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Singular value decomposition and its application to numerical inversion for ray transforms in 2D vector tomography

Evgeny Y. Derevtsov, Anton V. Efimov, Alfred K. Louis and Thomas Schuster

Abstract. The operators of longitudinal and transverse ray transforms acting on vector fields on the unit disc are considered in the paper. The goal is to construct SVD-decompositions of the operators and invert them approximately by means of truncated decomposition for the parallel scheme of data acquisition. The orthogonal bases in the initial spaces and the image spaces are constructed using harmonic, Jacobi and Gegenbauer polynomials. Based on the obtained decompositions inversion formulas are derived and the polynomial approximations for the inverse operators are obtained. Numerical tests for data sets with different noise levels of smooth and discontinuous fields show the validity of the approach for the reconstruction of solenoidal or potential parts of vector fields from their ray transforms.

Keywords. Vector tomography, vector field, Radon transform, ray transform, singular value decomposition, orthogonal polynomials.

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1 Introduction

The development both of mathematical approaches and of systems of data measurements and processing induced new mathematical models in tomography such as thermotomography, diffusion tomography, vector and tensor tomography. The models appear due to the necessity of the reconstruction of properties of media with different degree of complication. Thus the tomography of vector fields arises for the description of vector characteristics of currents of fluids, vectors of electromagnetic fields inside the conductor in inhomogeneous media, and many others.

We consider here a method of solving the vector tomography problem in a plane in the case of parallel scheme of observation. As the problem of scalar tomography

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consists in the inversion of the Radon transform for a function, the vector tomography problem is the problem of inversion of ray transform operators applying to vector fields. In other words, one has to solve operator equations $Af = g$ of the first kind. Here A is a linear, bounded operator, and coincides with one of the ray transform operators. There are longitudinal or transverse ray transforms. In the operator equation g is a known right hand-side (data of tomographic measurements), and f is an unknown vector field to be determined.

The method of singular value decomposition (SVD) is well known and often used for inversion of compact linear operators. The idea of the approach consists in representing the operator in a form of series of singular numbers and basic elements in the image space. Then the inverse operator is a similar series with the reciprocal of the singular numbers and pre-images of the bases elements. SVD-decomposition for the operator of Radon transform acting on functions given in $\mathbb{R}^n$ is well known ([2, 3, 10, 11, 17, 19]). The SVD-decomposition of the ray transform in $\mathbb{R}^n$, acting on functions, is constructed in [16]. In 2D the SVD-decomposition was used to analyze the ill-posedness of the limited angle problem in [12], and for the construction of inversion formulas for sparsely sampled data in [15]. In the 2D-case the SVD-decomposition for the operator of longitudinal ray transform, acting on symmetric tensor fields and reconstructing its solenoidal part, is known for fan-beam scheme of observation [8]. The polynomial bases for different subspaces of vector fields given in the ball of $\mathbb{R}^3$ are constructed in [5]. In [20,21] reconstruction kernels for the longitudinal ray transform have been computed by means of the SVD of the 2D-Radon transform.

Here we reformulate the SVD-decomposition for the Radon operator, acting on functions given on the unit disk, in a form useful for the construction of SVD for the operators of ray transform acting on vector fields.

For constructing the SVD-decompositions in the vector case we exploit the fact that every solenoidal v or potential u vector field in the plane can be defined by a function (potential) according to

$$v = \nabla^{\perp}\psi,$$

for the solenoidal field, and

$$u = \nabla\varphi,$$

for potential field. Similarly to the scalar case we construct the potentials using classical polynomials. The constructed potentials form a non-orthogonal system for the potentials, but the arising solenoidal or potential vector fields are orthogonal in $L_2(S^1(B))$.

The algorithm for numerical inversion of the operators of longitudinal and transverse ray transforms is based on the constructed SVD-decompositions for the mentioned operators. Numerical simulations show the validity of the approach leading to good results of vector field reconstruction.

2 Preliminary definitions, objects and results

Let $B = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$ be the unit disk with center in the origin of a rectangular Cartesian coordinate system. Its boundary, the unit circle, is denoted as $\partial B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. A notation $Z = \{(\alpha, s) \in \mathbb{R}^2 : \alpha \in [0, 2\pi]$, $s \in [-1, 1]\}$ for a cylinder $[-1, 1] \times [0, 2\pi]$ is used.

The function space $L_2(B)$ contains in B square integrable (in Lebesgue measure sense) functions. We use also Hilbert spaces of Sobolev $H^k(B)$, with integer $k \geq 0$, consisting of in B square integrable functions which have derivatives up to the order k in $L_2(B)$. The subspaces $H_0^k(B)$ are the subsets of functions from $H^k(B)$ vanishing on ∂B together with all derivatives up to the $(k-1)$th order. The inner product is defined as

$$(f, g)_{H^k(B)} = \sum_{0\leq|a|\leq k} \int_B D^a f D^a g \, dxdy,$$

where a is multi-index, and an operation D^a is the operator of derivation by multi-index. The norm is generated by inner product,

$$\|f\|_{H^k(B)} = \left(\sum_{0\leq|a|\leq k} \int_B |D^a f|^2 \, dxdy \right)^{1/2}.$$

Using weight functions $\rho > 0$ in B the space $L_2(Z, \rho)$, consists of with weight ρ square integrable functions, and the inner product is defined as

$$(f, g)_{L_2(Z,\rho)} = \int_Z f(x)g(x)\rho(x) \, dx$$

for these spaces. The norm then is defined by

$$\|f\|_{L_2(Z,\rho)} = \left(\int_Z |f(x)|^2 \rho(x) \, dx \right)^{1/2}.$$

In particular with $\rho \equiv 1$ we have the classical Hilbert space $L_2(Z)$.

2.1 Radon transform

Let on the plane a function $f \in L_2(B)$ with a support lying in B, supp $f \subset B$, $f : B \to \mathbb{R}$, be given. The *Radon transform* $\mathcal{R} : L_2(B) \to L_2(Z, (1-s^2)^{-1/2})$

of the function $f \in L_2(B)$ is defined by

$$(\mathcal{R}f)(\xi, s) \equiv (\mathcal{R}f)(\alpha, s) = \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} f(s\xi + t\xi^{\perp})\, dt, \tag{2.1}$$

where $(s\xi + t\xi^{\perp}) = (s\cos\alpha - t\sin\alpha, s\sin\alpha + t\cos\alpha)$, the unit vector $\xi = (\cos\alpha, \sin\alpha)$ specifies the direction, the orthogonal direction is defined by the vector $\eta = \xi^{\perp} = (-\sin\alpha, \cos\alpha)$. The fixed direction ξ generates parallel beams in the plane, and every line of the beam is determined by a parameter s. Thus the line $L_{\xi,s}$ has the parametric equations $x = s\cos\alpha - t\sin\alpha$, $y = s\sin\alpha + t\cos\alpha$. For fixed direction ξ we also use the notation

$$\mathcal{R}_{\xi} f(s) = \mathcal{R}f(\xi, s). \tag{2.2}$$

2.2 Ray transforms of vector fields

Let in B a vector field $w = (w_1, w_2)$, $w : B \to \mathbb{R}^2$ be given. The *transverse ray transform* of the vector field w is defined as

$$\begin{aligned}(\mathcal{P}^{\perp}w)(\xi, s) &= \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \langle w(s\xi + t\xi^{\perp}), \xi\rangle\, dt \\ &= \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} (w_1\cos\alpha + w_2\sin\alpha)\, dt.\end{aligned} \tag{2.3}$$

The *longitudinal ray transform* of w is determined by the formula

$$\begin{aligned}(\mathcal{P}w)(\xi, s) &= \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \langle w(s\xi + t\xi^{\perp}), \xi^{\perp}\rangle\, dt \\ &= \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} (-w_1\sin\alpha + w_2\cos\alpha)\, dt.\end{aligned} \tag{2.4}$$

For fixed direction ξ the transforms can be rewritten as

$$(\mathcal{P}^{\perp}w)(\xi, s) = \mathcal{R}_{\xi}(w^{\top}\xi)(s), \tag{2.5}$$

$$(\mathcal{P}w)(\xi, s) = \mathcal{R}_{\xi}(w^{\top}\xi^{\perp})(s). \tag{2.6}$$

