3+\omega$	—	—	6
	4	—	—	16
	5	—	—	-46
	max. dimension	0	0	0
		—	—	—
$\mathfrak{c}_2 = (\omega)$:		—	—	—
		—	—	—

Table 6.5: $F = \mathbb{Q}(\sqrt{3})$, $h^+ = 2$, $\omega = \sqrt{3}$, $\mathbf{k} = (3, 5)$

	$\mathfrak{n}$	(1)	$(1+\omega)$	$(3-2\omega)$
	$\mathcal{N}(\mathfrak{n})$	1	2	3
	max. dimension	0	1	1
	0	—	0	0
	1	—	16	83
	2	—	-64	?
	$2-\omega$	—	$2+\omega$	$83(2+\omega)$
$\mathfrak{c}_1 = (1)$:	$2+\omega$	—	$16(31-15\omega)$	$6(29-15\omega)$
	3	—	?	$-54(13-\omega)$
	$3-\omega$	—	$-64(3+\omega)$	$-36(18+5\omega)$
	$3+\omega$	—	?	?
	4	—	?	$-264(13-\omega)$
	5	—	-1760	?
	max. dimension	0	0	0
		—	—	—
$\mathfrak{c}_2 = (\omega)$:		—	—	—
		—	—	—

Table 6.6: $F = \mathbb{Q}(\sqrt{3})$, $h^+ = 2$, $\omega = \sqrt{3}$, $\mathbf{k} = (4, 4)$

	$\mathfrak{n}$	(1)	$(1+\omega)$		$(3-2\omega)$	
	$\mathcal{N}(\mathfrak{n})$	1	2		3	
	max. dimension	1	5		5	
	0	0	0	0	0	0
	1	1	1	0	1	0
	2	4	0	1	4	0
	$2-\omega$	1	1	0	1	0
$\mathfrak{c}_1 = (1)$:	$2+\omega$	1	1	0	1	0
	3	21	21	0	-1	-2
	$3-\omega$	-24	-16	-2	-2	1
	$3+\omega$	-24	-16	-2	-2	1
	4	-80	-64	-4	-80	0
	5	170	170	0	170	0
	max. dimension	0	1		1	
	0	—	0		0	
	1	—	0		0	
	2	—	0		0	
	$2-\omega$	—	0		0	
$\mathfrak{c}_2 = (\omega)$:	$2+\omega$	—	0		0	
	3	—	0		1	
	$3-\omega$	—	1		0	
	$3+\omega$	—	1		0	
	4	—	0		0	
	5	—	0		0	

Table 6.7: $F = \mathbb{Q}(\sqrt{3})$, $h^+ = 2$, $\omega = \sqrt{3}$, $\mathbf{k} = (5, 5)$

	$\mathfrak{n}$	(1)	$(1+\omega)$		$(3-2\omega)$		
	$\mathcal{N}(\mathfrak{n})$	1	2		3		
	max. dimension	1	5		10		
	0	0	0	0	0	0	0
	1	1	1	0	1	0	0
	2	16	0	1	0	1	0
	$2-\omega$	1	1	0	1	0	0
$\mathfrak{c}_1 = (1)$:	$2+\omega$	1	1	0	1	0	0
	3	81	-111	12	33	3	-2
	$3-\omega$	0	32	-2	0	0	1
	$3+\omega$	0	32	-2	0	0	1
	4	256	128	8	256	0	-24
	5	-1054	290	-84	-798	-16	0
	max. dimension	0	1		1		
	0	—	0		0		
	1	—	0		0		
	2	—	0		0		
	$2-\omega$	—	0		0		
$\mathfrak{c}_2 = (\omega)$:	$2+\omega$	—	0		0		
	3	—	0		1		
	$3-\omega$	—	1		0		
	$3+\omega$	—	1		0		
	4	—	0		0		
	5	—	0		0		

6.3.3 Theta series for $\mathbb{Q}(\sqrt{2})$

The narrow class number is $h^+ = 1$, so there will only be one space $S_{\mathbf{k}}\big(\Gamma_0(\mathfrak{d}, \mathfrak{n}), 1\big)$ for each level $\mathfrak{n}$, as in the case $\mathbb{Q}(\sqrt{5})$.