A vector field u is called *potential* if there exists a function φ (potential) such that $u = \nabla\varphi = (\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y})$. A vector field v is called *solenoidal* if its divergence is equal to 0,

$$\operatorname{div} v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0.$$

It is easy to check that in the 2D-case there exists a function ψ (potential) such that

$$v = \nabla^{\perp}\psi = \Big(-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x}\Big).$$

A Hilbert space of all square integrable vector fields given in B is denoted as $L_2(S^1(B))$. The inner product in the space is given as

$$(u, v)_{L_2(S^1(B))} = \int_B \langle u, v\rangle \, dxdy,$$

with associated norm

$$\|w\|_{L_2(S^1(B))} = \Big(\int_B (w_1^2 + w_2^2)\, dxdy\Big)^{1/2}.$$

It is well known ([9, 25]) that every vector field w can be decomposed uniquely into a sum of potential, solenoidal and harmonic parts (Helmholtz–Hodge decomposition),

$$w = \nabla\varphi + w^s + \nabla h, \quad \operatorname{div} w^s = 0, \quad \varphi|_{\partial B} = 0, \quad \langle w^s, \nu\rangle|_{\partial B} = 0,$$

where $\nabla\varphi$ is a potential vector field, $\varphi \in H_0^1(B)$, w^s is a solenoidal vector field, ∇h is a harmonic vector field, with h to be harmonic function, and ν is the outer normal to ∂B. It should be mentioned that the field ∇h is potential (by definition) as well as solenoidal, because $\operatorname{div} \nabla h = \Delta h = 0$.

We restrict our considerations on vector fields without harmonic vector part. For example these fields have potentials vanishing on the boundary or in infinity. See details in [4].

Lemma 2.1. *For potentials $\varphi \in H_0^1(B)$ the following relations hold:*

$$\mathcal{P}^{\perp}(\nabla\varphi) = \mathcal{P}(\nabla^{\perp}\varphi) = \frac{\partial}{\partial s} l(\mathcal{R}\varphi r)\,. \tag{2.7}$$

Proof. The proof easily follows from properties of the Radon transfrom. It can also be found in [4]. □

The fields w have the decomposition consisting of only two parts,

$$w = \nabla\varphi + w^s, \quad \varphi \in H_0^1(B), \quad \operatorname{div} w^s = 0, \quad \langle w^s, \nu\rangle|_{\partial B} = 0,$$

where the solenoidal field can be represented through a potential,

$$w^s = \nabla^{\perp}\psi, \quad \psi \in H_0^1(B).$$

The operator of transverse ray transform

$$\mathcal{P}^{\perp} : L_2(S^1(B)) \to L_2(Z, (1-s^2)^{-1/2})$$

has a nontrivial kernel consisting of solenoidal fields where the normal to the boundary ∂B is zero,

$$\mathcal{P}^{\perp} w^s \equiv 0, \quad \operatorname{div} w^s = 0, \quad \langle w^s, \nu \rangle |_{\partial B} = 0,$$

and this operator allows one to reconstruct only the potential part of any vector field w (cf. [6]). Thus the operator $\mathcal{P}^{\perp}$ may be treated as an operator mapping a potential vector field $u = \nabla\varphi$, $\varphi \in H_0^1(B)$, to its image $(\mathcal{P}^{\perp} u)(\alpha, s)$ corresponding to the transverse ray transform.

The operator of longitudinal ray transform

$$\mathcal{P} : L_2(S^1(B)) \to L_2(Z, (1-s^2)^{-1/2})$$

also has a nontrivial kernel. The kernel consists of potential fields vanishing on the boundary ∂B,

$$\mathcal{P} u \equiv 0, \quad u = \nabla\varphi, \quad \varphi|_{\partial B} = 0,$$

and from the longitudinal ray transform only the solenoidal part of a vector field w can be reconstructed (see [23]). In [23] also the corresponding inversion formulas can be found. The operator $\mathcal{P}$, similarly, can be treated as the mapping of a solenoidal vector field $v = \nabla^{\perp}\psi$, $\psi \in H_0^1(B)$, to its longitudinal ray transform value $(\mathcal{P} v)(\alpha, s)$.

For potentials vanishing on the boundary, both null-space results can be shown using (2.5) and (2.6).

The formulas (2.1)–(2.4) may be considered as integral equations,

$$\mathcal{R} f = g, \quad \mathcal{P}^{\perp} u = g, \quad \mathcal{P} v = g, \tag{2.8}$$

where $f \in L_2(B)$ is a function, and u, v are potential and solenoidal vector fields with potentials from $H_0^1(B)$, the right-hand side g is from $L_2(Z, (1-s^2)^{-1/2})$. The choice of the weight space $L_2(Z, (1-s^2)^{-1/2})$ as a space of images for investigating the operators is correct due to their continuity in the corresponding pairs of the spaces ([18]). Thus the three equations in (2.8) are treated as the operator equation $Aw = g$ of the first kind, A is one of the operators from (2.8). It should be mentioned also that due to data errors the right-hand side g is only known approximately as g_δ, and $\|g - g_\delta\|_{L_2(Z,(1-s^2)^{-1/2})} \le \delta$ with $g = Aw$.

The problem is to determine w (it may be the function f or vector fields u, v) for known g_δ. For this purpose we construct SVD-decompositions for the operators of transverse $\mathcal{P}^{\perp}$ and longitudinal $\mathcal{P}$ ray transforms. The constructed decompositions are then used for reconstruction of vector fields applying the truncated singular decompositions.

2.3 Method of SVD-decomposition

We apply the method of singular values decomposition for solving operator equations of the form (2.8). A brief description of the method is contained in this subsection. Let H, K be Hilbert spaces, and A be a linear bounded operator acting from H into K. The operator equation

$$Af = g \tag{2.9}$$

of the first kind is considered. For solving ill-posed problems like (2.9) the generalized inverse (Moore–Penrose inverse) A^+g is constructed. The operator

$$A^+ : K \to H$$

is defined on $\operatorname{Im} A + (\operatorname{Im} A)^\perp$ and known as the *operator of Moore–Penrose*. In general, i.e., if the range of A is infinite dimensional, the operator A^+ is not continuous. Therefore a family of continuous linear (regularizing) operators $(T_\gamma)_{\gamma>0}$ is considered. The operators $(T_\gamma)_{\gamma>0}$ are given on the whole space K, $T_\gamma : K \to H$, and the regularization property $\lim_{\gamma\to 0} T_\gamma g = A^+g$ is valid on the domain of definition of A^+. It is easy to see that if the operator A^+ is not bounded, then we have $\|T_\gamma\| \to \infty$ while $\gamma \to 0$. A regularizing family of operators allows to determine approximate solutions for (2.9) in the following sense. Let $g_\delta \in K$ be an approximation of g, i.e., $\|g_\delta - g\|_K \le \delta$. Let the function $\gamma(\delta)$ be such that if $\delta \to 0$, then

(i) $\gamma(\delta) \to 0$,

(ii) $\|T_{\gamma(\delta)}\|\delta \to 0$.

Hence (see [18]), for $\delta \to 0$ $\|T_{\gamma(\delta)}g_\delta - A^+g\|_H \to 0$. Thus the element $T_{\gamma(\delta)}g_\delta$ is close to A^+g if g_δ and g are close, and γ is the regularization parameter.

For operators admitting an SVD-decomposition one of the regularization methods is the truncated singular decomposition. The *singular value decomposition* (SVD) of an operator A consists of its representation

$$Af = \sum_{k=1}^{\infty} \sigma_k (f, f_k)_H g_k, \tag{2.10}$$

with normalized orthogonal bases (f_k), (g_k) of the spaces H and K, and the numbers $\sigma_k > 0$ are the *singular values* of the operator A. Let us assume that the sequence $\{\sigma_k\}$ is bounded. Then A is a continuous linear operator acting from H into K with the adjoint operator

$$A^*g = \sum_{k=1}^{\infty} \sigma_k (g, g_k)_K f_k.$$

The operators

$$A^*Af = \sum_{k=1}^{\infty} \sigma_k^2 (f, f_k)_H f_k, \quad AA^*f = \sum_{k=1}^{\infty} \sigma_k^2 (g, g_k)_K g_k$$

are self-adjoint in H and K respectively. The spectra of the operator A^*A contain the eigenvalues σ_k^2 corresponding to eigenfunctions f_k and in case of nontrivial null spaces the eigenvalue 0 with infinite multiplicity. The same proposition is valid for the operator AA^* with eigenfunctions g_k. The eigenfunctions are connected by the relations

$$A^* g_k = \sigma_k f_k, \quad Af_k = \sigma_k g_k. \tag{2.11}$$

Conversely, if (f_k), (g_k) are the normalized systems of eigenfunctions of the operators A^*A and AA^*, satisfying the relations (2.11), then the operator A admits the representation (2.10). In particular, compact operators admit always a SVD-decomposition (see [7]).

Theorem 2.2. *If an operator A admits singular decomposition* (2.10)*, then*

$$A^+ g = \sum_{\sigma_k > 0} \sigma_k^{-1} (g, g_k)_K f_k.$$

Proof. The proof of the theorem may be found, for example, in [18]. □

According to Theorem 2.2 the operator A^+ is unbounded if $\sigma_{k_j} \to 0$ for certain sequence $k_j \to \infty$. Then the operator A^+ can be regularized by means of *truncated singular value decomposition*,

$$T_\gamma g = \sum_{\sigma_k \geq \gamma} \sigma_k^{-1} (g, g_k)_K f_k, \quad \gamma > 0. \tag{2.12}$$

It follows from the theorem that $T_\gamma g \to A^+ g$ as $\gamma \to 0$, and $\|T_\gamma\| \leq \sup_{\sigma_k \geq \gamma} \sigma_k^{-1}$. The singular value decomposition also allows for constructing the approximate inverse where the solution is represented with the help of an a-priori calculated reconstruction kernel, see e.g. [22] and for directly calculating features of the solution see [14].

3 Construction of SVD-decompositions

This section is devoted to the construction of singular value decompositions for the Radon operator and the operators of transverse and longitudinal ray transforms acting on a vector field. Orthogonal normalized polynomials form the bases for the original spaces of the operators. Then their images are determined and the orthogonal normalized systems as well as the SVDs are constructed. Further, the inverse operator is constructed based on the obtained decompositions according to Theorem 2.2. Finally the polynomial approximation is constructed using the truncated singular values decomposition (2.12).

3.1 SVD-decomposition of the Radon transform

Restricting ourselves to the 2D-case we reformulate well-known (cf. [2, 3, 11, 17]) SVD-decompositions for the Radon transform $\mathcal{R} : L_2(B) \to L_2(Z, (1-s^2)^{-1/2})$ to a form more convenient for algorithm development and calculations. A family of functions

$$\Phi_{k,n}^{\cos,\sin}(x,y) = (1-x^2-y^2)^{\nu} H_k^{\cos,\sin}(x,y) P_n^{(k+1+2\nu,k+1)}(x^2+y^2), \quad k,n = 0,1,2,\dots, \quad \nu \geq -1/2. \tag{3.1}$$

with H_k defined in (6.1) and $P_N^{(\alpha,\beta)}$ the Jacobi polynomials for the interval [0, 1], is considered. In polar coordinates (r,φ) the functions have the form

$$\begin{Bmatrix} \tilde{\Phi}^{\cos} \\ \tilde{\Phi}^{\sin} \end{Bmatrix}_{k,n} (r,\varphi) = (1-r^2)^{\nu} r^k \begin{Bmatrix} \cos k\varphi \\ \sin k\varphi \end{Bmatrix} P_n^{(k+1+2\nu,k+1)}(r^2). \tag{3.2}$$

Proposition 3.1. *The functions* (3.1), (3.2) *form an orthogonal system in* $L_2(B)$ *with norms*

$$\|\Phi_{k,n}\|^2_{L_2(B)} = \frac{\pi k!}{2(k+2n+1)C^k_{n+k}} \frac{\Gamma(n+2\nu+1)}{\Gamma(k+n+2\nu+1)},$$

where $C^k_{n+k} = \frac{(n+k)!}{k!\,n!}$.

Proof. We fist prove that the following inner products are equal to 0,

$$\begin{aligned} \left(\Phi^{\cos}_{k,m}, \Phi^{\cos}_{l,n}\right)_{L_2(B)} &= 0, \\ \left(\Phi^{\cos}_{k,m}, \Phi^{\sin}_{l,n}\right)_{L_2(B)} &= 0, \\ \left(\Phi^{\sin}_{k,m}, \Phi^{\sin}_{l,n}\right)_{L_2(B)} &= 0 \end{aligned} \tag{3.3}$$

for all $k, m, l, n = 0, 1, 2, \dots$, except $k = l$, $m = n$ for the first and the third equalities. We remind the definition of inner product $(\cdot,\cdot)_{L_2(B)}$,

$$(f,g)_{L_2(B)} = \int_B f(x,y)g(x,y)\,dxdy = \int_0^1 \int_0^{2\pi} \tilde{f}(r,\varphi)\tilde{g}(r,\varphi) r\,dr d\varphi, \tag{3.4}$$

where the function with "tilde" are the functions depending on polar variables, $\tilde{f}(r,\varphi) = f(r\cos\varphi, r\sin\varphi)$. Below we use the representation (3.2) for our function system.

The second equality from equations (3.3) above holds for any $k, l \geq 1$ as we have $\int_0^{2\pi} \cos k\varphi \sin l\varphi \, d\varphi = 0$. The validity of the equality with $k = l = 0$ follows from the orthogonality of Jacobi polynomials (A.4) (the similar case will be considered in more details below).

We prove only the first formula from (3.3), as the third one is proved analogously. Let us write the inner product according to the formula (3.4),

$$I := \left(\Phi^{\cos}_{k,m}, \Phi^{\cos}_{l,n}\right)_{L_2(B)} = \frac{1}{2}\int_0^1\int_0^{2\pi} \tilde{\Phi}_{k,m}(r,\varphi)\tilde{\Phi}_{l,n}(r,\varphi)\, d(r^2)d\varphi$$
$$= \frac{1}{2}\int_0^1\int_0^{2\pi} \Big\{(1-r^2)^{2\nu} r^{k+l} P_n^{(k+1+2\nu,k+1)}(r^2) \times P_m^{(l+1+2\nu,l+1)}(r^2)\cos(k\varphi)\cos(l\varphi)d(r^2)\Big\}\, d\varphi.$$

We consider the two cases $k \neq l$ and $k = l, n \neq m$, separately:

- For $k \neq l$ it follows $I = 0$ as $\int_0^{2\pi} \cos(k\varphi)\cos(l\varphi)\, d\varphi = 0$.
- If $k = l, n \neq m$, then changing variables $t = r^2$, we obtain the expression
$$I = \frac{\pi}{2}\int_0^1 (1-t)^{2\nu} t^k P_n^{(k+1+2\nu,k+1)}(t) P_m^{(k+1+2\nu,k+1)}(t)\, dt.$$
The property of orthogonality of Jacobi polynomials (A.4) means that $I = 0$. The norms of Jacobi polynomials are well known ($k = l, n = m$), see [1] for example. □

We reformulate the next result which in fact is [10, Theorem 3.1] for the 2D-case, as it is fundamental for our further considerations.

Proposition 3.2. *Let $\mu > 0$, $k, n \geq 0$,*

$$\Psi(\varphi, s) = (1-s^2)^{\mu-1/2} C_{k+2n}^{(\mu)}(s) Y_k(\varphi)$$

with Gegenbauer polynomials $C_{k+2n}^{(\mu)}(s)$ and spherical harmonics $Y_k(\varphi)$ on the unit circle ∂B. Then $\Phi = \mathcal{R}^{-1}\Psi$ is given by

$$\Phi(\varphi, r) = c(k,n,\mu)(1-r^2)^{\mu-1} r^k P_n^{(k+2,k+1)}(r^2) Y_k(\varphi)$$

with Jacobi polynomials $P_n^{(p,q)}$ of degree n and indices p, q, and

$$c(k,n,\mu) = 2^{1-2\mu}\frac{\Gamma(k+2n+2\mu)\Gamma(n+1)}{\Gamma(k+2n+1)\Gamma(\mu)\Gamma(n+\mu)}.$$

The system of functions (3.1) (or (3.2)) with $\nu = 0$ is chosen as the basis in $L_2(B)$ for numerical simulation. The normalized orthogonal polynomial system in $L_2(B)$, in polar coordinates, is written as

$$\begin{Bmatrix} \tilde{F}^{\cos} \\ \tilde{F}^{\sin} \end{Bmatrix}_{k,n} (r,\varphi) = \sqrt{\frac{2(k+2n+1)}{\pi}} C_{n+k}^{k} r^k \begin{Bmatrix} \cos k\varphi \\ \sin k\varphi \end{Bmatrix} P_n^{(k+1,k+1)}(r^2), \qquad k,n = 0,1,2,\dots. \tag{3.5}$$

It should be mentioned that for $k = 0$ we have only one bases function, as

$$F_{0,n}^{\cos} = F_{0,n}^{\sin}.$$

For every fixed degree $N = k + 2n$ of polynomials there are exactly $N + 1$ bases polynomials, and the system (3.5) is complete in $L_2(B)$.