The ring of integers is $\mathfrak{o}_F = \mathbb{Z}[\omega]$ where $\omega = \sqrt{2}$, and a fundamental unit of $\mathfrak{o}_F^*$ is $\varepsilon = 1 + \omega$, which has a positive as well as a negative real conjugate. As in the case $\mathbb{Q}(\sqrt{5})$ we only need to consider weight vectors $\mathbf{k}$ that satisfy

$$k_1 \equiv k_2 \equiv 0 \bmod 2 \ .$$

We choose to restrict ourselves to

$$\mathbf{k} \in \{(4,4),\ (4,6),\ (6,6)\} \ .$$

Table 6.8: $F = \mathbb{Q}(\sqrt{2})$, $h^+ = 1$, $\omega = \sqrt{2}$, $\mathbf{k} = (4,4)$

$\mathfrak{n}$	(1)	(ω)		$(1-2\omega)$			
$\mathcal{N}(\mathfrak{n})$	1	2		7			
max. dimension	1	6		11			
0	0	0	0	0	0	0	0
1	1	1	0	1	0	0	0
$2+\omega$	-2	0	1	0	1	0	0
2	-4	-8	-2	0	2	0	0
$2-\omega$	-2	0	1	0	1	0	0
$3+\omega$	-8	-8	0	0	0	1	0
3	26	26	0	2	0	0	-3
$3-\omega$	-8	-8	0	0	0	0	1
4	-16	32	24	-48	0	0	-4
5	138	138	0	-70	0	0	-26

Table 6.9: $F = \mathbb{Q}(\sqrt{2})$, $h^+ = 1$, $\omega = \sqrt{2}$, $\mathbf{k} = (4, 6)$

$\mathfrak{n}$	(1)	(ω)
$\mathcal{N}(\mathfrak{n})$	1	2
max. dimension	0	1
0	—	0
1	—	1
$2+\omega$	—	$4-2\omega$
2	—	8
$2-\omega$	—	$4+2\omega$
$3+\omega$	—	$-24+40\omega$
3	—	-114
$3-\omega$	—	$24+40\omega$
4	—	64
5	—	178

Table 6.10: $F = \mathbb{Q}(\sqrt{2})$, $h^+ = 1$, $\omega = \sqrt{2}$, $\mathbf{k} = (6, 6)$

$\mathfrak{n}$	(1)		(ω)			
$\mathcal{N}(\mathfrak{n})$	1		2			
max. dimension	4		33			
0	0	0	0	0	0	0
1	1	0	1	0	0	0
$2+\omega$	0	1	0	1	0	0
2	40	-2	0	0	1	0
$2-\omega$	0	1	0	1	0	0
$3+\omega$	8	-16	0	0	0	1
3	234	32	250	0	0	-2
$3-\omega$	8	-16	0	0	0	1
4	-416	-104	-864	-144	12	-4
5	3194	-320	3034	0	0	20

6.3.4 Theta series for $\mathbb{Q}[X]/(X^3 - X^2 - 2X + 1)$

As a final example we consider a number field that is not quadratic but of degree 3. The cubic field of smallest discriminant that is moreover totally real is $\mathbb{Q}[X]/(X^3 - X^2 - 2X + 1)$. It has discriminant 49.

Denote by ζ a primitive element of F, i. e.

$$F = \mathbb{Q}(\zeta) \ , \qquad \text{where} \qquad \zeta^3 - \zeta^2 - 2\zeta + 1 = 0 \ .$$

We find that the unit group of F is

$$\mathfrak{o}_F^* = \{\pm 1\} \times \langle 1 + \zeta - \zeta^2 \rangle \times \langle 1 - \zeta \rangle \ .$$

With respect to the three real embeddings $F \hookrightarrow \mathbb{R}$, the elements $1 + \zeta - \zeta^2$ and $1 - \zeta$ have signs $(+--)$ and $(+-+)$, respectively. It follows that $\mathfrak{o}_F^*$ contains elements of every possible sign pattern. Hence the condition $\operatorname{sgn}(\delta)^{\mathbf{k}} = 1$ for all $\delta \in \mathfrak{o}_F^*$ is satisfied if and only if

$$k_1 \equiv k_2 \equiv k_3 \equiv 0 \bmod 2 \ .$$

We will contend ourselves with $\mathbf{k} = (4, 4)$.

Recall that forms of level 1 and their translates are not included in the spaces that we compute! (Cf. again Theorem 5.2.1, part *(ii)* and Remark 5.2.8.)

Table 6.11: $F = \mathbb{Q}(\zeta)$, $h^+ = 1$, $\mathfrak{o}_F = \mathbb{Z}[\zeta]$, $\mathbf{k} = (4,4)$

$\mathfrak{n}$	$(1+2\zeta-\zeta^2)$	(2)	
$\mathcal{N}(\mathfrak{n})$	7	8	
max. dimension	1	4	
0	0	0	0
1	1	1	0
ζ^2	1	−2	−17
$1+\zeta^2$	28	−24	−1
$1-\zeta+\zeta^2$	−7	1	0
$2-\zeta$	1	1	0
$2-\zeta+\zeta^2$	−110	110	1
2	23	0	2
$2+\zeta$	−7	−24	−1
3	154	180	25
$3-\zeta$	28	−2	−17
$3+\zeta-\zeta^2$	1	1	0
$3+\zeta$	−110	110	1
4	17	64	0
$4+\zeta-\zeta^2$	28	−2	−17
$4-\zeta^2$	−7	−24	−1
$5-\zeta^2$	−110	110	1
5	−1904	182	616

List of Symbols

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