According to Proposition 3.2 the images of the system (3.5) after application of Radon transform are the polynomials

$$\begin{aligned} &\left(\mathcal{R} \begin{Bmatrix} F^{\cos} \\ F^{\sin} \end{Bmatrix}_{k,n} (x,y) \right)(\alpha,s) \\ &\qquad = (-1)^n \frac{2\sqrt{2}}{\sqrt{\pi(k+2n+1)}} \sqrt{1-s^2}\, C_{k+2n}^{(1)}(s) \begin{Bmatrix} \cos k\alpha \\ \sin k\alpha \end{Bmatrix} \\ &\qquad \overset{\text{def}}{=} \begin{Bmatrix} \Psi^{\cos} \\ \Psi^{\sin} \end{Bmatrix}_{k,n} (\alpha,s). \end{aligned} \tag{3.6}$$

Hence the following system of functions,

$$\begin{aligned} G_{k,n}^{\cos,\sin}(\alpha,s) &:= \sqrt{\frac{k+2n+1}{4\pi}} \Psi_{k,n}^{\cos,\sin}(\alpha,s) \\ &= (-1)^n \frac{\sqrt{2}}{\pi} \sqrt{1-s^2}\, C_{k+2n}^{(1)}(s) \begin{Bmatrix} \cos k\alpha \\ \sin k\alpha \end{Bmatrix}, \end{aligned}$$

is orthogonal and normalized in the space of images $L_2(Z,(1-s^2)^{-1/2})$ of the Radon transform. Thus the representation

$$\big(\mathcal{R} F_{k,n}^{\cos,\sin}(x,y)\big)(\alpha,s) = \sigma_{k,n}\, G_{k,n}^{\cos,\sin}(\alpha,s), \quad k,n = 0,1,2,\dots,$$

is valid, where the numbers

$$\sigma_{k,n} = 2\sqrt{\frac{\pi}{k+2n+1}}$$

are the singular values of the operator $\mathcal{R}$. Hence we get the following result.

Theorem 3.3 (SVD-decomposition for the Radon transform).

(1) *The singular value decomposition for the Radon transform has the form*

$$\mathcal{R}f = \sum_{k,n=0}^{\infty} \sigma_{k,n}\Big(\big(f, F_{k,n}^{\cos}\big)_{L_2(B)} G_{k,n}^{\cos} + \delta_{k,0}\big(f, F_{k,n}^{\sin}\big)_{L_2(B)} G_{k,n}^{\sin}\Big),$$

where $\delta_{k,0}$ *is Kronecker symbol, and the numbers* $\sigma_{k,n}$ *are the singular values,*

$$\sigma_{k,n} = 2\sqrt{\frac{\pi}{k+2n+1}}.$$

(2) *The action of the the inverse operator can be described by the formula*

$$\mathcal{R}^{-1}g = \sum_{k,n=0}^{\infty} \sigma_{k,n}^{-1}\Big(\big(g, G_{k,n}^{\cos}\big)_{L_2(Z,(1-s^2)^{-1/2})} F_{k,n}^{\cos} + \delta_{k,0}\big(g, G_{k,n}^{\sin}\big)_{L_2(Z,(1-s^2)^{-1/2})} F_{k,n}^{\sin}\Big). \tag{3.7}$$

Remark 3.4. By means of the truncated singular value decomposition, restricting in (3.7) by finite series, we obtain polynomial approximation for the reconstructing function,

$$\begin{aligned} f(x,y) &= \big(\mathcal{R}^{-1}g(\alpha,s)\big)(x,y) \\ &\approx \sum_{k,n=0}^{k+2n\le N} \sigma_{k,n}^{-1}\Big(\big(g, G_{k,n}^{\cos}\big)_{L_2(Z,(1-s^2)^{-1/2})} F_{k,n}^{\cos}(x,y) \\ &\qquad + \delta_{k,0}\big(g,\ G_{k,n}^{\sin}\big)_{L_2(Z,(1-s^2)^{-1/2})} F_{k,n}^{\sin}(x,y)\Big). \end{aligned} \tag{3.8}$$

The maximal degree N of the polynomials is chosen in dependence of the noise level.

3.2 SVD-decomposition of the ray transforms

Here the SVD-decompositions for the operators of transverse, $\mathcal{P}^\perp$, and longitudinal, $\mathcal{P}$, ray transforms are constructed. The operator

$$\mathcal{P}^\perp : L_2(S^1(B)) \to L_2(Z, (1-s^2)^{-1/2})$$

restricted to potential vector fields is considered, since the solenoidal fields are in the null space. The operator $\mathcal{P} : L_2(S^1(B)) \to L_2(Z, (1-s^2)^{-1/2})$ accordingly acts on solenoidal fields only. We would like to remind, that the support of the vector fields lies in $B \cup \partial B$, and their potentials vanish on the boundary ∂B, hence Lemma 2.1 is applicable for sufficiently smooth potentials.

Bases vector fields in the original space $L_2(S^1(B))$ are constructed on the foundation of an approach which may be called conditionally as "the method of potentials". The main idea consists in the preliminary construction of (in $L_2(B)$ non-orthogonal) systems of polynomials. Then we apply the operator ∇ for construction of potential vector fields, and the operator $\nabla^\perp$ for construction of solenoidal fields. We choose the following system of polynomials as the potentials,

$$\Phi_{k,n}^{\cos,\sin}(x,y) = (1-x^2-y^2)H_k^{\cos,\sin}(x,y)P_n^{(k+2,k+1)}(x^2+y^2), \tag{3.9}$$

$k,n = 0,1,\dots$, and in polar coordinates

$$\begin{Bmatrix} \widetilde{\Phi}^{\cos} \\ \widetilde{\Phi}^{\sin} \end{Bmatrix}_{k,n} (r,\varphi) = (1-r^2)r^k \begin{Bmatrix} \cos k\varphi \\ \sin k\varphi \end{Bmatrix} P_n^{(k+2,k+1)}(r^2).$$

The polynomial factor $(1-r^2)$ appears due to the boundary conditions for potentials. An application of the operator ∇ leads to a set of potential vector fields

$$(\mathbf{\Phi}_{k,n}^{\cos,\sin})^{\mathrm{pot}}(x,y) := \nabla\Phi_{k,n}^{\cos,\sin}(x,y). \tag{3.10}$$

The operator $\nabla^\perp$ gives solenoidal vector fields

$$(\mathbf{\Phi}_{k,n}^{\cos,\sin})^{\mathrm{sol}}(x,y) := \nabla^\perp\Phi_{k,n}^{\cos,\sin}(x,y). \tag{3.11}$$

Proposition 3.5. *The systems of potential* (3.10) *and solenoidal* (3.11) *vector fields are orthogonal in $L_2(S^1(B))$ with norms*

$$\|(\mathbf{\Phi}_{k,n}^{\cos,\sin})^{\mathrm{pot}}\|^2_{L_2(S^1(B))} = \|(\mathbf{\Phi}_{k,n}^{\cos,\sin})^{\mathrm{sol}}\|^2_{L_2(S^1(B))} = \frac{2\pi(n+1)^2}{(k+2n+2)(C_{n+k}^k)^2}.$$

Proof. Let us consider the potentials

$$\widetilde{\Phi}_{k,n}^{\cos}(r,\varphi) = (1-r^2)r^k\cos(k\varphi)P_n^{(k+2,k+1)}(r^2)$$

and

$$\widetilde{\Phi}_{l,m}^{\cos}(r,\varphi) = (1-r^2)r^l\cos(l\varphi)P_m^{(l+2,l+1)}(r^2).$$

Immediate evaluations of $\langle\nabla\widetilde{\Phi}_{k,n}^{\cos}, \nabla\widetilde{\Phi}_{l,m}^{\cos}\rangle$ show that the integrand is

$$I = F(r^2)\cos(k\varphi)\cos(l\varphi) + G(r)\cos((k-l)\varphi),$$

where the function F depends on r^2, and G depends on r only. Hence after integration the obtained expression over φ from 0 to 2π we get zero for $k \neq l$. It means that the potential vector fields $\nabla\widetilde{\Phi}_{k,n}^{\cos}$ and $\nabla\widetilde{\Phi}_{l,m}^{\cos}$ are orthogonal for any $m,n,k \neq l$ in $L_2(S^1(B))$.

Considering the case $k = l$, $n \neq m$, at first we denote

$$P_n := P_n^{(k+2,k+1)}(r^2), \quad P_m := P_m^{(k+2,k+1)}(r^2), \quad H_k := r^k \cos(k\varphi).$$

After changing variables $t = r^2$, we obtain for the inner product $\langle \nabla \widetilde{\Phi}_{k,n}^{\cos}, \nabla \widetilde{\Phi}_{k,m}^{\cos} \rangle$ the expression for the integrand

$$\begin{aligned} I = (H^k)^2 \big(4t - 4k(1-t)\big) P_n P_m &+ k^2 (1-t)^2 t^{k-1} P_n P_m \\ &+ (H^k)^2 (1-t)\big(-4t + 2k(1-t)\big)\big(P_n (P_m)' + (P_n)' P_m\big) \\ &+ 4(H^k)^2 t (1-t)^2 (P_n)'(P_m)'. \end{aligned}$$

As $(H_k)^2 = t^k \cos^2(k\varphi)$ and the other terms are independent of the variable φ, it follows that

$$\int_0^{2\pi} (I/\pi)\, d\varphi = f(t) + 2k^2 (1-t)^2 t^{k-1} P_n P_m,$$

where

$$f(t) = f_1 + f_2 + f_3$$

with

$$\begin{aligned} f_1 &= t^k \big(4t - 4k(1-t)\big) P_n P_m, \\ f_2 &= t^k (1-t)\big(-4t + 2k(1-t)\big)\big(P_n P_m\big)', \\ f_3 &= 4t^{k+1}(1-t)^2 (P_n)'(P_m)'. \end{aligned}$$

Differentiation by parts of the integrand $f_2(t)$,

$$\begin{aligned} \int_0^1 f_2(t)\, dt &= \int_0^1 t^k (1-t)\big(-4t + 2k(1-t)\big)\big(P_n P_m\big)'\, dt \\ &= -\int_0^1 t^{k-1} \Big(4t^2 - 4kt(1-t) + 2k^2(1-t)^2 \\ &\qquad\qquad - (4k+4)t(1-t)\Big) P_n P_m\, dt, \end{aligned}$$

and adding the corresponding integrals with integrands $2k^2(1-t)^2 t^{k-1} P_n P_m$ and f_1 lead to

$$4(k+1) \int_0^1 (1-t) t^k P_n^{(k+2,k+1)}(t) P_m^{(k+2,k+1)}(t) dt. \tag{3.12}$$

For $n \neq m$ the integral is equal to zero as the Jacobi polynomials $P_n^{(p,q)}(t)$ and $P_m^{(p,q)}(t)$ are orthogonal on $(0, 1)$ with the weight $(1-t)^{p-q} t^{q-1}$.

Finally we consider the term $f_3(t) = 4t^{k+1}(1-t)^2(P_n)'(P_m)'$. Taking in mind the formula

$$\left(P_n^{(p,q)}\right)' = -\frac{n(n+p)}{q} P_{n-1}^{(p+2,q+1)},$$

we obtain for the corresponding integral

$$\frac{4nm(n+k+2)(m+k+2)}{(k+1)^2} \int_0^1 (1-t)^2 t^{k+1} P_{n-1}^{(k+4,k+2)}(t) P_{m-1}^{(k+4,k+2)}(t)\, dt. \tag{3.13}$$

The integral is equal to zero for $m \neq n$ due to the orthogonality of the Jacobi polynomials.

The orthogonality of solenoidal (3.11) vector fields follows from the property $\langle \nabla f, \nabla g \rangle = \langle \nabla^\perp f, \nabla^\perp g \rangle$ for any f, g.

For calculation of the norms we use (3.12), (3.13) with $m = n$ and the properties of Jacobi polynomials. □

Hence we obtain that the system of potentials

$$\begin{Bmatrix} \widetilde{F}^{\cos} \\ \widetilde{F}^{\sin} \end{Bmatrix}_{k,n} (r, \varphi) := \sqrt{\frac{k+2n+2}{2\pi} \frac{C_{n+k}^k}{n+1}}\, (1-r^2) r^k \begin{Bmatrix} \cos k\varphi \\ \sin k\varphi \end{Bmatrix} P_n^{(k+2,k+1)}(r^2) \tag{3.14}$$

forms an orthogonal and in the space $L_2(S^1(B))$ normalized system of potential vector fields

$$(\boldsymbol{F}_{k,n})^{\mathrm{pot}}(x, y) := \nabla F_{k,n}(x, y), \quad k, n = 0, 1, \ldots, \tag{3.15}$$

and solenoidal vector fields,

$$(\boldsymbol{F}_{k,n})^{\mathrm{sol}}(x, y) := \nabla^\perp F_{k,n}(x, y), \quad k, n = 0, 1, \ldots. \tag{3.16}$$

Proposition 3.6. *The functions*

$$\begin{aligned} \left(\mathcal{P}^\perp \begin{Bmatrix} \boldsymbol{F}^{\cos} \\ \boldsymbol{F}^{\sin} \end{Bmatrix}^{\mathrm{pot}}_{k,n} (x, y)\right)(\alpha, s) &= \left(\mathcal{P} \begin{Bmatrix} \boldsymbol{F}^{\cos} \\ \boldsymbol{F}^{\sin} \end{Bmatrix}^{\mathrm{sol}}_{k,n} (x, y)\right)(\alpha, s) \\ &= a(k,n)\sqrt{1-s^2}\, C_{k+2n+1}^{(1)}(s) \begin{Bmatrix} \cos k\alpha \\ \sin k\alpha \end{Bmatrix} \qquad (3.17) \\ &=: \begin{Bmatrix} \Psi^{\cos} \\ \Psi^{\sin} \end{Bmatrix}_{k,n} (\alpha, s), \end{aligned}$$

where

$$a(k,n) = (-1)^{n+1} \frac{2\sqrt{2}}{\pi(k+2n+2)},$$

form orthogonal system in the space $L_2(Z,(1-s^2)^{-1/2})$ *of images of transverse (longitudinal) ray transform. The norms are*

$$\|\Psi_{k,n}^{\cos,\sin}\|^2_{L_2(Z,(1-s^2)^{-1/2})} = \frac{4\pi}{k+2n+2}. \tag{3.18}$$

Proof. According to the Proposition 3.2 the images of the Radon transform of potentials (3.14), without taking in mind constants, are the functions

$$(1-s^2)^{3/2} C_{k+2n}^{(2)}(s) Y_k(\alpha). \tag{3.19}$$

As $F_{k,n}(x,y) = 0$ on ∂B, we get

$$\mathcal{P}^{\perp}\big(\nabla F_{k,n}\big) = \mathcal{P}\big(\nabla^{\perp} F_{k,n}\big) = \frac{\partial(\mathcal{R}F_{k,n})}{\partial s},$$

using Lemma 2.1. After differentiation of (3.19) by s and application of the formula for derivatives (A.9), we have

$$\begin{aligned} J_{k,n} &:= \frac{\partial}{\partial s}\Big((1-s^2)^{3/2} C_{k+2n}^{(2)}(s) Y_k(\alpha)\Big) \\ &= -(k+2n+3)(1-s^2)^{1/2}\big(s C_{k+2n}^{(2)}(s) - C_{k+2n-1}^{(2)}(s)\big) Y_k(\alpha). \end{aligned}$$

Then, using the recurrence formula (A.7), we get

$$J_{k,n} = -\frac{(k+2n+1)(k+2n+3)}{2}(1-s^2)^{1/2} C_{k+2n+1}^{(1)}(s) Y_k(\alpha).$$

Now we use the properties of orthogonality (A.6) on $(-1,1)$ of Gegenbauer polynomials $C_m^{(1)}$ with the weight $(1-s^2)^{-1/2}$, and of trigonometric functions $Y_k(\alpha)$ on $[0,2\pi]$. It follows that the inner products in $L_2(Z,(1-s^2)^{-1/2})$ of the functions $J_{k,n}$, $J_{l,m}$ with $k \neq l$ or $n \neq m$ are equal to zero.

The formulas for the constants $a(k,n)$ and (3.18) for the norms follow immediately from Propositions 3.2 and 3.5, (3.14), and the norms of the Gegenbauer polynomials. □

We conclude that the system of functions

$$\begin{aligned} G_{k,n}^{\cos,\sin}(\alpha,s) &:= \sqrt{\frac{k+2n+2}{4\pi}}\,\Psi_{k,n}^{\cos,\sin}(\alpha,s) \\ &= (-1)^{n+1}\frac{\sqrt{2}}{\pi}\sqrt{1-s^2}\, C_{k+2n+1}^{(1)}(s) \begin{Bmatrix} \cos k\alpha \\ \sin k\alpha \end{Bmatrix} \end{aligned}$$

is orthogonal and normalized in the space $L_2(Z,(1-s^2)^{-1/2})$. Thus the formulas

$$\begin{aligned}\Big(\mathcal{P}^{\perp}(\boldsymbol{F}_{k,n}^{\cos,\sin})^{\mathrm{pot}}(x,y)\Big)(\alpha,s) &= \Big(\mathcal{P}(\boldsymbol{F}_{k,n}^{\cos,\sin})^{\mathrm{sol}}(x,y)\Big)(\alpha,s)\\ &= \sigma_{k,n}\, G_{k,n}^{\cos,\sin}(\alpha,s)\end{aligned}$$

are valid, with $k,n=0,1,\ldots$. The numbers $\sigma_{k,n}=2\sqrt{\frac{\pi}{k+2n+2}}$ are the singular values of the operators $\mathcal{P}^{\perp}$ and $\mathcal{P}$.

Hence we obtain the following results.

Theorem 3.7. (1) *The singular value decomposition of the operator $\mathcal{P}^{\perp}$ has the form*

$$\begin{aligned}\mathcal{P}^{\perp}u = \sum_{k,n=0}^{\infty}\sigma_{k,n}\Big(&\big(u,(\boldsymbol{F}_{k,n}^{\cos})^{\mathrm{pot}}\big)_{L_2(S^1(B))}G_{k,n}^{\cos}\\ &+\delta_{k,0}\big(u,(\boldsymbol{F}_{k,n}^{\sin})^{\mathrm{pot}}\big)_{L_2(S^1(B))}G_{k,n}^{\sin}\Big),\end{aligned}$$

where $\sigma_{k,n}=2\sqrt{\frac{\pi}{k+2n+2}}$ are the singular values.

(2) *The required potential vector field u is calculated using the inverse operator by the formula*

$$\begin{aligned}u &= (\mathcal{P}^{\perp})^{-1}g\\ &= \sum_{k,n=0}^{\infty}\sigma_{k,n}^{-1}\Big(\big(g,G_{k,n}^{\cos}\big)_{L_2(Z,(1-s^2)^{-1/2})}(\boldsymbol{F}_{k,n}^{\cos})^{\mathrm{pot}}\\ &\qquad+\delta_{k,0}\big(g,G_{k,n}^{\sin}\big)_{L_2(Z,(1-s^2)^{-1/2})}(\boldsymbol{F}_{k,n}^{\sin})^{\mathrm{pot}}\Big).\end{aligned} \tag{3.20}$$

Theorem 3.8. (1) *The SVD-decomposition of the operator $\mathcal{P}$ is*

$$\begin{aligned}\mathcal{P}v = \sum_{k,n=0,1,2,\ldots}\sigma_{k,n}\Big(&\big(v,(\boldsymbol{F}_{k,n}^{\cos})^{\mathrm{sol}}\big)_{L_2(S^1(B))}G_{k,n}^{\cos}\\ &+\delta_{k,0}\big(v,(\boldsymbol{F}_{k,n}^{\sin})^{\mathrm{sol}}\big)_{L_2(S^1(B))}G_{k,n}^{\sin}\Big),\end{aligned}$$

where $\sigma_{k,n}=2\sqrt{\frac{\pi}{k+2n+2}}$ are the singular values.

(2) *The required solenoidal vector field is calculated using the inverse operator by the formula*

$$\begin{aligned}v &= \mathcal{P}^{-1}g\\ &= \sum_{k,n=0}^{\infty}\sigma_{k,n}^{-1}\Big(\big(g,G_{k,n}^{\cos}\big)_{L_2(Z,(1-s^2)^{-1/2})}(\boldsymbol{F}_{k,n}^{\cos})^{\mathrm{sol}}\\ &\qquad+\delta_{k,0}\big(g,G_{k,n}^{\sin}\big)_{L_2(Z,(1-s^2)^{-1/2})}(\boldsymbol{F}_{k,n}^{\sin})^{\mathrm{sol}}\Big).\end{aligned} \tag{3.21}$$

4 Numerical simulations

In this section algorithms for the approximate reconstruction of a scalar function by its Radon transform and vector fields by their ray transforms are developed. The results of numerical simulation are presented. The data for the tomographic problems (Radon or ray transforms) are calculated by means of Simpson formulas according to (2.1), (2.3), (2.4). The noise, besides the numerical approximations, is modeled using of pseudorandom uniformly distributed parameter of corresponding level in percents.

4.1 Reconstruction of functions by Radon transform

The subsection contains results of simulation not only with SVD-decomposition algorithms but also results obtained by least squares method (LSM) with a bases containing non-orthogonal polynomials of the form $x^k y^n$, and the orthogonal polynomial bases constructed in Section 3.1. We assume below that all the formulas for functions and vector fields are valid in the unit disk B, and outside they are extended by 0.

Test 1. Functions with different degrees of smoothness

At first we investigate an application of the SVD-decomposition algorithm to functions with different degrees of smoothness. We consider a function which is the sum of a discontinuous function, a function with discontinuous first derivatives, and C^1-function

$$f(x, y) = f_1(x, y) + f_2(x, y) + f_3(x, y),$$

where f_1 is a characteristic function of a disk with radius 0.3, f_2 is a part of the cone with discontinuous first derivatives, and f_3 has discontinuous second derivatives.

Quality and running time of the algorithm in dependence of the regularization parameter N is depicted in Figure 1. Figure 2 contains the visualized results for certain numbers N.

The discontinuous function in the model significantly influences the error. The best reconstruction is for $N = 260$ with relative error $\delta_{rel} = 7,92\%$. The quality of reconstruction is good, and we may conclude that SVD-algorithm is appropriate even for discontinuous functions.

Figure 3 contains the results obtained by the *Least-Squares-Method* (LSM) for the reconstruction of the same function. We conclude that the use of orthogonal polynomials in LSM allows for better results in comparison with non-orthogonal polynomials, but the results of SVD-algorithm are the best.

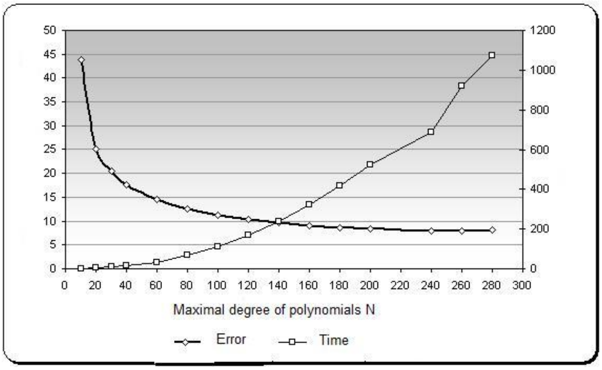

Figure 1. Reconstruction of functions with different degree of smoothness.

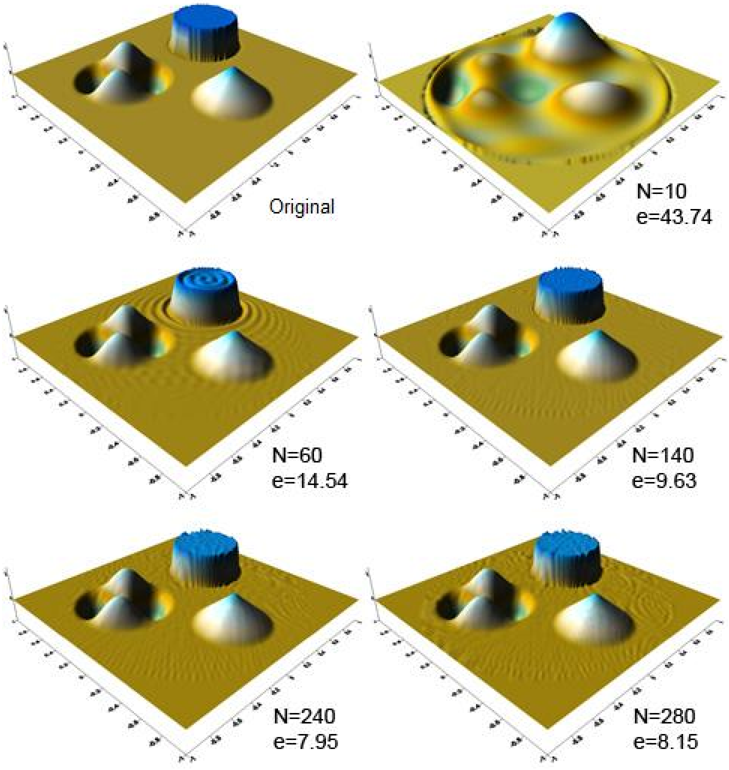

Figure 2. Original and its approximations for certain N. SVD-decomposition.

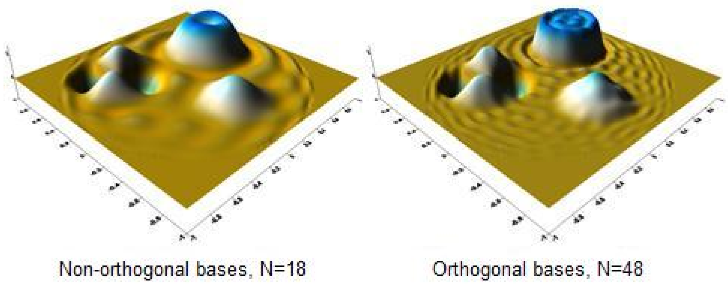

Figure 3. Reconstruction of the function by LSM-algorithm.

4.2 Reconstruction of vector fields

The behavior of the SVD-algorithm applied to vector fields reconstruction is similar to the reconstruction of a scalar function so we restrict ourselves to only one numerical test which is devoted to the influence of the noise level to the quality of reconstruction. The conclusions from analyzing the results are valid not only for vector fields reconstruction but for reconstruction of the functions, too.

We use a rather smooth function as potential,

$$\varphi(x, y) = (x + 0.2)^2(y + 0.1)^2(0.6^2 - (x + 0.2)^2 - (y + 0.1)^2)^2,$$

and construct the potential vector field $u(x, y) = (u_1, u_2) = \nabla\varphi(x, y)$. The vector field u has discontinuous first derivatives.

Five different levels of noise ranging from 0% to 10% with a step in 2.5% are used in the tests. The graphs of errors are presented in the Figure 4.

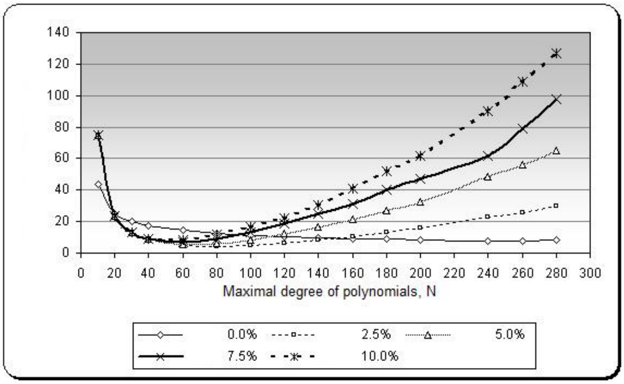

Figure 4. The influence of a noise on an accuracy of reconstruction of vector field.

In Figure 5 and Figure 6 the results of reconstruction of the first component of the vector field are displayed for $N = 60$ and $N = 200$. It is easy to see that the small level noise does not influence significantly the quality of reconstruction. But if the level of noise is more then 5%, the quality is worse, and there appear noticeable artifacts in the reconstructed vector field. The artifacts are more pronounced for series with large degrees of polynomials.

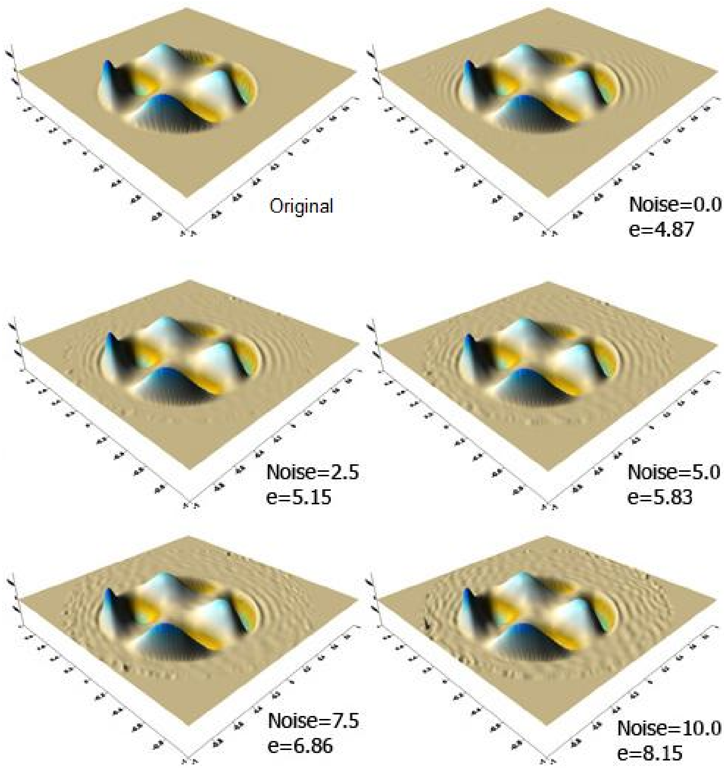

Figure 5. Original and approximations for vector field (first component) in dependence of the level of a noise, $N = 60$.

5 Conclusions

For the used resolution of 200×200 reconstruction points, the polynomials of degrees 60–80 are preferable for reconstruction of the objects from noisy data. For example, if $N = 200$ and the level of noise is 10.0% (Figure 6), then the

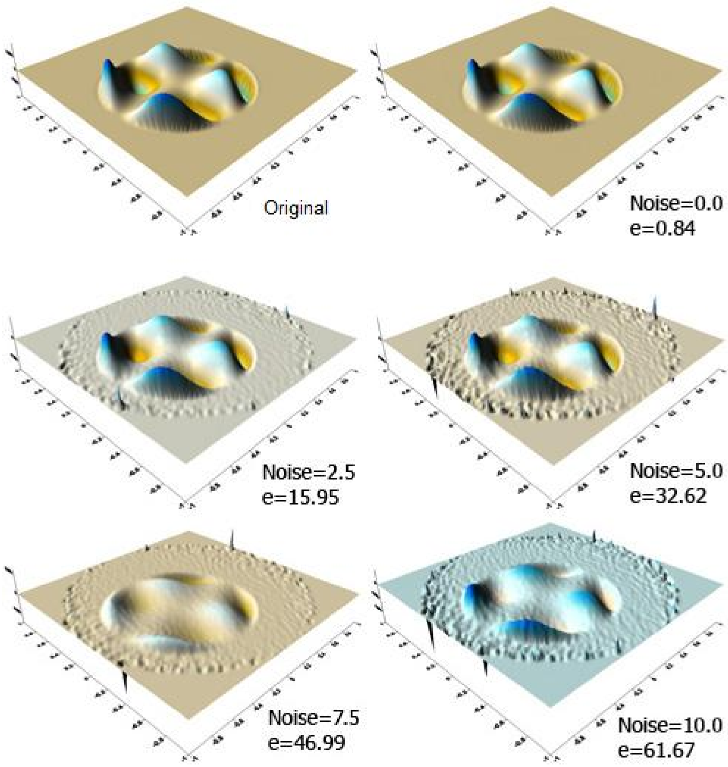

Figure 6. Original and approximations for vector field (first component) in dependence of the level of a noise, $N = 200$.

object is "lost" in background artifacts arising near the boundary of the unit disk B. The use of polynomials of higher degrees leads, as expected, to a completely bad quality of the reconstruction.

Such a behavior of the algorithm is caused by the circumstance that the support of the ray (or Radon) transforms becomes equal to the whole cylinder Z, and in particular even for s near ± 1 we have nonzero values of the ray transforms. The Gegenbauer polynomials of high degrees appearing in the formulas for calculation the coefficients for the truncated SVD-decompositions have at $s \to 1$ large oscillations. Thus the accuracy of calculations near the boundary is rather bad. If we add noise not to all values of s, $-1 \leq s \leq 1$, but, for example, only for s such that $0 \leq |s| \leq 0.98$, then we obtain much better results. In particular even for the polynomials of very large degrees ($N = 280$) the relative error is 41,27%. If we take the interval $[0, 0.99]$ for $|s|$, $N = 280$, then the error is 45,10%. In Figure 7 the

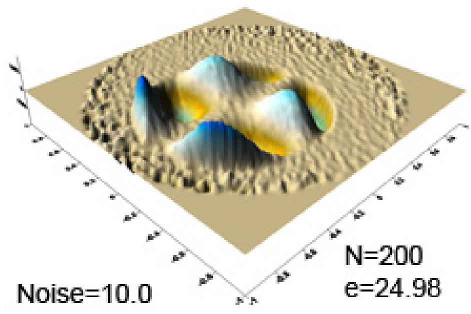

Figure 7. Reconstruction of the first component of vector field at addition of the noise from 0 to 0,98 by s.

first component of the reconstructed vector field is displayed. Noise on the level of 10.0% is added with $|s|$ changing from 0 to 0.98. We observe much better quality of reconstruction in comparison with those in Figure 6.

This shortage of the algorithm can be mended using an adaptive grid that is condensed as $|s|$ is near 1. Another possibility is a compression of the data with respect to s to a necessary level (as the numerical test showed it is enough to move from the boundary on 1–2%), and extend them by zero near the boundary. Then after finishing of the algorithm it is possible to return to the initial scale of the reconstructing object using properties of the ray transforms, such as linear change of the base, homogeneity, etc. Moreover, in real tomographic devices an object is located inside the domain of measurements and does not touch the boundaries where the sources and detectors are disposed. Hence the values of Radon or ray transforms are zero for $a < |s| < 1$ with appropriate a. The a priori information concerning the properties of "noise" of a concrete tomograph also is important, as this knowledge allows to apply the methods of multi-scale analysis and filtration before application of SVD-algorithm.

The numerical simulation showed that polynomials of high degree for the approximation leads to perturbations in the reconstruction which reduce the accuracy of the approximation. This feature is not only connected with SVD-approach, but is a natural characteristic of behavior of polynomials with high degree (large oscillations, especially near the boundary of Z). In addition the running time of the algorithm increases quadratically with the degree N. Moreover, the total accuracy can be improved by more precise calculations in intermediate calculations. For example, adaptive grids or formulas of higher accuracy can be used for the integration steps. But the main decision is how to choose the optimal parameter of regularization N for good results with minimal time of computations.

A Appendix: Classical orthogonal polynomials

For the construction of the bases as well as in original spaces as in the spaces of images of the operators of Radon and ray transform, we need special polynomials depending on one or two variables. They are harmonic polynomials, Jacobi and Gegenbauer polynomials. We list their main properties used in the paper.

A.1 Harmonic polynomials

Let us consider harmonic polynomials of two variables given in a unit disk B. By definition they are solutions of the Laplace equation

$$\triangle H(x,y) = 0.$$

In polar coordinate system (r,φ) they may be written in the form

$$\begin{aligned} H_k^{\cos}(r,\varphi) &= r^k \cos k\varphi, \\ H_k^{\sin}(r,\varphi) &= r^k \sin k\varphi, \end{aligned} \tag{A.1}$$

where k is a degree of the polynomial. For harmonics of degree zero ($k = 0$) we define

$$H_0^{\cos}(r,\varphi) = H_0^{\sin}(r,\varphi) \overset{\text{def}}{\equiv} \frac{1}{\sqrt{2}}. \tag{A.2}$$

The bases for vector fields are constructed by differentiation of potentials, containing harmonic polynomials therefor there are useful relations allowing to calculate their derivatives

$$\begin{aligned} \frac{\partial}{\partial x} H_k^{\cos} &= k H_{k-1}^{\cos}, & \frac{\partial}{\partial x} H_k^{\sin} &= k H_{k-1}^{\sin}, \\ \frac{\partial}{\partial y} H_k^{\cos} &= -k H_{k-1}^{\sin}, & \frac{\partial}{\partial y} H_k^{\sin} &= k H_{k-1}^{\cos}. \end{aligned} \tag{A.3}$$

A.2 Jacobi polynomials for the interval (0, 1)

The Jacobi polynomials given on the interval (0, 1) are calculated according to the formula

$$P_n^{(p,q)}(t) = 1 + \sum_{k=1}^{n} (-1)^k C_n^k \frac{(p+n)(p+n+1)\ldots(p+n+k-1)}{q(q+1)\ldots(q+k-1)} t^k.$$

On the interval (0, 1) the polynomials are orthogonal with a weight $t^{q-1}(1-t)^{p-q}$,

$$\int_0^1 t^{q-1}(1-t)^{p-q} P_n^{(p,q)}(t) P_m^{(p,q)}(t) dt = \frac{n!\Gamma(q)\Gamma(p-q+n+1)}{q(q+1)\dots(q+n-1)(p+2n)\Gamma(p+n)}\delta_{nm}, \tag{A.4}$$

where δ_{nm} is the Kronecker symbol.

The bases for vector fields are constructed by differentiation of potentials, containing not only harmonic polynomials, but Jacobi polynomials too. There are useful relations allowing to calculate their derivatives

$$\frac{d}{dt}\left(P_n^{(p,q)}(t)\right) = -\frac{n(n+p)}{q} P_{n-1}^{(p+2,q+1)}(t). \tag{A.5}$$

A.3 Gegenbauer polynomials for the interval $(-1, 1)$

The Gegenbauer polynomials $C_n^{(\mu)}(t)$ are the main tool for constructing bases in spaces of images of Radon and ray transforms. They are determined by the formula

$$C_n^{(\mu)}(t) = \frac{1}{\Gamma(\mu)} \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{\Gamma(\mu+n-k)}{k!(n-2k)!}(2t)^{n-2k}.$$

The polynomials are orthogonal with a weight $(1-t^2)^{\mu-1/2}$ on the interval $(-1, 1)$,

$$\int_{-1}^1 C_n^{(\mu)}(t) C_m^{(\mu)}(t)(1-t^2)^{\mu-1/2} dt = \frac{\pi 2^{1-2\mu}\Gamma(n+2\mu)}{n!(n+\mu)\Gamma^2(\mu)}\delta_{nm}. \tag{A.6}$$

Recurrence formulas are

$$(n+1)C_{n+1}^{(\mu)}(t) - 2\mu\left(tC_n^{(\mu+1)}(t) - C_{n-1}^{(\mu+1)}(t)\right) = 0, \tag{A.7}$$

$$(n+2\mu)C_n^{(\mu)}(t) - 2\mu\left(C_n^{(\mu+1)}(t) - tC_{n-1}^{(\mu+1)}(t)\right) = 0, \tag{A.8}$$

and the derivatives fulfill

$$(1-t^2)\frac{dC_n^{(\mu)}(t)}{dt} = (n+2\mu-1)C_{n-1}^{(\mu)}(t) - ntC_n^{(\mu)}(t) = (n+2\mu)tC_n^{(\mu)}(t) - (n+1)C_{n+1}^{(\mu)}(t). \tag{A.9}$$

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Author information

Evgeny Y. Derevtsov, Sobolev Institute of Mathematics,
Acad. Koptyug prosp., 4, 630090 Novosibirsk, Russia.
E-mail: dert@math.nsc.ru

Anton V. Efimov, Novosibirsk State University,
Pirogova St., 2, 630090 Novosibirsk, Russia.
E-mail: eprant@gmail.com

Alfred K. Louis, Institute of Applied Mathematics, Saarland University,
66041 Saarbrücken, Germany.
E-mail: louis@num.uni-sb.de

Thomas Schuster, Department for Mathematics,
Carl von Ossietzky University Oldenburg, 26129 Oldenburg, Germany.
E-mail: thomas.schuster@uni-oldenburg.de